

W13 Notes

Parametric curves

Videos, Organic Chemistry Tutor

- [Intro to parametric equations and graphing](#)

01 Theory

Parametric curves are curves traced by the path of a ‘moving’ point. An independent parameter, such as t for ‘time’, controls *both x and y* values through **Cartesian coordinate functions** $x(t)$ and $y(t)$. The coordinates of the moving point are $(x(t), y(t))$.

▣ Parametric curve

A **parametric curve** is a function from parameter space $\mathbb{R}$ to the plane $\mathbb{R}^2$ given in terms of coordinate functions:

$$t \mapsto (x(t), y(t))$$

△ Other notations

Be aware that sometimes the coordinate functions are written with f and g (or yet other letters) like this: $(x, y) = (f(t), g(t))$

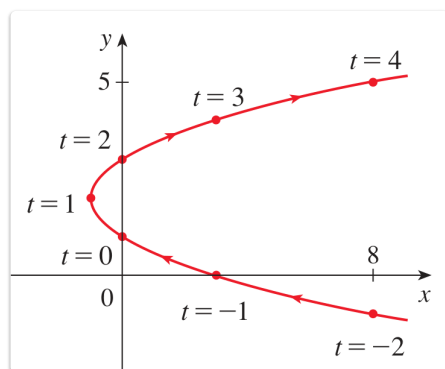
Or simply equating coordinate letters with functions: $x = f(t)$, $y = g(t)$

Sometimes a different parameter is used, like s or u .

For example, suppose:

$$x = t^2 - 2t, \quad y = t + 1$$

The curve traced out is a parabola that opens horizontally:



Given a parametric curve, we can create an equation satisfied by x and y variables by solving for t in either coordinate function (inverting either f or g) and plugging the result into the other function.

In the example:

$$y = t + 1 \quad \gg \gg \quad t = y - 1$$

$$\gg \gg \quad x = t^2 - 2t \quad \gg \gg \quad x = (y - 1)^2 - 2(y - 1)$$

$$\gg \gg \quad x = y^2 - 4y + 3 \quad \gg \gg \quad x = (y - 2)^2 - 1$$

This is the equation of a parabola centered at $(-1, 2)$ that opens to the right.

📐 Image of a parametric curve

The **image** of a parametric curve is the *set* of output points $(x(t), y(t))$ that are traversed by the moving point.

A parametric curve has *hidden information* that isn't contained in the image:

- The *time values* t when the moving point is found in various locations.
- The *speed* at which the curve is traversed.
- The *direction* in which the curve is traversed.

We can **reparametrize** a parametric curve to use a different parameter or different coordinate functions while leaving the *image unchanged*.

In the previous example, shift t by 1:

$$x = (t + 1)^2 - 2(t + 1), \quad y = (t + 1) + 1$$

$$\gg \gg \quad x = t^2 - 1, \quad y = t + 2$$

Since the parameter t and the parameter $t + 1$ both cover the same values for $t \in (-\infty, \infty)$, the same curve is traversed. But the moving point in the second, shifted version reaches any given location *one unit earlier* in time. (When $t = -1$ in the second version, the input to $x(t)$ and $y(t)$ is the same as when $t = 0$ in the first one.)

02 Illustration

☰ Example - Parametric circles

The standard equation of a circle of radius R centered at the point (h, k) :

$$(x - h)^2 + (y - k)^2 = R^2$$

This equation says that the *distance* from a point (x, y) on the circle to the center point (h, k) equals R . This fact defines the circle.

Parametric coordinates for the circle:

$$x = h + R \cos t, \quad y = k + R \sin t, \quad t \in [0, 2\pi)$$

For example, the unit circle $x^2 + y^2 = 1$ is parametrized by $x = \cos t$ and $y = \sin t$.

☰ Example - Parametric lines

Parametric coordinate functions for a line:

$$x = a + rt, \quad y = b + st, \quad t \in (-\infty, +\infty)$$

Compare this to the graph of linear function:

$$y = mx + b \quad \text{some } m, b$$

Vertical lines cannot be described as the graph of a function. We must use $x = a$.

Parametric lines can describe all lines equally well, including horizontal and vertical lines.

A vertical line $x = a$ is achieved by setting $s = 0$ and $r \neq 0$.

