Name: Solutions

Homework 7.2 – Trigonometric Integrals

Evaluate the following integrals without referring to tables or formulas. (LT 1c)

1) $\int \sin^3 t \cos^3 t dt$

$u = \sin t \quad du = \cos t\, dt$

$$\int \sin^3 t \cos^3 t\, dt = \int \sin^3 t \cos^2 t \cos t\, dt$$
$$= \int \sin^3 t (1 - \sin^2 t) \cos t\, dt$$
$$= \int u^3 (1 - u^2)\, du$$
$$= \int (u^3 - u^5)\, du$$
$$= \frac{u^4}{4} - \frac{u^6}{6} + C$$
$$= \frac{\sin^4 t}{4} - \frac{\sin^6 t}{6} + C$$

$$\boxed{\frac{\sin^4 t}{4} - \frac{\sin^6 t}{6} + C}$$

2) $\int_0^{2\pi} \sin^2 x dx$

$$\int_0^{2\pi} \sin^2 x\, dx = \int_0^{2\pi} \frac{1}{2}(1 - \cos 2x)\, dx$$
$$= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right)\Big|_0^{2\pi}$$
$$= \frac{1}{2}(2\pi)$$
$$= \pi$$

$$\boxed{\pi}$$

3) $\int \tan^2 x \sec^2 x dx$

$u = \tan x \quad dv = \sec^2 x\, dx$

$\int \tan^2 x \sec^2 x\, dx = \int u^2\, du$

$= \frac{u^3}{3} + C$

$= \frac{\tan^3 x}{3} + C$

$\boxed{\frac{\tan^3 x}{3} + C}$

4) $\int_0^{\frac{\pi}{2}} \sin^3 x dx$

$u = \cos x \quad du = -\sin x\, dx$

$\int_0^{\frac{\pi}{2}} \sin^3 x\, dx = \int_0^{\frac{\pi}{2}} \sin^2 x \sin x\, dx$

$= \int_0^{\frac{\pi}{2}} -(1-\cos^2 x)(-\sin x)\, dx$

$= \int_1^0 -(1-u^2)\, du$

$= -\left(u - \frac{u^3}{3}\right)\Big|_1^0$

$= -(-)\left(1-\frac{1}{3}\right)$

$= \frac{2}{3}$

$\boxed{\frac{2}{3}}$

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Worksheet 7.3a – Trigonometric Substitution

Evaluate the following integrals without referring to tables or formulas. (LT: 1f)

1) $\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}}\,dx$

$x = \sin\theta$

$dx = \cos\theta\, d\theta$

$$\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}}\,dx = \int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \cos\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{6}} \sin^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{2}(1-\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{\frac{\pi}{6}}$$

$$= \frac{1}{2}\left(\frac{\pi}{6} - \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\right) =$$

$$\boxed{\frac{\pi}{12} - \frac{\sqrt{3}}{8}}$$

2) $\int \frac{dx}{x^3\sqrt{x^2-4}}$

$x = 2\sec\theta$

$dx = 2\tan\theta\sec\theta\, d\theta$

$\sec\theta = \frac{x}{2}$

$\sin\theta = \frac{\sqrt{x^2-4}}{x}$

$\cos\theta = \frac{2}{x}$

$\sin 2\theta = 2\sin\theta\cos\theta = \frac{4\sqrt{x^2-4}}{x^2}$

$$\int \frac{dx}{x^3\sqrt{x^2-4}} = \int \frac{2\tan\theta\sec\theta\, d\theta}{(8\sec^3\theta)\sqrt{4\sec^2\theta-4}}$$

$$= \int \frac{1}{8\sec^2\theta}\, d\theta$$

$$= \int \frac{1}{8}\cos^2\theta\, d\theta$$

$$= \int \frac{1}{16}(1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{16}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$\frac{1}{16}\left(\sec^{-1}\frac{x}{2} + \frac{1}{2}\left(\frac{4\sqrt{x^2-4}}{x^2}\right)\right) + C$$

