W08 Notes

Simple divergence test

Videos, Math Dr. Bob

- Geometric series and SDT, again: Geometric series, Simple Divergence Test (aka "Limit Test")
- Integral test: Basics
- Integral test: p-series
    - Extra: Integral test: Further examples
    - Extra: Integral test: Estimations

01 Theory

Simple Divergence Test (SDT)

Applicability: *Any* series.

Test Statement:

$$\lim_{n\to\infty} a_n \neq 0 \qquad \Longrightarrow \qquad \sum_{n=1}^{\infty} a_n \quad \text{diverges}$$

- ⚠ The *converse is not valid*. For example, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges even though $\lim_{n\to\infty} \frac{1}{n} = 0$.

02 Illustration

Simple Divergence Test: examples

Consider: $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4n+1}$

- This diverges by the SDT because $a_n \to \frac{1}{4}$ and not 0.

Consider: $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n+1}$

- This diverges by the SDT because $\lim_{n\to\infty} a_n = \text{DNE}$.
- We can say the terms "converge to the pattern $+1,\ -1,\ +1,\ -1,\ \dots,$" but that is not a limit value.

Positive series

- Direct Comparison Test: Theory and basic examples
- Direct Comparison Test: Series $\frac{1}{\ln n}$
- Limit Comparison Test: Theory and basic examples
- Limit Comparison Test: Further examples

03 Theory

Positive series

A series is called **positive** when its individual terms are positive, i.e. $a_n > 0$ for all n.

The partial sum sequence S_N is *monotone increasing* for a positive series.

By the monotonicity test for convergence of sequences, S_N therefore converges whenever it is *bounded above*. If S_N is not bounded above, then $\sum_{n=1}^{\infty} a_n$ diverges to $+\infty$.

Another test, called the **integral test**, studies the terms of a series as if they represent rectangles with upper corner pinned to the graph of a continuous function.

To apply the test, we must convert the integer index variable n in a_n into a continuous variable x. This is easy when we have a formula for a_n (provided it doesn't contain factorials or other elements dependent on integrality).

Integral Test (IT)

Applicability:

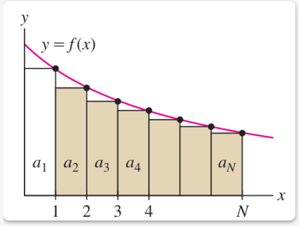
- (i) $f(x) > 0$
- (ii) $f(x)$ is continuous
- (iii) $f(x)$ is *monotone decreasing*

Test Statement:

$$\sum_{n=1}^{\infty} a_n \quad \frac{\text{converges}}{\text{diverges}} \qquad \Longleftrightarrow \qquad \int_1^{\infty} f(x)\,dx \quad \frac{\text{converges}}{\text{diverges}}$$

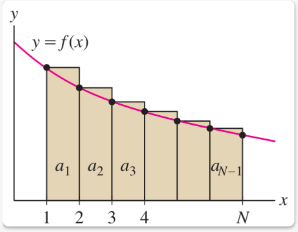
Extra - Integral test: explanation

To show that *integral convergence implies series convergence*, consider the diagram:



This shows that $\sum_{n=2}^{N} a_n \leq \int_1^N f(x)\,dx$ for any N. Therefore, if $\int_1^{\infty} f(x)\,dx$ converges, then $\int_1^N f(x)\,dx$ is bounded (independent of N) and so $\sum_{n=2}^{N} a_n$ is bounded by that inequality. But $\sum_{n=2}^{N} a_n = S_N - a_1$; so by adding a_1 to the bound, we see that S_N itself is bounded, which implies that $\sum_{n=1}^{\infty} a_n$ converges.

To show that *integral divergence implies series divergence*, consider a similar diagram:



This shows that $\sum_{n=1}^{N-1} a_n \geq \int_1^N f(x)\,dx$ for any N. Therefore, if $\int_1^\infty f(x)\,dx$ diverges, then $\int_1^N f(x)\,dx$ goes to $+\infty$ as $N \to \infty$, and so $\sum_{n=1}^{N-1} a_n$ goes to $+\infty$ as well. So $\sum_{n=1}^{\infty} a_n$ diverges.

- Notice: the picture shows $f(x)$ entirely above (or below) the rectangles.
    - This depends upon $f(x)$ being *monotone decreasing*, as well as $f(x) > 0$.
    - This explains the applicability conditions.

