11.7 Strategy for Testing Series

1. (a) $\sum\limits_{n=1}^{\infty} \frac{1}{5^n} = \sum\limits_{n=1}^{\infty} \left(\frac{1}{5}\right)^n$ is a geometric series with ratio $r = \frac{1}{5}$. Since $|r| = \frac{1}{5} < 1$, the series converges.

(b) $\frac{1}{5^n + n} < \frac{1}{5^n}$ for $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \frac{1}{5^n + n}$ converges by direct comparison with $\sum\limits_{n=1}^{\infty} \frac{1}{5^n}$, which converges because it is a geometric series with $|r| = \frac{1}{5} < 1$.

3. (a) $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\frac{n+1}{3^{n+1}} \cdot \frac{3^n}{n}\right| = \lim\limits_{n\to\infty} \frac{n+1}{3n} = \lim\limits_{n\to\infty} \left(\frac{n}{3n} + \frac{1}{3n}\right) = \lim\limits_{n\to\infty} \left(\frac{1}{3} + \frac{1}{3n}\right) = \frac{1}{3} < 1$,

so the series $\sum\limits_{n=1}^{\infty} \frac{n}{3^n}$ is absolutely convergent (and therefore convergent) by the Ratio Test.

(b) $\lim\limits_{n\to\infty} \frac{3^n}{n} \overset{\text{H}}{=} \lim\limits_{n\to\infty} \frac{3^n \ln 3}{1} = \infty$, so the series $\sum\limits_{n=1}^{\infty} \frac{3^n}{n}$ diverges by the Test for Divergence.

5. (a) Use the Limit Comparison Test with $a_n = \frac{n}{n^2+1}$ and $b_n = \frac{1}{n}$.

$\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{n^2}{n^2+1} = \lim\limits_{n\to\infty} \frac{1}{1 + 1/n^2} = 1 > 0$. Since $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ is the divergent harmonic series, the series $\sum\limits_{n=1}^{\infty} \frac{n^2}{n^2+1}$ also diverges.

(b) $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} \sqrt[n]{\left(\frac{n}{n^2+1}\right)^n} = \lim\limits_{n\to\infty} \frac{n}{n^2+1} = \lim\limits_{n\to\infty} \frac{1/n}{1+1/n^2} = \frac{0}{1} = 0 < 1$, so the series $\sum\limits_{n=1}^{\infty} \left(\frac{n}{n^2+1}\right)^n$ converges by the Root Test.

7. (a) Since $n! > n^2$ for $n \geq 4$, we have $\frac{1}{n+n!} < \frac{1}{n+n^2} < \frac{1}{n^2}$ for $n \geq 4$. Thus, $\sum\limits_{n=1}^{\infty} \frac{1}{n+n!}$ converges by direct comparison with $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$, which converges because it is a p-series with $p = 2 > 1$.

(b) $\frac{1}{n} + \frac{1}{n!} > \frac{1}{n}$ for $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \left(\frac{1}{n} + \frac{1}{n!}\right)$ diverges by direct comparison with $\sum\limits_{n=1}^{\infty} \frac{1}{n}$, which diverges because it is a p-series with $p = 1 \leq 1$.

9. Use the Limit Comparison Test with $a_n = \frac{n^2-1}{n^3+1}$ and $b_n = \frac{1}{n}$:

$\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{(n^2-1)n}{n^3+1} = \lim\limits_{n\to\infty} \frac{n^3-n}{n^3+1} = \lim\limits_{n\to\infty} \frac{1-1/n^2}{1+1/n^3} = 1 > 0$. Since $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ is the divergent harmonic series, the series $\sum\limits_{n=1}^{\infty} \frac{n^2-1}{n^3+1}$ also diverges.

11. $\sum\limits_{n=1}^{\infty}(-1)^n\dfrac{n^2-1}{n^3+1}=\sum\limits_{n=1}^{\infty}(-1)^n b_n$. Now $b_n=\dfrac{n^2-1}{n^3+1}>0$ for $n\geq 2$, $\{b_n\}$ is decreasing for $n\geq 2$, and $\lim\limits_{n\to\infty} b_n=0$, so the series $\sum\limits_{n=1}^{\infty}(-1)^n\dfrac{n^2-1}{n^3+1}$ converges by the Alternating Series Test. By Exercise 9, $\sum\limits_{n=1}^{\infty}\dfrac{n^2-1}{n^3+1}$ diverges, so the series $\sum\limits_{n=1}^{\infty}(-1)^n\dfrac{n^2-1}{n^3+1}$ is conditionally convergent.

13. $\lim\limits_{x\to\infty}\dfrac{e^x}{x^2}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{e^x}{2x}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{e^x}{2}=\infty$, so $\lim\limits_{n\to\infty}\dfrac{e^n}{n^2}=\infty$. Thus, the series $\sum\limits_{n=1}^{\infty}\dfrac{e^n}{n^2}$ diverges by the Test for Divergence.

