W06 Notes

Work

Videos, BlackPenRedPen:

- Work performed: pumping water from trough
- Work performed: pumping water from rectangular tank
- Work performed: pumping water from conical tank
- Work performed: pumping water from spherical tank

01 Theory

Work is a measure of energy expended to achieve some effect. According to physics:

$$\text{Work} = \text{Force} \times \text{Distance}$$

$$W = \int_a^b F(x)\,dx$$

To compute the work performed against gravity while *lifting some matter*, decompose the matter into *horizontal layers* at height y and thickness dy. Each layer is lifted some distance. The weight of the layer gives the force applied.

The work performed on each single layer is summed by an integral to determine the total work performed to lift all the layers:

Work performed

$$\text{Work to lift a layer} = g \times \text{density} \times A(y) \times \text{height raised} \times dy$$

$$\text{Total work} = \int_a^b \rho g\, h(y) A(y)\,dy$$

- $A(y) =$ area of layer
- $h(y) =$ height layer is lifted
- $\rho =$ mass density
- $g = 9.8\,\mathrm{m/s^2} =$ constant of gravitational acceleration

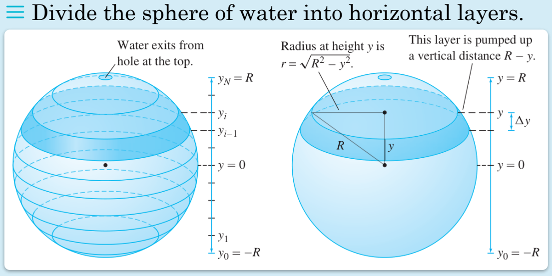
02 Illustration

Example - Pumping water from spherical tank

Calculate the work done pumping water out of a spherical tank of radius $5\,\mathrm{m}$.

Solution

1. Divide the sphere of water into horizontal layers.



   - Coordinate y is $y = 0$ at the center of the sphere.

2. Work done pumping out water is constant across any single layer.
3. Find formula for weight of a single layer.

- Area of the layer at y is $A(y) = \pi(5^2 - y^2)$ because its radius is $r = \sqrt{5^2 - y^2}$.
- Volume of the layer at y is then $\pi(5^2 - y^2)\,dy$.
- Weight of the layer is then $F(y)\,dy = g \cdot \rho \cdot \pi(5^2 - y^2)\,dy$.
  - Plug in:

$$g = 9.8\,\tfrac{m}{s^2}, \quad \rho = 1000\,\tfrac{kg}{m^3} \qquad \gg\gg \qquad F(y)\,dy = \left(9800\,\tfrac{kg}{m^2 s^2}\right)\pi(5^2 - y^2)\,dy$$

4. Find formula for vertical distance a given layer is lifted.
   - Layer at y must be lifted by $5 - y$ to the top of the tank.
5. Work per layer is the product.
   - Product of weight times height lifted:

$$\left(9800\,\tfrac{kg}{m^2 s^2}\right)\pi(5^2 - y^2)(5 - y)\,dy$$

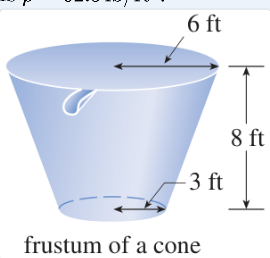
6. Total work done is the integral.
   - Integrate over the layers:

$$W = \int_{-5}^{+5} \left(9800\,\tfrac{kg}{m^2 s^2}\right)\pi(5^2 - y^2)(5 - y)\,dy \quad \approx \quad 2.6 \times 10^7\,\text{J}$$

7. ⚠ Supplement: what if the spigot sits 2m above the tank?
   - Increase the height function from $5 - y$ to $7 - y$.
8. ⚠ Supplement: what if the tank starts at just 3m of water depth?
   - Integrate the water layers only: change bounds to $\int_{-5}^{-2}$.

Example - Water pumped from a frustum

Find the work required to pump water out of the frustum in the figure. Assume the weight of water is $\rho = 62.5\,\text{lb/ft}^3$.