Next we use the integral test to evaluate the family of ***p*-series**, and later we can use p-series in comparison tests without repeating the work of the integral test.

p-series

A ***p*-series** is a series of this form: $$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

Convergence properties:

$$p > 1 : \text{series converges} \qquad\qquad p \leq 1 : \text{series diverges}$$

Extra - Proof of p-series convergence

To verify the convergence properties of p-series, apply the integral test:

1. Applicability: verify it's continuous, positive, decreasing.
    - Convert n to x to obtain the function $f(x) = \frac{1}{x^p}$.
    - Indeed $\frac{1}{x^p}$ is continuous and positive and decreasing as x increases.
2. Apply the integral test.
    - Integrate, assuming $p \neq 1$:

$$\int_1^\infty \frac{1}{x^p}\,dx \quad \gg\gg \quad \lim_{R\to\infty} \left.\frac{x^{p-1}}{p-1}\right|_1^R$$

$$\gg\gg \quad \lim_{R\to\infty} \left(\frac{R^{-p+1}}{-p+1} - \frac{1^{-p+1}}{-p+1}\right)$$

- When $p > 1$ we have $\lim_{R\to\infty} \frac{R^{-p+1}}{-p+1} = 0$
- When $p < 1$ we have $\lim_{R\to\infty} \frac{R^{-p+1}}{-p+1} = \infty$

- When $p = 1$, integrate a second time:

$$\int_1^\infty \frac{1}{x}\,dx \quad \gg\gg \quad \lim_{R\to\infty} \ln x\Big|_1^R$$

$$\gg\gg \quad \lim_{R\to\infty} \ln R - \ln 1 \quad \gg\gg \quad \infty$$

3. Conclude: the integral converges when $p > 1$ and diverges when $p \leq 1$.

- Supplement: we could instead immediately refer to the convergence results for *p-integrals* instead of reproving them here.

04 Illustration

p-series examples

By finding p and applying the p-series convergence properties:

We see that $\sum_{n=1}^\infty \frac{1}{n^{1.1}}$ converges: $p = 1.1$ so $p > 1$

But $\sum_{n=1}^\infty \frac{1}{\sqrt{n}}$ diverges: $p = 1/2$ so $p \leq 1$

Integral test - pushing the envelope of convergence

Does $\displaystyle\sum_{n=2}^\infty \frac{1}{n \ln n}$ converge?

Does $\displaystyle\sum_{n=2}^\infty \frac{1}{n(\ln n)^2}$ converge?

Notice that $\ln n$ grows *very slowly* with n, so $\frac{1}{n \ln n}$ is just a *little* smaller than $\frac{1}{n}$ for large n, and similarly $\frac{1}{n(\ln n)^2}$ is just a little smaller still.

Solution

1. The two series lead to the two functions $f(x) = \frac{1}{x \ln x}$ and $g(x) = \frac{1}{x(\ln x)^2}$.
2. Check applicability.
   - Clearly $f(x)$ and $g(x)$ are both continuous, positive, decreasing functions on $x \in [2, \infty]$.
3. Apply the integral test to $f(x)$.
   - Integrate $f(x)$:

$$\int_2^\infty \frac{1}{x \ln x}\,dx \quad \gg\gg \quad \int_{u=\ln 2}^\infty \frac{1}{u}\,du$$

$$\gg\gg \quad \lim_{R\to\infty} \ln u\Big|_{\ln 1}^R \quad \gg\gg \quad \infty$$

4. Conclude: $\sum_{n=2}^\infty \frac{1}{n \ln n}$ *diverges*.
5. Apply the integral test to $g(x)$.

- Integrate $g(x)$:

$$\int_2^\infty \frac{1}{x(\ln x)^2}\,dx \quad \gg\gg \quad \int_{u=\ln 2}^\infty \frac{1}{u^2}\,du$$

$$\gg\gg \quad \lim_{R\to\infty} -u^{-1}\Big|_{\ln 2}^{R} \quad \gg\gg \quad \frac{1}{\ln 2}$$

6. ≡ Conclude: $\sum_{n=2}^\infty \frac{1}{n(\ln n)^2}$ *converges*.

05 Theory

Direct Comparison Test (DCT)

Applicability: Both series are positive: $a_n > 0$ and $b_n > 0$.

Test Statement: Suppose $a_n \leq b_n$ for large enough n.
(Meaning: for $n \geq N$ with some given N.) Then:

- Smaller pushes up bigger:

$$\sum_{n=1}^\infty a_n \quad \text{diverges} \qquad \Longrightarrow \qquad \sum_{n=1}^\infty b_n \quad \text{diverges}$$

- Bigger controls smaller:

$$\sum_{n=1}^\infty b_n \quad \text{converges} \qquad \Longrightarrow \qquad \sum_{n=1}^\infty a_n \quad \text{converges}$$

06 Illustration

Direct comparison test: rational functions

The series $\displaystyle\sum_{n=1}^\infty \frac{1}{\sqrt{n}\,3^n}$ *converges* by the DCT.

