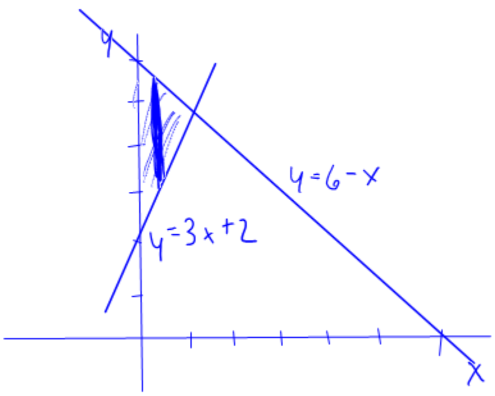
Name: Solutions

Worksheet 6.3 – Volumes by Cylindrical Shells

For problems 1 and 2, compute the volume obtained by rotating the region enclosed by the given curves about the given axis. Be sure to sketch the region. (LT 2c)

1) $y = 3x + 2,\ y = 6 - x,\ x = 0$; about y-axis



intersection points: $3x+2=6-x$

$4x=4$

$x=1$

$V_i \approx 2\pi x_i\left(6-x_i-(3x_i+2)\right)\Delta x$

$V=\int_0^1 2\pi x\left(6-x-(3x+2)\right)dx$

$=\int_0^1 2\pi x(4-4x)\,dx$

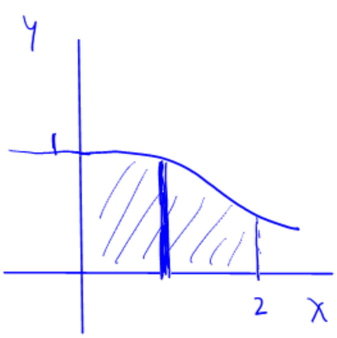
$=\int_0^1 2\pi(4x-4x^2)\,dx$

$=2\pi\left(2x^2-\frac{4x^3}{3}\right)\Big|_0^1$

$=2\pi\left(2-\frac{4}{3}\right)=2\pi\left(\frac{2}{3}\right)=\frac{4\pi}{3}$

$\boxed{\frac{4\pi}{3}}$

2) $y = \frac{1}{\sqrt{x^2+1}},\ x = 0,\ x = 2,\ y = 0$; about y-axis



$V_i = 2\pi x_i\left(\frac{1}{\sqrt{x_i^2+1}}\right)\Delta x$

$V=\int_0^2 2\pi x\left(\frac{1}{\sqrt{x^2+1}}\right)dx$

$u=x^2+1 \quad du=2x\,dx$

$V=\int_1^5 \pi\,\frac{du}{\sqrt{u}}$

$=\frac{\pi u^{1/2}}{\left(\frac{1}{2}\right)}\Big|_1^5$

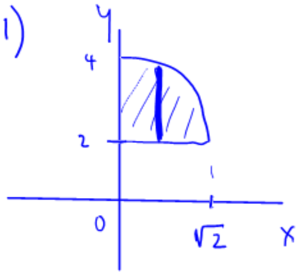
$=2\pi(\sqrt{5}-1)$

$\boxed{2\pi(\sqrt{5}-1)}$

3) Consider the region in the first quadrant, R, bounded by $y = 4 - x^2$, $y = 2$, $x = 0$. (LT 2c).

a) Set up an integral to find the volume of revolution of region R about the x-axis.

Option 1)

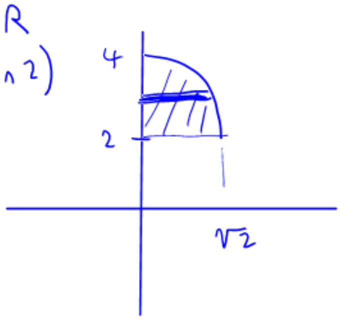


$$V_i = \pi\left((4 - x_i^2)^2 - 2^2\right)\Delta x$$ (washer)

$$V_x = \int_0^{\sqrt{2}} \pi\left((4-x^2)^2 - 2^2\right) dx$$

OR

option 2)



$r = y \quad h = \sqrt{4-y}$ ← $y = 4 - x^2$, $x = \sqrt{4-y}$

$$V_i = 2\pi y_i\left(\sqrt{4-y_i}\right)\Delta y$$ (cylindrical shells)

$$V_x = \int_2^4 2\pi y\sqrt{4-y}\, dy$$

b) Set up an integral to find the volume of revolution of region R about the y-axis.

option 1) $V_y = \int_0^{\sqrt{2}} 2\pi x\left((4-x^2)-2\right) dx$ (cylindrical shells)

OR

option 2) $V_y = \int_2^4 \pi\left(\sqrt{4-y}\right)^2 dy$ (disks)

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Worksheet 7.1a – Integration by Parts

Evaluate the integral using Integration by Parts (IBP)

1) $\int (2x+9)e^x dx$ Hint: Use $u = 2x+9, dv = e^x dx$.