Solution

1. Find weight of a horizontal slice.
   - Coordinate $y = 0$ at top, increasing downwards.
   - Use $r(y)$ for radius of cross-section circle.
   - Linear decrease in r from $r(0) = 6$ to $r(8) = 3$:

$$r(y) = 6 - \frac{3}{8}y$$

   - Area is πr^2:

$$\text{Area}(y) = \pi\left(6 - \tfrac{3}{8}y\right)^2$$

   - Weight $=$ density $\times$ area $\times$ thickness:

$$\text{weight of layer} = \rho\pi\left(6 - \tfrac{3}{8}y\right)^2 dy$$

2. Find work to pump out a horizontal layer.
   - Layer at y is raised a distance of y.

- Work to raise layer at y:

$$\rho\pi y\left(6-\tfrac{3}{8}y\right)^2\,dy$$

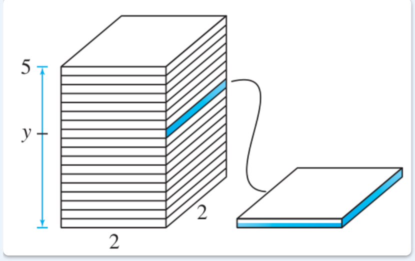
3. Integrate over all layers.
   - Integrate from top to bottom of frustum:

$$\begin{aligned}\int_0^8 \rho\pi y\left(6-\tfrac{3}{8}y\right)^2\,dy &= 528\pi\rho \\ &= 528\pi\cdot 62.5 \\ &\approx 1.04\times 10^5\text{ ft-lb}\end{aligned}$$

   - Final answer is 1.04×10^5 ft-lb.

Example - Raising a building

Find the work done to raise a cement columnar building of height $5\,\text{m}$ and square base $2\,\text{m}$ per side. Cement has a density of $1500\,\text{kg}/\text{m}^3$.



Solution

1. Divide the building into horizontal layers.
   - Work done raising up the layers is constant for each layer.
2. Find formula for weight of each layer.
   - Volume $=$ area $\times$ thickness $= 4\,dy$
   - Mass $=$ density $\times$ volume $= 1500\times 4\,dy = 6000\,dy$
   - Weight of layer $= g\times$ mass $= 9.8\times 6000\,dy = 58800\,dy$
3. Find formula for work performed lifting one layer into place.
   - Work $=$ weight $\times$ distance lifted $= 58800\times y\,dy$
4. Find total work as integral over the layers.
   - Total work $= \int_0^5 58000y\,dy = 735\,\text{kJ}$

Example - Raising a chain

An $80\,\text{ft}$ chain is suspended from the top of a building. Suppose the chain has weight density $0.5\,\text{lb}/\text{ft}$. What is the total work required to reel in the chain?

Solution

1. Divide the chain into horizontal layers.
   - Each layer has vertical thickness dy.
   - Each layer has weight $0.5\,dy$ in lb.
2. Find formula for distance each layer is raised.
   - Each layer is raised from y to $80\,\text{ft}$, a distance of $(80-y)\,\text{ft}$.
3. Compute total work.

- Work to raise each layer is weight times distance raised:

$$\text{Work to raise layer} = (80 - y) \cdot 0.5\, dy$$

- Add the work over all layers:

$$\int_0^{80} (80 - y) \cdot 0.5\, dy = 1600\,\text{ft-lb}$$

Example - Raising a leaky bucket

Suppose a bucket is hoisted by a cable up an $80\,\text{ft}$ tower. The bucket is lifted at a constant rate of $2\,\text{ft/sec}$ and is leaking water weight at a constant rate of $0.2\,\text{lb/sec}$. The initial weight of water is $50\,\text{lb}$. What is the total work performed against gravity in lifting the water? (Ignore the bucket itself and the cable.)