- Choose: $a_n = \frac{1}{\sqrt{n}\,3^n}$ and $b_n = \frac{1}{3^n}$
- Check: $0 < \frac{1}{\sqrt{n}\,3^n} \leq \frac{1}{3^n}$
- Observe: $\sum \frac{1}{3^n}$ is a convergent geometric series

The series $\displaystyle\sum_{n=1}^\infty \frac{\cos^2 n}{n^3}$ *converges* by the DCT.

- Choose: $a_n = \frac{\cos^2 n}{n^3}$ and $b_n = \frac{1}{n^3}$.
- Check: $0 \leq \frac{\cos^2 n}{n^3} \leq \frac{1}{n^3}$
- Observe: $\sum \frac{1}{n^3}$ is a convergent p-series

The series $\displaystyle\sum_{n=1}^\infty \frac{n}{n^3+1}$ *converges* by the DCT.

- Choose: $a_n = \frac{n}{n^3+1}$ and $b_n = \frac{1}{n^2}$

- Check: $0 \leq \frac{n}{n^3+1} \leq \frac{1}{n^2}$ (notice that $\frac{n}{n^3+1} \leq \frac{n}{n^3}$)
- Observe: $\sum \frac{1}{n^2}$ is a convergent p-series

The series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n-1}$ *diverges* by the DCT.

- Choose: $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n-1}$
- Check: $0 \leq \frac{1}{n} \leq \frac{1}{n-1}$
- Observe: $\sum \frac{1}{n}$ is a divergent p-series

07 Theory

Some series can be compared using the DCT after applying certain manipulations and tricks.

For example, consider the series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$. We suspect convergence because $a_n \approx \frac{1}{n^2}$ for *large* n. But unfortunately, $a_n > \frac{1}{n^2}$ always, so we cannot apply the DCT.

We could make some *ad hoc* arguments that do use the DCT, eventually:

- Trick Method 1:
  - Observe that for $n > 1$ we have $\frac{1}{n^2-1} \leq \frac{10}{n^2}$. (Check it!)
  - But $\sum \frac{10}{n^2}$ converges, indeed its value is $10 \cdot \sum \frac{1}{n^2}$, which is $\frac{10\pi^2}{6}$.
  - So the series $\sum \frac{1}{n^2-1}$ converges.
- Trick Method 2:
  - Observe that we can change the letter n to $n+1$ by starting the new n at $n = 1$.
  - Then we have:

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1} \quad = \quad \sum_{n=1}^{\infty} \frac{1}{(n+1)^2-1} \quad = \quad \sum_{n=1}^{\infty} \frac{1}{n^2+2n}$$

  - This last series has terms smaller than $\frac{1}{n^2}$ so the DCT with $b_n = \frac{1}{n^2}$ (a convergent p-series) shows that the original series converges too.

These convoluted arguments suggest that a more general version of Comparison is possible.

Indeed, it is sufficient to compare the *limiting behavior* of two series. The limit of *ratios* (limit of 'comparison') links up the convergence / divergence of $\sum a_n$ and $\sum b_n$.

Limit Comparison Test (LCT) - "Limiting Ratio Test"

Applicability: Both series are positive: $a_n > 0$ and $b_n > 0$.

Test Statement: Suppose that $\lim_{n\to\infty} \dfrac{a_n}{b_n} = L$. Then:

- If $0 < L < \infty$:

$$\sum a_n \;\; \frac{\text{converges}}{\text{diverges}} \qquad \Longleftrightarrow \qquad \sum b_n \;\; \frac{\text{converges}}{\text{diverges}}$$

If $L = 0$ or $L = \infty$, we can still draw an inference, but in only one direction:

- If $L = 0$:

$$\sum b_n \quad \text{converges} \qquad \Longrightarrow \qquad \sum a_n \quad \text{converges}$$

- If $L = \infty$:

$$\sum b_n \quad \text{diverges} \qquad \Longrightarrow \qquad \sum a_n \quad \text{diverges}$$

Extra - Limit Comparison Test: Theory

Suppose $a_n/b_n \to L$ and $0 < L < \infty$. Then for n sufficiently large, we know $a_n/b_n < L + 1$.