$\int u\,dv = uv - \int v\,du$

$u = 2x+9 \quad dv = e^x dx$

$du = 2dx \quad v = e^x$

$\int (2x+9)e^x dx = (2x+9)e^x - \int 2e^x dx$

$= (2x+9)e^x - 2e^x + C$

$= (2x+7)e^x + C$

$\boxed{(2x+7)e^x + C}$

2) $\int x^3 \ln x\, dx$ Hint: Use $u = \ln x, dv = x^3 dx$

$u = \ln x \quad dv = x^3 dx$

$du = \frac{1}{x} dx \quad v = \frac{x^4}{4}$

$\int u\,dv = uv - \int v\,du$

$\int x^3 \ln x\, dx = \frac{x^4}{4}\ln x - \int \frac{x^3}{4} dx$

$= \frac{x^4}{4}\ln x - \frac{x^4}{16} + C$

$\boxed{\frac{x^4}{4}\ln x - \frac{x^4}{16} + C}$

3) $\int_0^3 xe^{4x} dx$

$u = x \quad dv = e^{4x} dx$

$du = dx \quad v = \frac{1}{4}e^{4x}$

$\int_0^3 xe^{4x} dx = \frac{x}{4}e^{4x}\Big|_0^3 - \int_0^3 \frac{1}{4}e^{4x} dx$

$= \left[\frac{x}{4}e^{4x} - \frac{1}{16}e^{4x}\right]_0^3$

$= e^{4x}\left(\frac{x}{4} - \frac{1}{16}\right)\Big|_0^3$

$= e^{12}\left(\frac{3}{4} - \frac{1}{16}\right) - 1\left(-\frac{1}{16}\right)$

$= \frac{11}{16}e^{12} + \frac{1}{16}$

$\boxed{\frac{11}{16}e^{12} + \frac{1}{16}}$

Name: Solutions

Worksheet 7.1b – Integration by Parts

Evaluate the following integrals: (LT 1a, 1b)

1) $\int x^2 \sin x dx$

$u = x^2 \quad dv = \sin x\, dx$

$du = 2x\, dx \quad v = -\cos x$

$\int x^2 \sin x\, dx = -x^2 \cos x + \int 2x \cos x\, dx$

$u = 2x \quad dv = \cos x\, dx$

$du = 2dx \quad v = \sin x$

$= -x^2 \cos x + \left[2x \sin x - \int 2 \sin x\, dx\right]$

$= -x^2 \cos x + 2x \sin x + 2\cos x + C$

$\boxed{-x^2 \cos x + 2x \sin x + 2\cos x + C}$

2) $\int x \cos\left(x^2\right) dx$

$u = x^2 \quad du = 2x\, dx$

$\int x \cos(x^2)\, dx = \int \frac{1}{2} \cos u\, du$

$= \frac{1}{2} \sin u + C$

$= \frac{1}{2} \sin(x^2) + C$

$\boxed{\frac{1}{2} \sin(x^2) + C}$

3) $\int \tan^{-1} x dx$

$u = \tan^{-1} x \quad dv = dx$

$du = \frac{1}{1+x^2} dx \quad v = x$

$\int \tan^{-1} x \, dx = x \tan^{-1} x - \int \frac{x}{1+x^2} dx$

$= x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C$

$\int \frac{x}{1+x^2} dx$

$u = 1+x^2 \quad du = 2x dx$

$= \int \frac{1}{2} \frac{du}{u}$

$= \frac{1}{2} \ln|u| + C$

$= \frac{1}{2} \ln|1+x^2| + C$

$= \frac{1}{2} \ln(1+x^2) + C$

$\boxed{x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C}$

Supplement: Evaluate $\int \sqrt{x} e^{\sqrt{x}} dx$ using IBP, substitution, or both. (LT 1a, 1b)

first substitute: $a = \sqrt{x}$

$a^2 = x$

$2a \, da = dx$

$\int \sqrt{x} e^{\sqrt{x}} dx = \int a e^a (2a) da$

$= \int 2a^2 e^a da$

$u = 2a^2 \quad dv = e^a da$

$du = 4a \, da \quad v = e^a$

$= 2a^2 e^a - \int 4a e^a da$

$u = 4a \quad dv = e^a da$

$du = 4 da \quad v = e^a$

$= 2a^2 e^a - \left[4a e^a - \int 4 e^a da\right]$

$= 2a^2 e^a - 4a e^a + 4e^a + C$

$= 2e^a (a^2 - 2a + 2) + C$

$= 2e^{\sqrt{x}} (x - 2\sqrt{x} + 2) + C$

$\boxed{2e^{\sqrt{x}} (x - 2\sqrt{x} + 2) + C}$