Solution

1. Convert to static format.
   - Compute rate of water weight loss per unit of vertical height:

$$\frac{\text{rate of leak}}{\text{rate of lift}} = \text{leaked weight per foot}, \qquad \frac{0.2\,\text{lb/sec}}{2\,\text{ft/sec}} = 0.1\,\text{lb/ft}$$

   - Choose coordinate $y = 0$ at base, $y = 80$ at top.
   - Compute water weight at each height y:

$$\text{water weight} = (50 - 0.1y)\,\text{lb}$$

2. ⚠ Work formula.
   - Total work is integral of force times infinitesimal distance:

$$\text{work} = \int_a^b F(y)\, dy$$

3. Integrate weight times dy.
   - Plug in weight as force:

$$\text{work} = \int_0^{80} (50 - 0.1y)\, dy$$

   - Compute integral:

$$\gg\gg \qquad 50y - 0.05y^2 \Big|_0^{80} = 3680\,\text{ft-lb}$$

- In the last example, the same bucket passes through each height, it is not sliced into layers.
  - This integral adds work done to lift matter through each dy as if in parallel, and dy is thus representing *distance lifted*.
  - Previous examples have integral adding work done to lift matter through some y or $5 - y$; the matter was sliced into layers with dy featuring in the *weight* of a portion.

Improper integrals

Videos, Math Dr. Bob:

- Improper integrals: Infinite limits
- Improper integrals: Vertical asymptote
- Improper integrals: $\int \frac{1}{x^p}\, dx$

- Improper integrals: $\int \frac{1}{x^2} e^{-1/x}\, dx$

03 Theory

Improper integrals are those for which either a *bound* or the *integrand* itself become *infinite* somewhere on the interval of integration.

Examples:

$$(a) \quad \int_1^\infty \frac{1}{x^2}\, dx, \qquad (b) \quad \int_0^2 \frac{1}{x}\, dx, \qquad (c) \quad \int_{-1}^{+1} \frac{1}{x^2}\, dx$$

- (a) the upper bound is ∞
- (b) the integrand goes to ∞ as $x \to 0^+$
- (c) the integrand is ∞ at the point $0 \in [-1, 1]$

The limit interpretation of (a) is this:

$$\int_1^\infty \frac{1}{x^2}\, dx \quad = \quad \lim_{R\to\infty} \int_1^R \frac{1}{x^2}\, dx$$

The limit interpretation of (b) is this:

$$\int_0^2 \frac{1}{x}\, dx \quad = \quad \lim_{R\to 0^+} \int_R^2 \frac{1}{x}\, dx$$

The limit interpretation of (c) is this:

$$\begin{aligned}\int_{-1}^{+1} \frac{1}{x^2}\, dx \quad &= \quad \int_{-1}^{0} \frac{1}{x^2}\, dx + \int_0^{+1} \frac{1}{x^2}\, dx \\ &= \quad \lim_{R\to 0^-} \int_{-1}^{R} \frac{1}{x^2}\, dx + \lim_{R\to 0^+} \int_R^{+1} \frac{1}{x^2}\, dx\end{aligned}$$

These limits are evaluated using familiar methods.

An improper integral is said to be **convergent** or **divergent** according to whether it may be assigned a finite value through the appropriate *limit interpretation*.

For example, (a) converges while (b) diverges.

04 Illustration

Example - Improper integral - infinite bound

Show that the improper integral $\int_2^\infty \frac{dx}{x^3}$ converges. What is its value?

Solution

1. Replace infinity with a new symbol R.
   - Compute the integral:

$$\int_2^R \frac{dx}{x^3} = -\frac{1}{2}x^{-2}\Big|_2^R = \frac{1}{8} - \frac{1}{2R^2}$$

2. Take limit as $R \to \infty$.
   - Find limit:

$$\lim_{R\to\infty} \frac{1}{8} - \frac{1}{2R^2} = \frac{1}{8}$$

3. Apply definition of improper integral.
   - By definition:

$$\int_2^\infty \frac{dx}{x^3} = \lim_{R\to\infty} \int_2^R \frac{dx}{x^3} = \frac{1}{8}$$

4. Conclude that $\int_2^\infty \frac{dx}{x^3}$ converges and equals $\frac{1}{8}$.

Improper integral - infinite integrand

Show that the improper integral $\int_0^9 \frac{dx}{\sqrt{x}}$ converges. What is its value?