Doing some algebra, we get $a_n < (L+1)b_n$ for n large.

If $\sum b_n$ converges, then $\sum (L+1)b_n$ also converges (constant multiple), and then the DCT implies that $\sum a_n$ converges.

Conversely: we also know that $b_n/a_n \to 1/L$, so $b_n < (1/L + 1)a_n$ for all n sufficiently large. Thus if $\sum a_n$ converges, $\sum (1/L+1)a_n$ also converges, and by the DCT again $\sum b_n$ converges too.

The cases with $L = 0$ or $L = \infty$ are handled similarly.

08 Illustration

Limit Comparison Test examples

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ *converges* by the LCT.

- Choose: $a_n = \frac{1}{2^n-1}$ and $b_n = \frac{1}{2^n}$.
- Compare in the limit:

$$\lim_{n\to\infty} \frac{a_n}{b_n} \quad \gg\gg \quad \lim_{n\to\infty} \frac{2^n}{2^n - 1} \quad \gg\gg \quad 1 \ =: \ L$$

- Observe: $\sum \frac{1}{2^n}$ is a convergent geometric series

The series $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$ *diverges* by the LCT.

- Choose: $a_n = \frac{2n^2+3n}{\sqrt{5+n^5}}$, $b_n = n^{-1/2}$
- Compare in the limit:

$$\lim_{n\to\infty} \frac{a_n}{b_n} \quad \gg\gg \quad \lim_{n\to\infty} \frac{(2n^2+3n)\sqrt{n}}{\sqrt{5+n^5}}$$

$$\frac{(2n^2+3n)\sqrt{n}}{\sqrt{5+n^5}} \quad \xrightarrow{n\to\infty} \quad \frac{2n^{5/2}}{n^{5/2}} \to 2 \ =: \ L$$

- Observe: $\sum n^{-1/2}$ is a divergent p-series

The series $\sum_{n=2}^{\infty} \frac{n^2}{n^4 - n - 1}$ *converges* by the LCT.

- Choose: $a_n = \frac{n^2}{n^4-n-1}$ and $b_n = n^{-2}$
- Compare in the limit:

$$\lim_{n\to\infty} \frac{a_n}{b_n} \quad \gg\gg \quad \lim_{n\to\infty} \frac{n^4}{n^4 - n - 1} \quad \gg\gg \quad 1 \; =: \; L$$

- Observe: $\sum_{n=2}^{\infty} n^{-2}$ is a converging p-series

Alternating series

Videos, Math Dr. Bob:

- Alternating Series Test: Theory and basic examples
- Alternating Series Test: Remainder estimates
- Alternating Series Test: Further remainder estimates

09 Theory

Consider these series:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots \quad = \quad \infty$$

$$-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \frac{1}{7} - \cdots \quad = \quad -\infty$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots \quad = \quad \ln 2$$

$$1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \cdots \quad = \quad ?$$

The absolute values of terms are the same between these series, only the signs of terms change.

The first is a **positive series** because there are no negative terms.

The second series is the negation of a positive series – the study of such series is equivalent to that of positive series, just add a negative sign everywhere. These signs can be factored out of the series. (For example $\sum -\frac{1}{n} = -\sum \frac{1}{n}$.)

The third series is an **alternating series** because the signs alternate in a strict pattern, every other sign being negative.

The fourth series is *not* alternating, nor is it positive, nor negative: it has a mysterious or unknown pattern of signs.

A series with any negative signs present, call it $\sum_{n=1}^{\infty} a_n$, **converges absolutely** when the positive series of absolute values of terms, namely $\sum_{n=1}^{\infty} |a_n|$, converges.

THEOREM: Absolute implies ordinary

If a series $\sum_{n=1}^{\infty} a_n$ converges absolutely, then it also converges as it stands.

A series might converge due to the presence of negative terms and yet *not* converge absolutely:

A series $\sum_{n=1}^{\infty} a_n$ is said to be **converge conditionally** when the series converges as it stands, but the series produced by inserting absolute values, namely $\sum_{n=1}^{\infty} |a_n|$, diverges.

The alternating harmonic series above, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2$, is therefore conditionally convergent. Let us see why it converges. We can group the terms to create new sequences of *pairs*, each pair being a positive term. This can be done in two ways. The first creates an increasing sequence, the second a decreasing sequence:

$$\text{increasing from 0:} \quad \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{7} - \frac{1}{8}\right) + \cdots$$

$$\text{decreasing from 1:} \quad 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \left(\frac{1}{6} - \frac{1}{7}\right) - \cdots$$

Suppose S_N gives the sequence of partial sums of the original series. Then S_{2N} gives the first sequence of pairs, namely $S_2, S_4, S_6, \ldots$. And S_{2N-1} gives the second sequence of pairs, namely $S_1, S_3, S_5, \ldots$.