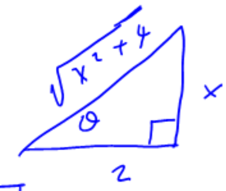
$$\boxed{\frac{1}{16}\left(\sec^{-1}\frac{x}{2} + \frac{2\sqrt{x^2-4}}{x^2}\right) + C}$$

3) $\int \frac{dx}{\sqrt{x^2+4}}$

$x = 2\tan\theta$

$dx = 2\sec^2\theta \, d\theta$

$\tan\theta = \frac{x}{2}$



$\sec\theta = \frac{\sqrt{x^2+4}}{2}$

$$\int \frac{dx}{\sqrt{x^2+4}} = \int \frac{2\sec^2\theta}{\sqrt{4+4\tan^2\theta}}\, d\theta$$

$$= \int \sec\theta \, d\theta$$

$$= \ln|\tan\theta + \sec\theta| + C$$

$$= \ln\left|\frac{x}{2} + \frac{\sqrt{x^2+4}}{2}\right| + C$$

$$= \ln\left|x + \sqrt{x^2+4}\right| \underbrace{- \ln 2 + C}_{\text{combine}}$$

$$\boxed{\ln\left|x + \sqrt{x^2+4}\right| + C}$$

Supplement: $\int \frac{dx}{\sqrt{x^2+4x+13}}$ Hint: Complete the square first.

$x^2+4x+13 = x^2+4x+4+9$

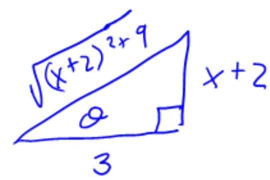
$= (x+2)^2+9$

$$\int \frac{dx}{\sqrt{x^2+4x+13}} = \int \frac{dx}{\sqrt{(x+2)^2+9}}$$

$x+2 = 3\tan\theta$

$dx = 3\sec^2\theta \, d\theta$

$\tan\theta = \frac{x+2}{3}$



$$= \int \frac{3\sec^2\theta \, d\theta}{\sqrt{9\tan^2\theta + 9}}$$

$$= \int \sec\theta \, d\theta$$

$$= \ln|\tan\theta + \sec\theta| + C$$

$$= \ln\left|\frac{x+2}{3} + \frac{\sqrt{(x+2)^2+9}}{3}\right| + C$$

$$= \ln\left|x+2 + \sqrt{(x+2)^2+9}\right| \underbrace{- \ln 3 + C}_{\text{combine}}$$

$$\boxed{\ln\left|x+2 + \sqrt{(x+2)^2+9}\right| + C}$$

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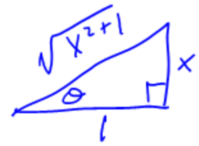
Worksheet 7.3b – Trigonometric Substitution

1) Evaluate $\int \frac{x^2}{(x^2+1)^{\frac{3}{2}}}dx$. (LT: 1f) $x = \tan\theta \quad dx = \sec^2\theta\, d\theta$



$\tan\theta = x$

$\sec\theta = \sqrt{x^2+1}$

$\sin\theta = \frac{x}{\sqrt{x^2+1}}$

$$\int \frac{x^2}{(x^2+1)^{3/2}}dx = \int \frac{\tan^2\theta}{(\tan^2\theta+1)^{3/2}}(\sec^2\theta)\,d\theta$$

$$= \int \frac{\tan^2\theta\sec^2\theta}{(\sec^2\theta)^{3/2}}$$

$$= \int \frac{\tan^2\theta}{\sec\theta}\,d\theta$$

$$= \int \frac{\sec^2\theta - 1}{\sec\theta}\,d\theta$$

$$= \int (\sec\theta - \cos\theta)\,d\theta$$

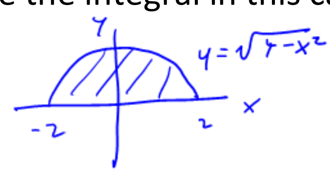
$$= \ln|\tan\theta + \sec\theta| - \sin\theta + C$$

$$= \ln|x + \sqrt{x^2+1}| - \frac{x}{\sqrt{x^2+1}} + C$$

$$\boxed{\ln|x+\sqrt{x^2+1}| - \frac{x}{\sqrt{x^2+1}} + C}$$

2) Find the average height of a point on the semicircle, $y = \sqrt{4-x^2}$, for $-2 \le x \le 2$. Remember, the average value of a function, $f(x)$, between x = a and x = b is $f_{AVE} = \frac{1}{(b-a)}\int_a^b f(x)dx$.

