W07 Notes

Sequences and series basics

01 Theory

A **sequence** is a rule that defines a **term** for each natural number $n \in \mathbb{N}$:

$$a_0,\ a_1,\ a_2,\ a_3,\ a_4,\ \dots$$

So a sequence is a function from $\mathbb{N}$ to $\mathbb{R}$.

Geometric sequence

A sequence is called **geometric** if the *ratio of consecutive terms is some constant r*, independent of n:

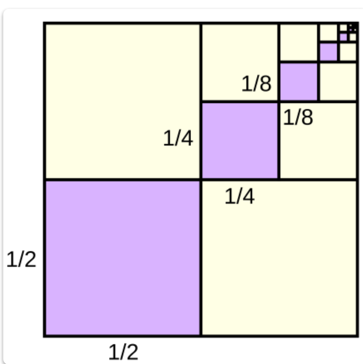
$$\frac{a_{n+1}}{a_n} = r \quad \text{for every } n$$

The defining relation of a geometric sequence is equivalent to $a_{n+1} = a_n \cdot r$.

By plugging $a_1 = a_0 \cdot r$ into $a_2 = a_1 \cdot r$, we have $a_2 = (a_0 \cdot r) \cdot r = a_0 \cdot r^2$. This plugging can be repeated n-times to get a formula for the n^{th} term:

$$a_n = a_{n-1} \cdot r = a_{n-2} \cdot r^2 = a_{n-3} \cdot r^3 = \cdots = a_1 \cdot r^{n-1} = a_0 \cdot r^n$$

Therefore $a_n = a_0 \cdot r^n$, and we have a formula for the **general term** of the sequence (the term with index n).



Starting point of a sequence

Note that sometimes the index (variable) of a sequence starts somewhere other than 0. Most common is 1 but any other starting point is allowed, even negative numbers.

Sometimes c is used instead of a_0 in the formula for the general term of a sequence, thus $a_n = cr^n$. The 'c' notation is useful when the sequence starts from $n \neq 0$.

Extra - Fibonacci sequence

The **Fibonacci** sequence goes like this:

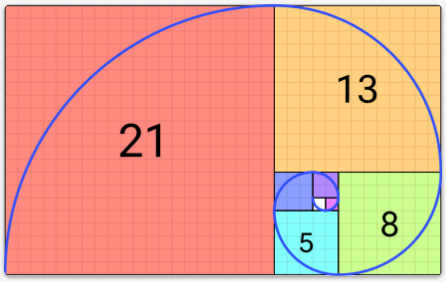
$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \dots$$

The pattern is:

$$F_n = F_{n-1} + F_{n-2}$$

This formula is a **recursion relation**, which means that terms are defined using the values of prior terms.

The Fibonacci sequence is perhaps the most famous sequence of all time. It is related to the Golden Ratio and the Golden Spiral:



02 Illustration

Geometric sequence: revealing the format

Find a_0 and r and a_n (written in the geometric sequence format) for the following geometric sequences:

(a) $a_n = \left(-\frac{1}{2}\right)^n$ (b) $b_n = -3\left(\frac{2^{n+1}}{5^n}\right)$ (c) $c_n = e^{5+7n}$

Solution

(a)
Plug in $n = 0$ to obtain $a_0 = 1$. Notice that $a_{n+1}/a_n = -1/2$ and so therefore $r = -1/2$. Then the 'general term' is $a_n = a_0 \cdot r^n = 1 \cdot (-1/2)^n$.

(b)
Rewrite the fraction:

$$\frac{2^{n+1}}{5^n} \quad \gg\gg \quad 2 \cdot \left(\frac{2}{5}\right)^n$$

Plug that in and observe $b_n = -6 \cdot (2/5)^n$. From this format we can *read off* $b_0 = -6$ and $r = 2/5$.

(c)
Rewrite:

$$c_n \quad \gg\gg \quad e^5 \cdot e^{7n} \quad \gg\gg \quad e^5 \cdot \left(e^7\right)^n$$

From this format we can *read off* $c_0 = e^5$ and $r = e^7$.