The second sequence shows that S_N is bounded above by 1, so S_{2N} is monotone increasing and bounded above, so it converges. Similarly S_{2N-1} is monotone decreasing and bounded below, so it converges too, and of course they must converge to the same thing.

The fact that the terms were *decreasing in magnitude* was an essential ingredient of the argument for convergence. This fact ensured that the parenthetical pairs were positive numbers.

Alternating Series Test (AST) - "Leibniz Test"

Applicability: Alternating series only: $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ with $a_n > 0$

Test Statement:
If:

- (1) a_n are *decreasing*, so $a_1 > a_2 > a_3 > a_4 > \cdots > 0$
- (2) $a_n \to 0$ as $n \to \infty$ (i.e. it passes the SDT)

Then:

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n \quad \text{converges}$$

Furthermore, partial sum *errors* are bounded by "subsequent terms":

$$|S - S_N| \leq a_{N+1}$$

Extra - Alternating Series Test: Theory

Just as for the alternating harmonic series, we can form *positive* paired-up series because the terms are decreasing:

$$(a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) + \cdots$$

$$a_1 - (a_2 - a_3) - (a_4 - a_5) - (a_6 - a_7) - \cdots$$

The first sequence S_{2N} is monotone increasing from 0, and the second S_{2N-1} is decreasing from a_1. The first is therefore also bounded above by a_1. So it converges. Similarly, the second converges. Their difference at any point is $S_{2N} - S_{2N-1}$ which is equal to $-a_{2N}$, and this goes to zero. So the two sequences must converge to the same thing.

By considering these paired-up sequences and the effect of adding each new term one after the other, we obtain the following order relations:

$$0 < S_2 < S_4 < S_6 < \ \cdots \ < S < \ \cdots \ < S_5 < S_3 < S_1 = a_1$$

Thus, for *any even* $2N$ and *any odd* $2M-1$:

$$S_{2N} < S < S_{2M-1}$$

Now set $M = N$ and subtract S_{2N-1} from both sides:

$$S_{2N} - S_{2N-1} < S - S_{2N-1} < 0$$

$$\gg\gg \quad -a_{2N} < S - S_{2N-1} < 0$$

Now set $M = N+1$ and subtract S_{2N} from both sides:

$$0 < S - S_{2N} < S_{2N+1} - S_{2N}$$

$$\gg\gg \quad 0 < S - S_{2N} < a_{2N+1}$$

This covers both even cases ($n = 2N$) and odd cases ($n = 2N-1$). In either case, we have:

$$|S - S_n| < a_{n+1}$$

10 Illustration

Alternating Series Test: Basic illustration

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ converges by the AST.

- Notice that $\sum \frac{1}{\sqrt{n}}$ diverges as a p-series with $p = 1/2 < 1$.
- Therefore the first series converges *conditionally*.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2}$ converges by the AST.

- Notice the funny notation: $\cos n\pi = (-1)^n$.
- This series converges *absolutely* because $\left|\frac{\cos n\pi}{n^2}\right| = \frac{1}{n^2}$, which is a p-series with $p = 2 > 1$.

Approximating π

The Taylor series for $\tan^{-1} x$ is given by:

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Use this series to approximate π with an error less than 0.001.

Solution

The main idea is to use $\tan \frac{\pi}{4} = 1$ and thus $\tan^{-1} 1 = \frac{\pi}{4}$. Therefore:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

and thus:

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots$$

Write E_n for the error of the approximation, meaning $E_n = S - S_n$.

By the AST error formula, we have $|E_n| < a_{n+1}$.

We desire n such that $|E_n| < 0.001$. Therefore, calculate n such that $a_{n+1} < 0.001$, and then we will know:

$$|E_n| < a_{n+1} < 0.001$$

The general term is $a_n = \frac{4}{2n-1}$. Plug in $n+1$ in place of n to find $a_{n+1} = \frac{4}{2n+1}$. Now solve:

$$a_{n+1} = \frac{4}{2n+1} < 0.001$$

$$\ggg \qquad \frac{4}{0.001} < 2n + 1$$

$$\ggg \qquad 3999 < 2n$$

$$\ggg \qquad 2000 \leq n$$

We conclude that at least 2000 terms are necessary to be confident (by the error formula) that the approximation of π is accurate to within 0.001.