A horizontal line $y = b$ is achieved by setting $r = 0$ and $s \neq 0$.

A non-vertical line $y = mx + b$ may be achieved by setting $s = m$ and $r = 1$, and $a = 0$.

Assuming that $r \neq 0$, the parametric coordinate functions describe a line satisfying:

$$y = b + s \left(\frac{x - a}{r} \right)$$

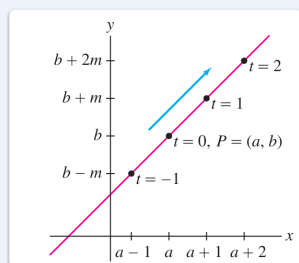
$$\gg \gg \quad y = \frac{s}{r} \cdot x + \left(b - \frac{s}{r} \cdot a \right)$$

and therefore the slope is $m = \frac{s}{r}$ and the y -intercept is $b - \frac{s}{r} \cdot a$.

The point-slope construction of a line has a parametric analogue:

point-slope line:

$$y - a = m(x - b) \quad (x, y) = (a + t, b + mt)$$



Example - Parametric ellipses

The general equation of an ellipse centered at (h, k) with half-axes a and b is:

$$\left(\frac{x - h}{a} \right)^2 + \left(\frac{y - k}{b} \right)^2 = 1$$

This equation represents a *stretched unit circle*:

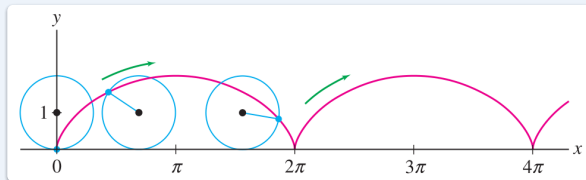
- by a in the x -axis
- by b in the y -axis

Parametric coordinate functions for the general ellipse:

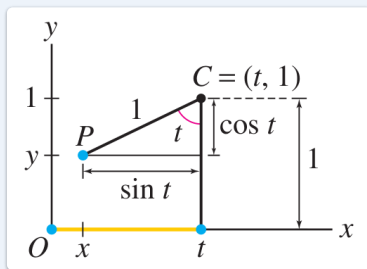
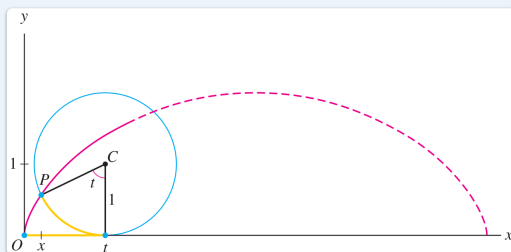
$$x = h + a \cos t, \quad y = k + b \sin t, \quad t \in [0, 2\pi)$$

Example - Parametric cycloids

The cycloid is the curve traced by a pen attached to the rim of a wheel as it rolls.



It is easy to describe the cycloid parametrically. Consider the geometry of the situation:



The center C of the wheel is moving rightwards at a constant speed of 1, so its position is $(t, 1)$. The angle is revolving at the same constant rate of 1 (in *radians*) because the *radius* is 1.

The triangle shown has base $\sin t$, so the x coordinate is $t - \sin t$. The y coordinate is $1 - \cos t$.

So the coordinates of the point $P = (x, y)$ are given parametrically by:

$$x = t - \sin t, \quad y = 1 - \cos t, \quad t > 0$$

If the circle has another radius, say R , then the parametric formulas change to:

$$x = Rt - R \sin t, \quad y = R - R \cos t, \quad t > 0$$

Calculus with parametric curves

03 Theory - Slope, concavity

We can use $x(t)$ and $y(t)$ data to compute the slope of a parametric curve in terms of t .

Slope formula

Given a parametric curve $(x(t), y(t))$, its slope satisfies:

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} \quad (\text{where } x'(t) \neq 0)$$

Concavity formula

Given a parametric curve $(x(t), y(t))$, its concavity satisfies the formula:

$$\frac{d^2y}{dx^2} = \frac{d}{dt} \left(\frac{y'(t)}{x'(t)} \right) \cdot \frac{1}{x'(t)} \quad (\text{where } x'(t) \neq 0)$$

Extra - Derivation of slope and concavity formulas

For both derivations, it is necessary to view t as a function of x through the inverse parameter function. For example if $x = f(t)$ is the parametrization, then $t = f^{-1}(x)$ is the inverse parameter function.