15. Let $f(x)=\dfrac{1}{x\sqrt{\ln x}}$. Then f is positive, continuous, and decreasing on $[2,\infty)$, so we can apply the Integral Test.

Since $\displaystyle\int\frac{1}{x\sqrt{\ln x}}\,dx\ \begin{bmatrix}u=\ln x,\\ du=dx/x\end{bmatrix}=\int u^{-1/2}\,du=2u^{1/2}+C=2\sqrt{\ln x}+C$, we find

$\displaystyle\int_2^{\infty}\frac{dx}{x\sqrt{\ln x}}=\lim_{t\to\infty}\int_2^{t}\frac{dx}{x\sqrt{\ln x}}=\lim_{t\to\infty}\left[2\sqrt{\ln x}\right]_2^t=\lim_{t\to\infty}\left(2\sqrt{\ln t}-2\sqrt{\ln 2}\right)=\infty$. Since the integral diverges, the given series $\sum\limits_{n=2}^{\infty}\dfrac{1}{n\sqrt{\ln n}}$ diverges.

17. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=\lim\limits_{n\to\infty}\left|\dfrac{\pi^{2n+2}}{(2n+2)!}\cdot\dfrac{(2n)!}{\pi^{2n}}\right|=\lim\limits_{n\to\infty}\dfrac{\pi^2}{(2n+2)(2n+1)}=0<1$, so the series $\sum\limits_{n=0}^{\infty}(-1)^n\dfrac{\pi^{2n}}{(2n)!}$ is absolutely convergent (and therefore convergent) by the Ratio Test.

19. $\sum\limits_{n=1}^{\infty}\left(\dfrac{1}{n^3}+\dfrac{1}{3^n}\right)=\sum\limits_{n=1}^{\infty}\dfrac{1}{n^3}+\sum\limits_{n=1}^{\infty}\left(\dfrac{1}{3}\right)^n$. The first series converges since it is a p-series with $p=3>1$ and the second series converges since it is geometric with $|r|=\frac{1}{3}<1$. The sum of two convergent series is convergent.

21. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=\lim\limits_{n\to\infty}\left|\dfrac{3^{n+1}(n+1)^2}{(n+1)!}\cdot\dfrac{n!}{3^n n^2}\right|=\lim\limits_{n\to\infty}\dfrac{3(n+1)^2}{(n+1)n^2}=3\lim\limits_{n\to\infty}\dfrac{n+1}{n^2}=0<1$, so the series $\sum\limits_{n=1}^{\infty}\dfrac{3^n n^2}{n!}$ converges by the Ratio Test.

23. $a_k=\dfrac{2^{k-1}3^{k+1}}{k^k}=\dfrac{2^k 2^{-1}3^k 3^1}{k^k}=\dfrac{3}{2}\left(\dfrac{2\cdot 3}{k}\right)^k$. By the Root Test, $\lim\limits_{k\to\infty}\sqrt[k]{\left(\dfrac{6}{k}\right)^k}=\lim\limits_{k\to\infty}\dfrac{6}{k}=0<1$, so the series $\sum\limits_{k=1}^{\infty}\left(\dfrac{6}{k}\right)^k$ converges. It follows from Theorem 8(i) in Section 11.2 that the given series, $\sum\limits_{k=1}^{\infty}\dfrac{2^{k-1}3^{k+1}}{k^k}=\sum\limits_{k=1}^{\infty}\dfrac{3}{2}\left(\dfrac{6}{k}\right)^k$, also converges.

25. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=\lim\limits_{n\to\infty}\left|\dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)(2n+1)}{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-1)(3n+2)}\cdot\dfrac{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-1)}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}\right|=\lim\limits_{n\to\infty}\dfrac{2n+1}{3n+2}$

$=\lim\limits_{n\to\infty}\dfrac{2+1/n}{3+2/n}=\dfrac{2}{3}<1,$

so the series $\sum\limits_{n=1}^{\infty}\dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-1)}$ converges by the Ratio Test.

27. Let $f(x) = \dfrac{\ln x}{\sqrt{x}}$. Then $f'(x) = \dfrac{2 - \ln x}{2x^{3/2}} < 0$ when $\ln x > 2$ or $x > e^2$, so $\dfrac{\ln n}{\sqrt{n}}$ is decreasing for $n > e^2$.

By l'Hospital's Rule, $\lim\limits_{n\to\infty} \dfrac{\ln n}{\sqrt{n}} = \lim\limits_{n\to\infty} \dfrac{1/n}{1/\left(2\sqrt{n}\right)} = \lim\limits_{n\to\infty} \dfrac{2}{\sqrt{n}} = 0$, so the series $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{\ln n}{\sqrt{n}}$ converges by the Alternating Series Test.