Also, be clever (by sketching the curve) about how you evaluate the integral in this case. (LT: 1e)



$$f_{ave} = \frac{1}{2-(-2)}\int_{-2}^{2}\sqrt{4-x^2}\,dx$$

$$= \frac{1}{4}\left(\frac{4\pi}{2}\right)$$

$$= \frac{\pi}{2}$$

$$\boxed{\frac{\pi}{2}}$$

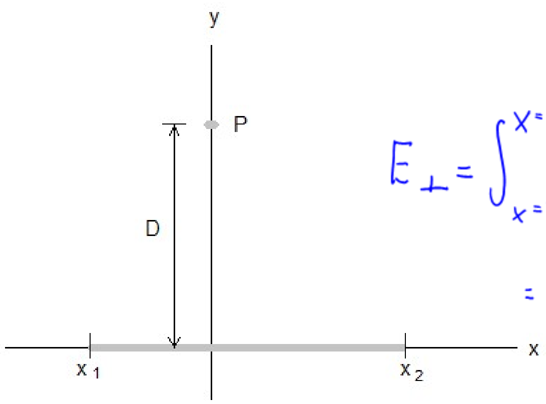
3) A charged wire creates an electric field at a point P located at a distance D from the wire. The component $E_{\perp}$ of the field perpendicular to the wire (in N/C) is

$$E_{\perp} = \int_{x_1}^{x_2} \frac{k\lambda D}{\left(x^2 + D^2\right)^{\frac{3}{2}}} dx$$

Where λ is the charge density (in C/m), $k = 8.99 \times 10^9\, N - m^2/C^2$ and x_1 and x_2 are as in the figure. Suppose that $\lambda = 6 \times 10^{-4}\, C/m$, and D = 3m. Find $E_{\perp}$ if $x_1 = -15m$ and $x_2 = 15m$. (Use symbols until the very end. Then you may use a calculator.) (LT: 1f)

$x = D \tan\theta$

$dx = D \sec^2\theta \, d\theta$



$$E_{\perp} = \int_{x=x_1}^{x=x_2} \frac{K\lambda D}{(D^2\tan^2\theta + D^2)^{3/2}} D\sec^2\theta \, d\theta$$

$$= \int_{x=x_1}^{x=x_2} \frac{K\lambda D^2 \sec^2\theta}{D^3 \sec^3\theta} d\theta$$

$$= \int_{x=x_1}^{x=x_2} \frac{K\lambda}{D\sec\theta} d\theta$$

$$= \int_{x=x_1}^{x=x_2} \frac{K\lambda}{D} \cos\theta \, d\theta$$

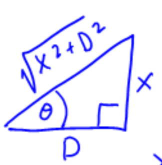
$$= \frac{K\lambda}{D} \sin\theta \Big|_{x=x_1}^{x=x_2}$$

$$= \frac{K\lambda}{D} \left(\frac{x}{\sqrt{x^2+D^2}}\right) \Big|_{x_1}^{x_2}$$

$$= \frac{2K\lambda}{D} \left(\frac{x_2}{\sqrt{x_2^2+D^2}}\right)$$

$$= \frac{2\left(8.99\times10^9 \frac{N\text{-}m^2}{C^2}\right)\left(6\times10^{-4}\frac{C}{m}\right)(15\,m)}{3m\sqrt{15^2+3^2\, m^2}}$$

$\tan\theta = \frac{x}{D}$



$\sin\theta = \frac{x}{\sqrt{x^2+D^2}}$

$\boxed{3.53\times10^6\ N/C}$