03 Theory

A **series** is an infinite sum that is created by successive additions without end. The terms are not added up "all at once" but rather they are added up "as n increases" or "as $n \to \infty$."

$$a_0 + a_1 + a_2 + a_3 + \dots \qquad = \quad \sum_{n=0}^{\infty} a_n$$

Three of the most famous series are the Leibniz series and the geometric series:

$$\text{Leibniz series:} \qquad 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^n}{2n+1} + \cdots \qquad = \quad \frac{\pi}{4}$$

$$\text{Geometric series:} \qquad 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \left(\frac{1}{2}\right)^n + \cdots \qquad = \quad 2$$

Partial sum sequence of a series

The **partial sum sequence** of a series is the *sequence* whose terms are the sums up to the given index:

$$S_N = a_0 + a_1 + \cdots + a_N \qquad = \quad \sum_{n=0}^{N} a_n$$

These S_N terms themselves form a sequence:

$$S_0,\ S_1,\ S_2,\ S_3,\ \dots$$

04 Illustration

Example - Geometric series

The geometric series *total sum S* can be calculated using a *"shift technique"* as follows:

1. Compare S and rS:

$$\begin{aligned} S \quad &= \quad a_0 + a_0 r + a_0 r^2 + a_0 r^3 + \cdots \\ \overset{\times r}{\ggg} \qquad rS \quad &= \quad a_0 r + a_0 r^2 + a_0 r^3 + a_0 r^4 + \cdots \end{aligned}$$

2. Subtract second line from first line, many cancellations:

$$\begin{aligned} S \quad &= \quad a_0 + a_0 r + a_0 r^2 + a_0 r^3 + \cdots \\ -\Big(rS \quad &= \quad a_0 r + a_0 r^2 + a_0 r^3 + a_0 r^4 + \cdots \Big) \\ \hline S - rS \quad &= \quad a_0 \end{aligned}$$

3. Solve to find S:

$$S \quad = \quad \frac{a_0}{1-r}$$

- ⚠ Note: this calculation *assumes* that S exists, i.e. that the series *converges*.

The geometric series *partial sums* can be calculated similarly, as follows:

1. Compare S and rS:

$$\begin{aligned} S_N \quad &= \quad a_0 + a_0 r + a_0 r^2 + \cdots + a_0 r^N \\ \overset{\times r}{\ggg} \qquad rS_N \quad &= \quad a_0 r + a_0 r^2 + \cdots + a_0 r^N + a_0 r^{N+1} \end{aligned}$$

2. Subtract second line from first line, many cancellations:

$$\begin{aligned} S_N &= a_0 + a_0 r + a_0 r^2 + \cdots + a_0 r^N \\ -\Big(r S_N &= a_0 r + a_0 r^2 + \cdots + a_0 r^N + a_0 r^{N+1} \Big) \\ \hline S_N - r S_N &= a_0 - a_0 r^{N+1} \end{aligned}$$

3. Solve to find S_N:

$$S_N = a_0 \frac{1 - r^{N+1}}{1 - r}$$

$$= \frac{a_0}{1-r} - \frac{a_0}{1-r} r^{N+1} = S - S r^{N+1}$$

- The last formula is revealing in its own way. Here is what it means in terms of terms:

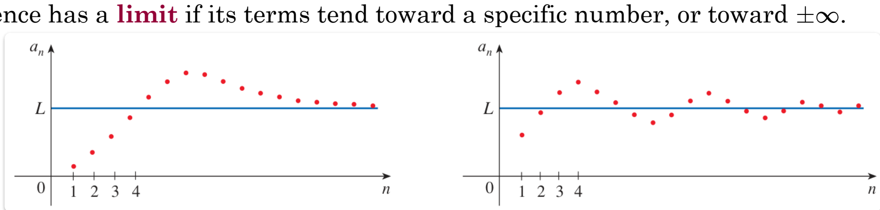
$$a_0 + a_0 r + \cdots + a_0 r^N =$$

$$a_0 + a_0 r + a_0 r^2 + \cdots$$

$$- \left(a_0 r^{N+1} + a_0 r^{N+2} + \cdots \right)$$

05 Theory

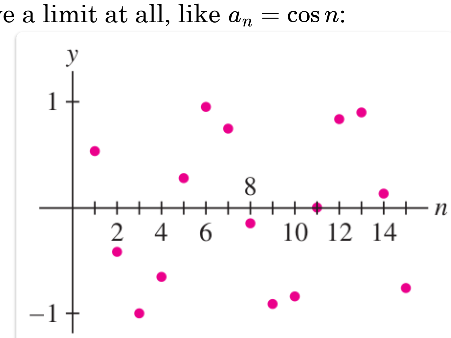
A sequence has a **limit** if its terms tend toward a specific number, or toward $\pm\infty$.



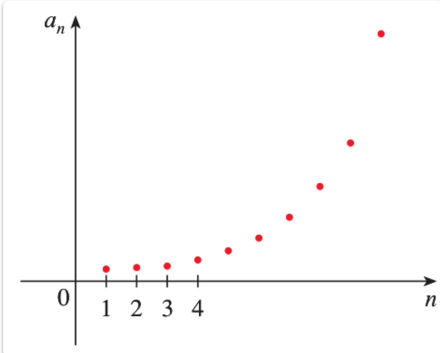
When this happens we can write "$\lim_{n\to\infty} a_n = L$" with some number $L \in \mathbb{R}$ or $L = \pm\infty$. We can also write "$a_n \to L$ as $n \to \infty$".