29. $\lim\limits_{n\to\infty} |a_n| = \lim\limits_{n\to\infty} \left|(-1)^n \cos(1/n^2)\right| = \lim\limits_{n\to\infty} \left|\cos(1/n^2)\right| = \cos 0 = 1$, so the series $\sum\limits_{n=1}^{\infty} (-1)^n \cos(1/n^2)$ diverges by the Test for Divergence.

31. Using the Limit Comparison Test with $a_n = \tan\left(\dfrac{1}{n}\right)$ and $b_n = \dfrac{1}{n}$, we have

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{\tan(1/n)}{1/n} = \lim\limits_{x\to\infty} \dfrac{\tan(1/x)}{1/x} \stackrel{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{\sec^2(1/x) \cdot (-1/x^2)}{-1/x^2} = \lim\limits_{x\to\infty} \sec^2(1/x) = 1^2 = 1 > 0$. Since $\sum\limits_{n=1}^{\infty} b_n$ is the divergent harmonic series, $\sum\limits_{n=1}^{\infty} a_n$ is also divergent.

33. $\dfrac{4 - \cos n}{\sqrt{n}} \geq \dfrac{4 - 1}{\sqrt{n}} = \dfrac{3}{n^{1/2}}$ for $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{4 - \cos n}{\sqrt{n}}$ diverges by direct comparison with $\sum\limits_{n=1}^{\infty} \dfrac{3}{n^{1/2}}$, which diverges because it is a constant multiple of a p-series with $p = \frac{1}{2} \leq 1$.

35. Use the Ratio Test. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{(n+1)!}{e^{(n+1)^2}} \cdot \dfrac{e^{n^2}}{n!}\right| = \lim\limits_{n\to\infty} \dfrac{(n+1)n! \cdot e^{n^2}}{e^{n^2+2n+1} n!} = \lim\limits_{n\to\infty} \dfrac{n+1}{e^{2n+1}} = 0 < 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{n!}{e^{n^2}}$ converges.

37. $\displaystyle\int_2^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty} \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^t$ [using integration by parts] $\stackrel{\text{H}}{=} 1$. So $\sum\limits_{n=1}^{\infty} \dfrac{\ln n}{n^2}$ converges by the Integral Test, and since $\dfrac{k \ln k}{(k+1)^3} < \dfrac{k \ln k}{k^3} = \dfrac{\ln k}{k^2}$, the given series $\sum\limits_{k=1}^{\infty} \dfrac{k \ln k}{(k+1)^3}$ converges by the Direct Comparison Test.

39. $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} (-1)^n \dfrac{1}{\cosh n} = \sum\limits_{n=1}^{\infty} (-1)^n b_n$. Now $b_n = \dfrac{1}{\cosh n} > 0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n = 0$, so the series converges by the Alternating Series Test.

Or: Write $\dfrac{1}{\cosh n} = \dfrac{2}{e^n + e^{-n}} < \dfrac{2}{e^n}$ and $\sum\limits_{n=1}^{\infty} \dfrac{1}{e^n}$ is a convergent geometric series, so $\sum\limits_{n=1}^{\infty} \dfrac{1}{\cosh n}$ is convergent by the Direct Comparison Test. So $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{1}{\cosh n}$ is absolutely convergent and therefore convergent.

41. $\lim\limits_{k\to\infty} a_k = \lim\limits_{k\to\infty} \dfrac{5^k}{3^k + 4^k} =$ [divide by 4^k] $\lim\limits_{k\to\infty} \dfrac{(5/4)^k}{(3/4)^k + 1} = \infty$ since $\lim\limits_{k\to\infty} \left(\dfrac{3}{4}\right)^k = 0$ and $\lim\limits_{k\to\infty} \left(\dfrac{5}{4}\right)^k = \infty$.

Thus, $\sum\limits_{k=1}^{\infty} \dfrac{5^k}{3^k + 4^k}$ diverges by the Test for Divergence.

43. $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} \left(\dfrac{n}{n+1}\right)^{n^2/n} = \lim\limits_{n\to\infty} \dfrac{1}{[(n+1)/n]^n} = \dfrac{1}{\lim\limits_{n\to\infty} (1+1/n)^n} = \dfrac{1}{e} < 1$, so the series $\sum\limits_{n=1}^{\infty} \left(\dfrac{n}{n+1}\right)^{n^2}$ converges by the Root Test.

45. $a_n = \dfrac{1}{n^{1+1/n}} = \dfrac{1}{n \cdot n^{1/n}}$, so let $b_n = \dfrac{1}{n}$ and use the Limit Comparison Test. $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{1}{n^{1/n}} = 1 > 0$ (see Exercise 6.8.63 [ET 4.4.63]), so the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{1+1/n}}$ diverges by comparison with the divergent harmonic series.

47. $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} (2^{1/n} - 1) = 1 - 1 = 0 < 1$, so the series $\sum\limits_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)^n$ converges by the Root Test.