Solution

1. Replace the 0 where $\frac{1}{\sqrt{x}}$ diverges with a new symbol a.
   - Compute the integral:

$$\int_a^9 \frac{dx}{\sqrt{x}} = \int_a^9 x^{-1/2}\,dx = 2x^{+1/2}\Big|_a^9 = 6 - 2\sqrt{a}$$

2. Take limit as $a \to 0^+$.
   - Find limit:

$$\lim_{a\to 0^+} 6 - 2\sqrt{a} = 6$$

3. Apply definition of improper integral.
   - By definition:

$$\int_a^9 \frac{dx}{\sqrt{x}} = \lim_{a\to 0^+} \int_a^9 \frac{dx}{\sqrt{x}} = 6$$

4. Conclude that $\int_0^9 \frac{dx}{\sqrt{x}}$ converges to 6.

Example - Improper integral - infinity inside the interval

Does the integral $\int_{-1}^{+1} \frac{1}{x}\,dx$ converge or diverge?

Solution

It is *tempting* to compute the integral *incorrectly*, like this:

$$\int_{-1}^{+1} \frac{1}{x}\,dx = \ln|x|\Big|_{-1}^{+1} = \ln|2| - \ln|-2| = 0$$

But this is wrong. There is an infinite integrand at $x = 0$. We must instead break it into parts.

1. ⚠ Identify infinite integrand at $x = 0$.
   - Integral becomes:

$$\int_{-1}^{+1} \frac{1}{x}\,dx = \int_{-1}^{0} \frac{1}{x}\,dx + \int_{0}^{+1} \frac{1}{x}\,dx$$

2. Interpret improper integrals.
   - Limit interpretations:

$$\lim_{R\to 0^-} \int_{-1}^{R} \frac{1}{x}\,dx + \lim_{R\to 0^+} \int_{R}^{+1} \frac{1}{x}\,dx$$

3. Compute integrals.
   - Using $\int \frac{1}{x}\,dx = \ln|x| + C$:

$$\int_{-1}^{R} \frac{1}{x}\,dx = \ln|R| - \ln|-1| = \ln|R|, \qquad \int_{R}^{+1} \frac{1}{x}\,dx = \ln|1| - \ln|R| = -\ln R$$

4. Take limits.

- We have:

$$\lim_{R\to 0^-} \ln|R| = -\infty, \qquad \lim_{R\to 0^+} -\ln R = +\infty$$

- *Neither* limit is finite. For $\int_{-1}^{+1} \frac{1}{x}\,dx$ to exist we'd need *both* of these limits to be finite.

05 Theory

Two tools allow us to determine convergence of a large variety of integrals. They are the **comparison test** and the **p-integral cases**.

The comparison test says:

- When an integral converges, every *smaller* integral converges.
- When an integral diverges, every *bigger* integral diverges.

Here, smaller and bigger are comparisons of the *integrand* (accounting properly for signs), and the bounds are assumed to be the same.

For example, since $\int_2^\infty \frac{dx}{x^3}$ converges, and $x^4 > x^3$ implies $\frac{1}{x^4} < \frac{1}{x^3}$ (when $x > 1$), the comparison test implies that $\int_2^\infty \frac{dx}{x^4}$ also converges.

p-integral cases

Assume $p > 0$ and $a > 0$. We have:

$$\begin{array}{llll} p > 1: & \displaystyle\int_a^\infty \frac{dx}{x^p} \quad \text{converges} & \text{and} & \displaystyle\int_0^a \frac{dx}{x^p} \quad \text{diverges} \\ p < 1: & \displaystyle\int_a^\infty \frac{dx}{x^p} \quad \text{diverges} & \text{and} & \displaystyle\int_0^a \frac{dx}{x^p} \quad \text{converges} \\ p = 1: & \displaystyle\int_a^\infty \frac{dx}{x} \quad \text{diverges} & \text{and} & \displaystyle\int_0^a \frac{dx}{x} \quad \text{diverges} \end{array}$$

Proving the p-integral cases

It is easy to prove the convergence / divergence of each p-integral case using the limit interpretation and the power rule for integrals. (Or for $p = 1$, using $\int \frac{1}{x}\,dx = \ln x + C$.)