The sequence is said to **converge** if it has a *finite limit* $L \in \mathbb{R}$.

Some sequences don't have a limit at all, like $a_n = \cos n$:



Or $a_n = e^n$:



These sequences **diverge**.
In the second case, there is a limit $L = \infty$, so we say it **diverges to** $+\infty$.

- The difference between *converging* and *having a limit* is that a limit could 'exist', namely at $+\infty$ or $-\infty$, yet we still say the sequence diverges.

Extra - Convergence definition

The precise meaning of convergence is this. We have $a_n \to L$ as $n \to \infty$ if, given any proposed error $\varepsilon > 0$, it is possible to find N such that for all $n > N$ we have $|a_n - L| < \varepsilon$.

When $L = \infty$, convergence means that given any $B > 0$, we can find N such that for all $n > N$ we have $a_n > B$.

Similarly for $L = -\infty$.

If the general term a_n is a continuous function of n, we can replace n with the continuous variable x and compute the continuous limit instead:

$$\lim_{n\to\infty} a_n \quad = \quad \lim_{x\to\infty} a_x$$

If a_x would be a *differentiable* function, and we discover an *indeterminate form*, then we can apply L'Hopital's Rule to find the limit value. For example, if the indeterminate form is $0 \cdot \infty$, we can convert it to $\frac{\infty}{1/0} = \frac{\infty}{\infty}$ and apply L'Hopital.

06 Illustration

L'Hopital's Rule for sequence limits

(a) What is the limit of $a_n = \dfrac{\ln n}{n}$?
(b) What is the limit of $b_n = \dfrac{(\ln n)^2}{n}$?
(c) What is the limit of $c_n = n\left(\sqrt{n^2+1} - \sqrt{n}\right)$?

Solution
(a)
Identify indeterminate form $\frac{\infty}{\infty}$. Change from n to x and apply L'Hopital:

$$\lim_{x\to\infty} \frac{\ln x}{x} \quad \overset{\frac{d}{dx}}{\ggg} \quad \lim_{x\to\infty} \frac{1/x}{1} = 0$$

(b)

Identify indeterminate form $\frac{\infty}{\infty}$. Change from n to x and apply L'Hopital:

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x} \quad \overset{\frac{d}{dx}}{\ggg} \quad \lim_{x\to\infty} \frac{2\ln x \cdot \frac{1}{x}}{1} = 2 \lim_{x\to\infty} \frac{\ln x}{x} \quad \overset{\text{(by } a_n \text{ result)}}{=} \quad 0$$

(c)

Identify form $\infty \cdot 0$ and rewrite as $\frac{\infty}{\infty}$:

$$n\left(\sqrt{n^2+1} - \sqrt{n}\right) \quad \gg\gg \quad \frac{\sqrt{n^2+1} - \sqrt{n}}{1/n}$$

Change from n to x and apply L'Hopital:

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1} - \sqrt{x}}{1/x} \quad \gg\gg \quad \frac{\frac{1}{2}\left(x^2+1\right)^{-1/2}(2x) - \frac{1}{2}x^{-1/2}}{-1/x^2}$$

Simplify:

$$\gg\gg \quad \frac{-2x^3}{\sqrt{x^2+1}} + x^{3/2} = \frac{-2x^3 + x^{3/2}\sqrt{x^2+1}}{\sqrt{x^2+1}}$$

Consider the limit:

$$\frac{-2x^3 + x^{3/2}\sqrt{x^2+1}}{\sqrt{x^2+1}} \quad \overset{x\to\infty}{\longrightarrow} \quad \frac{-2x^3 + x^{3/2}x}{x} \longrightarrow \frac{-2x^3}{x} \longrightarrow -\infty$$

Extra - Squeeze theorem

Use the squeeze theorem to show that $\frac{4^n}{n!} \to 0$ as $n \to \infty$.

Solution

We will squeeze the given general term above 0 and below a sequence b_n that we must devise:

$$0 \le \frac{4^n}{n!} \le b_n$$

We need b_n to satisfy $b_n \to 0$ and $\frac{4^n}{n!} \le b_n$. Let us study $\frac{4^n}{n!}$.

$$\frac{4^n}{n!} = \frac{4 \cdot 4 \cdot \ \cdots \ \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4}{n(n-1)\cdots 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

Now for the trick. Collect factors in the middle bunch:

$$\frac{4^n}{n!} = \frac{4}{n}\left(\frac{4}{n-1} \cdot \frac{4}{n-2} \cdot \ \cdots \ \cdot \frac{4}{7} \cdot \frac{4}{6} \cdot \frac{4}{5}\right)\frac{4 \cdot 4 \cdot 4 \cdot 4}{4 \cdot 3 \cdot 2 \cdot 1}$$

Each factor in the middle bunch is < 1 so the entire middle bunch is < 1. Therefore:

$$\frac{4^n}{n!} < \frac{4}{n} \cdot \frac{4^4}{4!} = \frac{1024}{24n}$$

Now we can easily see that $1024/24n \to 0$ as $n \to \infty$, so we set $b_n = 1024/24n$ and we are done.

