W03 - Examples

Partial fractions

09 - Partial fractions with repeated factor

Find the PFD:

$$\frac{3x-9}{x^3+3x^2-4}$$

Solution

1. Check! Numerator is smaller than denominator (degree-wise).
2. Factor the denominator.
    - Rational roots theorem: $x=1$ is a zero.
    - Divide by $x-1$:

    $$\frac{x^3+3x^2-4}{x-1}=x^2+4x+4$$

    - Factor again:

    $$x^2+4x+4=(x+2)^2$$

    - Final factored form:

    $$x^3+3x^2-4 \qquad \ggg \qquad (x-1)(x+2)^2$$

3. Write the generic PFD.
    - Allow all lower powers:

    $$\frac{3x-9}{(x-1)(x+2)^2}=\frac{A}{x-1}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$$

4. Solve for A, B, and C.
    - Multiply across by the common denominator:

    $$3x-9=A(x+2)^2+B(x-1)(x+2)+C(x-1)$$

    - For A, set $x=1$, obtain:

    $$\begin{aligned} & 3\cdot 1-9=A(1+2)^2+B\cdot 0+C\cdot 0 \\ \ggg \qquad & -6=9A \\ \ggg \qquad & A=-2/3 \end{aligned}$$

    - For C, set $x=-2$, obtain:

    $$\begin{aligned} & 3\cdot(-2)-9=A\cdot 0+B\cdot 0+C\cdot(-3) \\ \ggg \qquad & -15=-3C \\ \ggg \qquad & C=5 \end{aligned}$$

    - For B, insert prior results and solve.
        - Plug in A and C:

        $$3x-9=-\frac{2}{3}(x+2)^2+B(x-1)(x+2)+5(x-1)$$

- Now plug in another convenient x, say $x = 3$:

$$0 = -\frac{2}{3} \cdot 5^2 + B \cdot 2 \cdot 5 + 5 \cdot 2$$

$$\frac{50}{3} - 10 = 10B \qquad \gg\gg \qquad B = \frac{2}{3}$$

5. Plug in A, B, C for the final answer.
   - Final answer:

$$\frac{3x-9}{x^3+3x^2-4} = \frac{-2/3}{x-1} + \frac{2/3}{x+2} + \frac{5}{(x+2)^2}$$

10 - Partial fractions - repeated quadratic, linear tops

Compute the integral:

$$\int \frac{x^3+1}{(x^2+4)^2}\,dx$$

Solution

1. Compute the partial fraction decomposition.
   - Check that numerator degree is lower than denominator. ✓
   - Factor denominator completely. ✓ (No real roots.)
   - Write generic PFD:

$$\frac{x^3+1}{(x^2+4)^2} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2}$$

   - Notice "linear over quadratic" in first term.
   - ⚠ Notice repeated factor: sum with incrementing powers up to 2.
   - Common denominators and solve:

$$x^3+1 = (Ax+B)(x^2+4) + Cx + D$$

$$\gg\gg \quad x^3+1 = Ax^3 + Bx^2 + (4A+C)x + 4B + D$$

$$\gg\gg \quad A = 1,\; B = 0$$

$$\gg\gg \quad C = -4,\; D = 1$$

   - Therefore:

$$\frac{x^3+1}{(x^2+4)^2} = \frac{x}{x^2+4} + \frac{-4x+1}{(x^2+4)^2}$$

2. Integrate by terms.
   - Integrate the first term using $u = x^2+4$:

$$\int \frac{x}{x^2+4}dx \overset{u=x^2+4}{\gg\gg} \frac{1}{2}\int \frac{du}{u}$$

$$\gg\gg \quad \frac{1}{2}\ln|u| + C \quad \gg\gg \quad \frac{1}{2}\ln\left|x^2+4\right| + C$$

   - Break up the second term:

$$\frac{-4x+1}{(x^2+4)^2} \quad \gg\gg \quad \frac{-4x}{(x^2+4)^2} + \frac{1}{(x^2+4)^2}$$

- Integrate the first term of RHS:

$$\int \frac{-4x}{(x^2+4)^2}\,dx \quad \gg\gg \quad -2\int \frac{du}{u^2}$$

$$\gg\gg \quad \frac{2}{u}+C \quad \gg\gg \quad \frac{2}{x^2+4}+C$$

- Integrate the second term of RHS:

$$\int \frac{dx}{(x^2+4)^2} \quad \overset{x=2\tan\theta}{\gg\gg} \quad \int \frac{2\sec^2\theta\,d\theta}{16\sec^4\theta}$$

$$\gg\gg \quad \frac{1}{8}\int \cos^2\theta\,d\theta \quad \gg\gg \quad \frac{1}{16}\theta+\frac{1}{32}\sin(2\theta)+C$$

Simpson's Rule

11 - Simpson's Rule on the Gaussian distribution

The function e^{x^2} is very important for probability and statistics, but it cannot be integrated analytically.

Apply Simpson's Rule to approximate the integral:

$$\int_0^1 e^{x^2}\,dx$$

with $\Delta x = 0.1$ and $n = 10$. What error bound is guaranteed for this approximation?

Solution

We need a table of values of x_i and $y_i = f(x_i)$:

x_i :	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$f(x_i)$:	$e^{0.0^2}$	$e^{0.1^2}$	$e^{0.2^2}$	$e^{0.3^2}$	$e^{0.4^2}$	$e^{0.5^2}$	$e^{0.6^2}$	$e^{0.7^2}$	$e^{0.8^2}$	$e^{0.9^2}$	$e^{1.0^2}$
$\approx$	1.000	1.010	1.041	1.094	1.174	1.284	1.433	1.632	1.896	2.248	2.718

These can be plugged into the Simpson Rule formula to obtain our desired approximation:

$$S_{10} = \frac{1}{3}\cdot 0.1\cdot \Big(1.000+4\cdot 1.010+2\cdot 1.041+4\cdot 1.094+\cdots+2\cdot 1.896+4\cdot 2.248+2.718\Big)$$

$$\approx 1.463$$

To find the error bound we need to find the smallest number we can manage for K_4.

Take four derivatives and simplify:

$$f^{(4)}(x) = (12+48x^2+16x^4)e^{x^2}$$

On the interval $x \in [0,1]$, this function is maximized at $x = 1$. Use that for the optimal K_4:

$$f^{(4)}(1.000) = 206.589$$

Finally we plug this into the error bound formula:

$$\frac{K_4(b-a)^5}{180n^4} = \frac{206.589\cdot 1.000^5}{180\cdot 10^4} \approx 0.0001$$

$$\gg\gg \quad \text{Error}(S_{10}) \le 0.0001$$