Additional improper integral types

The improper integral $\int_{-\infty}^a f(x)\,dx$ also has a limit interpretation:

$$\int_{-\infty}^a f(x)\,dx \quad = \quad \lim_{R\to-\infty} \int_R^a f(x)\,dx$$

The **double improper** integral $\int_{-\infty}^\infty f(x)\,dx$ has this limit interpretation:

$$\int_{-\infty}^\infty f(x)\,dx \quad = \quad \lim_{R\to-\infty} \int_R^a f(x)\,dx + \lim_{R\to\infty} \int_a^R f(x)\,dx$$

Where a is any finite number. This double integral does not exist if either limit does not exist for any value of a.

⚠ Double improper is not simultaneous!

Watch out! This may happen:

$$\int_{-\infty}^{\infty} f(x)\,dx \quad \neq \quad \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$$

This simultaneous limit might exist only due to internal cancellation, in a case where the separate individual limits do not exist!

06 Illustration

Example - Comparison to p-integrals

Determine whether the integral converges:

- (a) $\int_2^\infty \frac{x^3}{x^4-1}\,dx$
- (b) $\int_1^\infty \frac{1}{x^2+x+1}\,dx$

Solution
(a)

1. ⚠ Integrand tends toward $1/x$ for large x.
   - Consider large x values:

$$\frac{x^3}{x^4-1} \quad\longrightarrow\quad \frac{x^3}{x^4} \quad \text{for} \quad x\to\infty, \qquad \text{and} \quad \frac{x^3}{x^4} = \frac{1}{x}$$

2. Try comparison to $1/x$.
   - Comparison attempt:

$$\frac{x^3}{x^4-1} \overset{?}{>} \frac{1}{x}$$

   - Validate. Notice $x^4-1>0$ and $x>0$ when $x\geq 2$.

$$\frac{x^3}{x^4-1} \overset{?}{>} \frac{1}{x} \qquad \gg\gg \qquad x^3\cdot x \overset{?}{>} 1\cdot(x^4-1) \qquad \gg\gg \qquad x^4 \overset{\checkmark}{>} x^4-1$$

3. Apply comparison test.
   - We know:

$$\frac{x^3}{x^4-1} > \frac{1}{x}$$

   - We know:

$$\int_2^\infty \frac{1}{x}\,dx \qquad \text{diverges}$$

   - We conclude:

$$\int_2^\infty \frac{x^3}{x^4-1}\,dx \qquad \text{diverges}$$

(b)

1. ⚠ Integrand tends toward $1/x^2$ for large x.
   - Consider large x values:

$$\frac{1}{x^2+x+1} \quad\longrightarrow\quad \frac{1}{x^2} \quad \text{for} \quad x\to\infty$$

2. Try comparison to $1/x^2$.

- Comparison attempt:

$$\frac{1}{x^2+x+1} \overset{?}{<} \frac{1}{x^2}$$

- Validate. Notice $x^2+x+1>0$ and $x^2>0$ when $x \geq 1$.

$$\frac{1}{x^2+x+1} \overset{?}{<} \frac{1}{x^2} \qquad \gg\gg \qquad 1 \cdot x^2 \overset{?}{<} 1 \cdot (x^2+x+1) \qquad \gg\gg \qquad x^2 \overset{\checkmark}{<} x^2+x+1$$

3. Apply comparison test.
   - We know:

$$\frac{1}{x^2+x+1} < \frac{1}{x^2}$$

   - We know:

$$\int_1^\infty \frac{1}{x^2}\,dx \qquad \text{converges}$$

   - We conclude:

$$\int_1^\infty \frac{1}{x^2+x+1}\,dx \qquad \text{converges}$$