We will need the derivative $\frac{dt}{dx}$ in terms of t . For this we use the formula for derivative of inverse functions:

$$\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$$

Given all this, both formulas are simple applications of the chain rule.

For the slope:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{dt}{dx} && \gg \gg && y'(t) \cdot \frac{1}{dx/dt} \\ &&& \gg \gg && \frac{y'(t)}{x'(t)} \end{aligned}$$

For the concavity:

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx} \left(\frac{dy}{dx} \right) && \gg \gg && \frac{d}{dt} \left(\frac{dy}{dx} \right) \cdot \frac{dt}{dx} \\ &&& \gg \gg && \frac{d}{dt} \left(\frac{y'(t)}{x'(t)} \right) \cdot \frac{1}{x'(t)} \end{aligned}$$

(In the second step we inserted the formula for $\frac{dy}{dx}$ from the slope.)

Pure vertical, Pure horizontal movement

In view of the formula $\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$, we see:

- Pure vertical: when $x'(t) = 0$ and yet $y'(t) \neq 0$
- Pure horizontal: when $y'(t) = 0$ and yet $x'(t) \neq 0$

When $x'(t_0) = y'(t_0) = 0$ for the same $t = t_0$, we have a **stationary point**, which might subsequently progress into pure vertical, pure horizontal, or neither.

04 Illustration

≡ Example - Tangent to a cycloid

Find the tangent line (described parametrically) to the cycloid $(4t - 4\sin t, 4 - 4\cos t)$ when $t = \pi/4$.

Solution

Compute x' and y' .

Find $x'(t)$:

$$x(t) = 4t - 4\sin t \quad \gg \gg \quad x'(t) = 4 - 4\cos t$$

Find $y'(t)$:

$$y(t) = 4 - 4\cos t \quad \gg \gg \quad y'(t) = 4\sin t$$

Plug in $t = \pi/4$:

$$x'(\pi/4) \quad \gg \gg \quad 4 - 4\cos(\pi/4) \quad \gg \gg \quad 4 - 2\sqrt{2}$$

Plug in $t = \pi/4$:

$$y'(\pi/4) \quad \gg \gg \quad 4\sin(\pi/4) \quad \gg \gg \quad 2\sqrt{2}$$

Apply formula: $\frac{dy}{dx} = \frac{y'}{x'}$:

Calculate $\frac{dy}{dx}$ at $t = \pi/4$:

$$\frac{dy}{dx}(\pi/4) = \frac{y'(\pi/4)}{x'(\pi/4)} \quad \gg \gg \quad \frac{2\sqrt{2}}{4 - 2\sqrt{2}}$$

Simplify:

$$\begin{aligned} &\gg \gg \quad \frac{2\sqrt{2}}{4 - 2\sqrt{2}} \cdot \frac{4 + 2\sqrt{2}}{4 + 2\sqrt{2}} \\ &\gg \gg \quad \frac{8\sqrt{2} + 8}{16 - 8} \quad \gg \gg \quad \sqrt{2} + 1 \end{aligned}$$

So:

$$\left. \frac{dy}{dx} \right|_{t=\pi/4} = \sqrt{2} + 1$$

This is the slope m for our line.

Need the point P for our line. Find (x, y) at $t = \pi/4$.

Plug $t = \pi/4$ into parametric formulas:

$$\begin{aligned} (x(t), y(t)) \Big|_{t=\pi/4} &\gg \gg \quad \left(4\frac{\pi}{4} - 4\sin(\pi/4), 4 - 4\cos(\pi/4) \right) \\ &\gg \gg \quad (\pi - 2\sqrt{2}, 4 - 2\sqrt{2}) \end{aligned}$$

Point-slope formulation of tangent line:

$$x = a + t, \quad y = b + mt$$

Inserting our data:

$$x = (\pi - 2\sqrt{2}) + t, \quad y = (4 - 2\sqrt{2}) + (\sqrt{2} + 1)t$$

Example - Vertical and horizontal tangents of the circle

Consider the circle parametrized by $x = \cos t$ and $y = \sin t$. Find the points where the tangent lines are vertical or horizontal.