07 Theory

Monotone sequences

A sequence is called **monotone increasing** if $a_{n+1} \geq a_n$ for every n.

A sequence is called **monotone decreasing** if $a_{n+1} \leq a_n$ for every n.

In this context, 'monotone' just means it preserves the increasing or decreasing modality for *all* terms.

Monotonicity Theorem

If a sequence is monotone increasing, and **bounded above** by B, then it *must converge* to some limit L, and $L \leq B$.

If a sequence is monotone decreasing, and **bounded below** by B, then it *must converge* to some limit L, and $L \geq B$.

Terminology:

- *Bounded above by B* means that $a_n \leq B$ for every n
- *Bounded below by B* means that $B \leq a_n$ for every n

Notice!

- The Monotonicity Theorem says that a limit L *exists*, but it does not *provide* the limit value.

08 Illustration

Monotonicity theorem

Show that $a_n = \sqrt{n+1} - \sqrt{n}$ converges.

Solution

1. Observe that $a_n > 0$ for all n.
    - Because $n+1 > n$, we know $\sqrt{n+1} > \sqrt{n}$.
    - Therefore $\sqrt{n+1} - \sqrt{n} > 0$
2. Change n to x and show a_x is decreasing.
    - New formula: $a_x = \sqrt{x+1} - \sqrt{x}$ considered as a *differentiable* function.
    - Take derivative to show decreasing.
        - Derivative of a_x:

$$\frac{d}{dx} a_x = \frac{1}{2\sqrt{x+1}} - \frac{1}{2\sqrt{x}}$$

- Simplify:

$$\gg\gg \qquad \frac{2\left(\sqrt{x}-\sqrt{x+1}\right)}{4\sqrt{x}\sqrt{x+1}}$$

- Denominator is >0. Numerator is <0. So $\frac{d}{dx}a_x < 0$ and a_x is monotone decreasing.

3. Therefore a_n is monotone decreasing as $n \to \infty$.

09 Theory

Series convergence

We say that a *series* converges when its *partial sum sequence converges*:

$$\text{“}\sum_{n=0}^{\infty} a_n \quad \text{converges”} \qquad \text{MEANS:} \qquad \text{“}S_N \quad \text{converges as } N \to \infty\text{”}$$

Let us apply this to the geometric series. Recall our formula for the partial sums:

$$S_N = a_0 \frac{1-r^{N+1}}{1-r}$$

Rewrite this formula:

$$\gg\gg \qquad S_N = \frac{a_0}{1-r} - \frac{a_0}{1-r} r^{N+1}$$

Now take the limit as $N \to \infty$:

$$\lim_{N\to\infty} S_N \quad = \quad \text{“}\frac{a_0}{1-r} - \frac{a_0}{1-r} r^{\infty+1}\text{”} \quad = \quad \frac{a_0}{1-r}$$

- ⚠ So we see that S_N converges exactly when $|r| < 1$. It converges to $\frac{a_0}{1-r}$.

(If $|r| = 1$ then the denominator is 0, and if $|r| > 1$ then the factor $r^{\infty+1}$ does not converge.)

Furthermore, we have the limit value:

$$\sum_{n=0}^{\infty} a_n \quad = \quad \lim_{N\to\infty} S_N = \frac{a_0}{1-r} = S$$

This result confirms the formula we derived for the total S for a geometric series. This time we did not start by assuming S exists, on the contrary we *proved* that S exists. (Provided that $|r| < 1$.)

Extra - Aspects of S and S_N from the geometric series

Notice that we always have the rule:

$$S_N = S - r^{N+1} S$$

$$S_N = \frac{a_0}{1-r} - \frac{a_0}{1-r} r^{N+1}$$

This rule can be viewed as coming from partitioning the full series into a finite part S_N

and the remaining infinite part:

$$S = \underbrace{a_0 + a_0 r + \cdots + a_0 r^N}_{S_N} + \underbrace{a_0 r^{N+1} + a_0 r^{N+2} + \dots}_{S - S_N}$$

We can remove a factor r^{N+1} from the infinite part:

$$S - S_N = r^{N+1}\left(a_0 + a_0 r + a_0 r^2 \ \dots\right)$$

The parenthetical expression is equal to S, so we have the formula $S_N = S - r^{N+1} S$ given above.