Summary of Sequence and Series

Sequences

- The sequence $\{a_n\}$:
  - converges to L if $\lim_{n\to\infty} a_n = L$ (where L is finite)
  - diverges to ∞ (or $-\infty$) if L is ∞ (or $-\infty$).
- The sequence $\{r^n\}$:
  - converges to 0 for $-1 < r < 1$,
  - converges to 1 for $r = 1$,
  - diverges otherwise.
- Monotonically increasing: $a_{n+1} \geq a_n$ for all n.
- Monotonically decreasing: $a_{n+1} \leq a_n$ for all n.
- Bounded above: if there exists a $M > 0$ such that $a_n \leq M$ for all n
- Bounded bellow: if there exists a $N > 0$ such that $N \leq a_n$ for all n
- Bounded sequence: if there exists a $M > 0$ and a $N > 0$ such that $N \leq a_n \leq M$ for all n
- **Theorem**: If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.
- **Theorem**: A monotonic bounded sequence converges.
- **Squeeze Theorem**: If $a_n \leq b_n \leq c_n$ for all $n \geq n_0$, and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L,$$

  then $\lim_{n\to\infty} b_n = L$.

Series

- A series is the sum of the terms of an infinite sequence
- Partial sum , $s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$

- **Note:** Series have two associated sequences:
  - The sequence of terms, $\{a_n\}$.
  - The sequence of partial sums, $\{s_n\}$.
- If $\lim_{n\to\infty} s_n = s$ where s is a real number $\implies$ The series $\sum a_n$ converges. Otherwise, diverges.
- $\sum_{n=1}^{\infty} a_n = s$ (sum of the series).
- If $\sum_{n=1}^{\infty} a_n$ converges $\implies$ $\lim_{n\to\infty} a_n = 0$.
- **Absolutely convergence:** Both $\sum a_n$ and $\sum |a_n|$ converges
- **Conditional convergence:** $\sum a_n$ converges but $\sum |a_n|$ diverges
- $\sum |a_n|$ **converges** $\implies$ $\sum a_n$ **converge**
- Harmonic series $\sum \frac{1}{n}$ diverges but alternating harmonic series $\sum (-1)^n \frac{1}{n}$ converges conditionally.

Alternating Series Estimation Theorem:

If $s = \sum_{n=1}^{\infty} (-1)^n b_n$ convergent by alternating series test then

$$|Error| = |s - s_n| \leq b_{n+1}$$

In other words, the absolute error is less than or equal to the first omitted term in the series.

Function growth rate comparison chart at $n \to \infty$

$$constant < log(n) < n^k < a^n < n! < f(n)^n < a^{b^n}$$

where constants $a > 1, b > 1$ and $k > 0$.

Table 1: Series Tests and Convergence Criteria

Test Name	**Conditions**	**Result**	**Comments**
Divergence Test	$\sum a_n$	$\lim_{n\to\infty} a_n \neq \mathbf{0} \implies \sum a_n$ diverges	Cannot ever be used to determine convergence, only divergence.
Geometric Series	$\sum_{n=0}^{\infty} ar^n$	$\lvert \boldsymbol{r} \rvert < \mathbf{1} \implies$ converges. $\lvert \boldsymbol{r} \rvert \geq \mathbf{1} \implies$ diverges. Sum, $\boldsymbol{s} = \frac{\boldsymbol{a}}{\mathbf{1}-\boldsymbol{r}}$	
p-series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$\boldsymbol{p} > \mathbf{1} \implies$ converges. $\boldsymbol{p} \leq \mathbf{1} \implies$ diverges.	Useful in conjunction with comparison tests.
Integral Test	$a_n > 0$, continuous and decreasing	Converges or diverges with $\int_1^{\infty} a_n dn$	Don't use if a_n difficult to integrate
Comparison Test (CT)	$a_n \geq 0$	If $0 \leq a_n \leq b_n$, if $\sum b_n$ converges $\implies \sum a_n$ converges if $\sum a_n$ diverges $\implies \sum b_n$ diverges	Series that closely resemble p-series or geometric series
Limit Comparison Test (LCT)	$a_n \geq 0$	If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ then: If $c > 0$ and finite, both series converge or both diverge.	$\sum b_n \rightarrow$ series of your choice (usually p-series or geometric), for which you already know convergence or divergence.
Alternating Series Test	Alternating series: $\sum_{n=1}^{\infty} (-1)^n b_n$, $b_n > 0$ and decreasing for all n	If $\lim_{n\to\infty} b_n = 0 \implies$ converges.	Useful if the series converges but does **not** converge absolutely. Other tests may be easier if series converges absolutely.
Ratio Test	Series involving $n!\,, n^n, a^n$ or expressions like $1 \cdot 3 \cdot 5 \cdots (2n-1)$	If $\lim_{n\to\infty} \frac{\lvert a_{n+1} \rvert}{\lvert a_n \rvert} = L$ then: If $L < 1 \implies$ converges **absolutely**. If $L > 1$ *or* $\infty \implies$ diverges. $L = 1 \implies$ no conclusion (Test fails)	If test fails use another test. If the series has terms alternating in sign, try the Alternating Series Test.
Root Test	$\lvert a_n \rvert = [f(n)]^n$	If $\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} = L$ then: If $L < 1 \implies$ converges **absolutely**. If $L > 1$ *or* $\infty \implies$ diverges. $L = 1 \implies$ no conclusion (Test fails)	If test fails use another test.