Solution

For the points with vertical tangent line, we find where the moving point has $x'(t) = 0$ (purely vertical motion):

$$\begin{aligned} x'(t) &= -\sin t, \\ x'(t) = 0 &\gg \gg -\sin t = 0 \\ &\gg \gg t = 0, \pi \end{aligned}$$

The moving point is at $(1, 0)$ when $t = 0$, and at $(-1, 0)$ when $t = \pi$.

For the points with horizontal tangent line, we find where the moving point has $y'(t) = 0$ (purely horizontal motion):

$$\begin{aligned} y'(t) &= \cos t, \\ y'(t) = 0 &\gg \gg \cos t = 0 \\ &\gg \gg t = \frac{\pi}{2}, \frac{3\pi}{2} \end{aligned}$$

The moving point is at $(0, 1)$ when $t = \pi/2$, and at $(0, -1)$ when $t = 3\pi/2$.

Example - Finding the point with specified slope

Consider the parametric curve given by $(x, y) = (t^2, t^3)$. Find the point where the slope of the tangent line to this curve equals 5.

Solution

Compute the derivatives:

$$x'(t) = 2t, \quad y'(t) = 3t^2$$

Therefore the slope of the tangent line, in terms of t :

$$\begin{aligned} m &= \frac{dy}{dx} = \frac{y'(t)}{x'(t)} \\ &\gg \gg \frac{3t^2}{2t} \gg \gg \frac{3}{2}t \end{aligned}$$

Set up equation:

$$m = 5$$

$$\frac{3}{2}t = 5$$

Solve. Obtain $t = \frac{10}{3}$.

Find the point:

$$(x, y)|_{t=10/3} \gg \gg \left(\frac{100}{9}, \frac{1000}{27} \right)$$

05 Theory - Arclength

📏 Arclength formula

The **arclength** of a parametric curve with coordinate functions $x(t)$ and $y(t)$ is:

$$L = \int_a^b \sqrt{(x')^2 + (y')^2} dt$$

This formula assumes the curve is traversed one time as t increases from a to b .

⚠ Counts total traversal

This formula applies when the curve image is traversed *one time* by the moving point.

Sometimes a parametric curve traverses its image with repetitions. The arclength formula would add length from each repetition!

📖 Extra - Derivation of arclength formula

The arclength of a parametric curve is calculated by integrating the infinitesimal arc element:

$$ds = \sqrt{dx^2 + dy^2}$$

$$L = \int_a^b ds$$

In order to integrate ds in the t variable, as we must for parametric curves, we convert ds to a function of t :

$$ds = \sqrt{dx^2 + dy^2} \gg \gg \sqrt{\frac{1}{dt^2} \cdot (dx^2 + dy^2) \cdot dt^2}$$

$$\gg \gg \sqrt{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}} \cdot \sqrt{dt^2} \gg \gg \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$\gg \gg ds = \sqrt{x'(t)^2 + y'(t)^2} dt$$

So we obtain $ds = \sqrt{(x')^2 + (y')^2} dt$ and the arclength formula follows from this:

$$L = \int_a^b \sqrt{(x')^2 + (y')^2} dt$$

06 Illustration

Example - Perimeter of a circle

The perimeter of the circle $(R \cos t, R \sin t)$ is easily found. We have $(x', y') = (-R \sin t, R \cos t)$, and therefore:

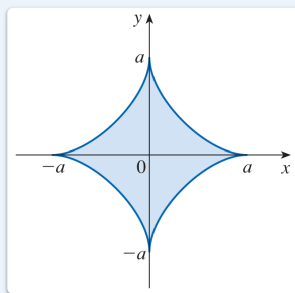
$$\begin{aligned} (x')^2 + (y')^2 &= (-R \sin t)^2 + (R \cos t)^2 \\ &\gg \gg R^2 \sin^2 t + R^2 \cos^2 t \gg \gg R^2 \\ ds &= \sqrt{(x')^2 + (y')^2} dt = R dt \end{aligned}$$

Integrate around the circle:

$$\begin{aligned} \text{Perimeter} &= \int_0^{2\pi} ds \gg \gg \int_0^{2\pi} R dt \\ &\gg \gg Rt \Big|_0^{2\pi} = 2\pi R \end{aligned}$$

Example - Perimeter of an asteroid

Find the perimeter length of the ‘asteroid’ given parametrically by $(x, y) = (a \cos^3 \theta, a \sin^3 \theta)$ for $a = 2$.



Solution

Notice: Throughout this problem we use the parameter θ instead of t . This does *not* mean we are using polar coordinates!

Compute the derivatives in θ :

$$(x', y') = (3a \cos^2 \theta \sin \theta, 3a \sin^2 \theta \cos \theta)$$

Compute the infinitesimal arc element.

$$\begin{aligned}
 (x')^2 + (y')^2 &\gg \gg 9a^2 \cos^4 \theta \sin^2 \theta + 9a^2 \sin^4 \theta \cos^2 \theta \\
 &\gg \gg 9a^2 \sin^2 \theta \cos^2 \theta (\cos^2 \theta + \sin^2 \theta) \\
 &\gg \gg 9a^2 \sin^2 \theta \cos^2 \theta
 \end{aligned}$$

Plug into the arc element, simplify:

$$\begin{aligned}
 ds &= \sqrt{(x')^2 + (y')^2} d\theta \\
 &\gg \gg \sqrt{9a^2 \sin^2 \theta \cos^2 \theta} d\theta \\
 &\gg \gg ds = 3a |\sin \theta \cos \theta| d\theta
 \end{aligned}$$

Bounds of integration?

Easiest to use $\theta \in [0, \pi/2]$. This covers one edge of the asteroid. Then multiply by 4 for the final answer.

On the interval $\theta \in [0, \pi/2]$, the factor $3a \sin \theta \cos \theta$ is *positive*. So we can drop the absolute value and integrate directly.

⚠ Absolute values matter!

If we tried to integrate on the whole range $\theta \in [0, 2\pi]$, then $3a \sin \theta \cos \theta$ really does change sign.

To perform integration properly with these absolute values, we'd need to convert to a piecewise function by adding appropriate minus signs.

Integrate the arc element:

$$\begin{aligned}
 \int_0^{\pi/2} ds &\gg \gg \int_0^{\pi/2} 3a \sin \theta \cos \theta d\theta \\
 &\gg \gg 3a \int_{u=0}^1 u du && (u = \sin \theta) \\
 &\gg \gg 3a \left. \frac{u^2}{2} \right|_0^1 && \gg \gg \frac{3a}{2}
 \end{aligned}$$

Finally, multiply by 4 to get the total perimeter: **L=6a**

07 Theory - Distance, speed

📏 Distance function

The **distance function** $s(t)$ returns the total distance traveled by the particle from a chosen starting time t_0 up to the (input) time t :

$$s(t) = \int_{t_0}^t ds = \int_{t_0}^t \sqrt{x'(u)^2 + y'(u)^2} du$$

We need the dummy variable u so that the integration process does not conflict with t in the upper bound.

Speed function

The **speed** of a moving particle is the *rate of change of distance*:

$$v(t) = s'(t) = \sqrt{x'(t)^2 + y'(t)^2}$$

This formula can be explained in either of two ways:

1. Apply the Fundamental Theorem of Calculus to the integral formula for $s(t)$.
2. Consider $ds = \sqrt{x'(t)^2 + y'(t)^2} dt$ for a small change dt : so the *rate of change* of arclength is $\frac{ds}{dt}$, in other words $s'(t)$.

08 Illustration

Example - Speed, distance, displacement

The parametric curve $(t, \frac{2}{3}t^{3/2})$ describes the position of a moving particle (t measuring seconds).

(a) What is the speed function?

Suppose the particle travels for 8 seconds starting at $t = 0$.

(b) What is the total distance traveled?

(c) What is the total displacement?

Solution

(a)

Compute *derivatives*:

$$(x', y') = (1, t^{1/2})$$

Now compute the *speed*.

Find sum of squares:

$$(x')^2 + (y')^2 = 1 + (t^{1/2})^2 = 1 + t$$

Get the speed function:

$$v(t) = \sqrt{(x')^2 + (y')^2} = \sqrt{1+t}$$

(b)

Distance traveled by using *speed*.

Compute total distance traveled function:

$$s(t) = \int_{u=0}^t \sqrt{1+u} \, du$$

Integrate.

Substitute $w = 1 + u$ and $dw = du$.

New bounds are 1 and $1 + t$.

Calculate:

$$\begin{aligned} & \gg \gg \int_1^{1+t} \sqrt{w} \, dw \\ & \gg \gg \left. \frac{2}{3} w^{3/2} \right|_1^{1+t} \gg \gg \frac{2}{3} \left((1+t)^{3/2} - 1 \right) \end{aligned}$$

Insert $t = 8$ for the answer.

The distance traveled up to $t = 8$ is:

$$s(8) = \frac{2}{3} \left(9^{3/2} - 1 \right) \gg \gg \frac{2}{3} (27 - 1) \gg \gg \frac{52}{3}$$

This is our final answer.

(c)

Displacement formula: $d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$

Pythagorean formula for distance between given points.

Compute starting and ending points.

For starting point, insert $t = 0$:

$$(x(t), y(t)) \Big|_{t=0} \gg \gg \left(t, \frac{2}{3} t^{3/2} \right) \Big|_{t=0} \gg \gg (0, 0)$$

For ending point, insert $t = 8$:

$$\begin{aligned} & (x(t), y(t)) \Big|_{t=8} \gg \gg \left(t, \frac{2}{3} t^{3/2} \right) \Big|_{t=8} \\ & \gg \gg \left(8, \frac{2}{3} 8^{3/2} \right) \gg \gg \left(8, \frac{32\sqrt{2}}{3} \right) \end{aligned}$$

Plug points into distance formula.

Insert $(0, 0)$ and $\left(8, 32\sqrt{2}/3 \right)$:

$$\begin{aligned} \sqrt{8^2 + \left(\frac{32\sqrt{2}}{3}\right)^2} &\gg \gg \sqrt{64 + \frac{2048}{9}} \\ &\gg \gg \frac{\sqrt{2624}}{3} \end{aligned}$$

This is our final answer.

09 Theory - Surface area of revolutions

📐 Surface area of a surface of revolution: thin bands

Suppose a parametric curve $(x(t), y(t))$ is revolved around the x -axis or the y -axis.

The surface area is:

$$A = \int_a^b 2\pi R(t) \sqrt{(x')^2 + (y')^2} dt$$

The radius $R(t)$ should be the distance to the axis:

$$\begin{aligned} R(t) &= y(t) && \text{revolution about } x\text{-axis} \\ R(t) &= x(t) && \text{revolution about } y\text{-axis} \end{aligned}$$

This formula adds the areas of thin bands, but the bands are demarcated using parametric functions instead of input values of a graphed function.

The formula assumes that the curve is traversed one time as t increases from a to b .

10 Illustration

☰ Example - Surface of revolution - parametric circle

By revolving the unit upper semicircle about the x -axis, we can compute the surface area of the unit sphere.

The parametrization of the unit upper semicircle is: $(x, y) = (\cos t, \sin t)$.

The derivative is: $(x', y') = (-\sin t, \cos t)$.

Therefore, the arc element:

$$\begin{aligned} ds &= \sqrt{(x')^2 + (y')^2} dt \\ &\gg \gg \sqrt{(-\sin t)^2 + (\cos t)^2} dt \gg \gg dt \end{aligned}$$

Now for R we choose $R = y(t) = \sin t$ because we are revolving about the x -axis.

Plugging all this into the integral formula and evaluating gives:

$$A = \int_0^\pi 2\pi \sin t dt \gg \gg -2\pi \cos t \Big|_0^\pi \gg \gg 4\pi$$

Notice: This method is a little easier than the method using the graph $y = \sqrt{1 - x^2}$.

≡ Example - Surface of revolution - parametric curve

Set up the integral which computes the surface area of the surface generated by revolving about the x -axis the curve $(t^3, t^2 - 1)$ for $0 \leq t \leq 1$.

Solution

For revolution about the x -axis, we set $R = y(t) = t^2 - 1$.

Then compute ds :

$$\begin{aligned} ds &= \sqrt{(x')^2 + (y')^2} \gg \gg \sqrt{(3t^2)^2 + (2t)^2} \gg \gg \sqrt{9t^4 + 4t^2} \\ &\gg \gg \sqrt{t^2(9t^2 + 4)} \gg \gg t\sqrt{9t^2 + 4} \end{aligned}$$

Therefore the desired integral is:

$$A = \int_0^1 2\pi R ds \gg \gg \int_0^1 2\pi(t^2 - 1)t\sqrt{9t^2 + 4} dt$$