W02 Notes

Trig power products

Videos, Math Dr. Bob:

- Trig power products: $\int \cos^m x \sin^n x\, dx$
- Trig differing frequencies: $\int \cos mx \sin nx\, dx$
- Trig tan and sec: $\int \tan^m x \sec^n x\, dx$
- Secant power: $\int \sec^5 x\, dx$

Videos, Organic Chemistry Tutor:

- Trig power product techniques
- Trig substitution

01 Theory

Review: trig identities

- $\sin^2 x + \cos^2 x = 1$
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

Trig power product: sin / cos

A sin / cos power product has this form:

$$\int \cos^m x \cdot \sin^n x\, dx$$

for some integers m and n (even negative!).

To compute these integrals, use a sequence of these techniques:

- **Swap an even bunch.**
- **u-sub for power-one.**
- **Power-to-frequency conversion.**

- Memorize these three techniques!

Examples of trig power products:

- $\displaystyle\int \sin x \cdot \cos^7 x\, dx$
- $\displaystyle\int \sin^3 x\, dx$
- $\displaystyle\int \sin^2 x \cdot \cos^2 x\, dx$

Swap an even bunch

If *either* $\cos^m x$ or $\sin^n x$ is an *odd* power, use

$$\sin^2 x \quad \gg\gg \quad 1-\cos^2 x$$

$$\text{OR} \quad \cos^2 x \quad \gg\gg \quad 1-\sin^2 x$$

(maybe repeatedly) to convert an **even bunch** to the opposite trig type.

An **even bunch** is *all but one* from the odd power.

For example:

$$\begin{aligned}
\sin^5 x \cdot \cos^8 x \quad &\gg\gg \quad \sin x\,(\sin^2 x)^2 \cdot \cos^8 x \\
&\gg\gg \quad \sin x\,(1-\cos^2 x)^2 \cdot \cos^8 x \\
&\gg\gg \quad \sin x\,\left(1-2\cos^2 x+\cos^4 x\right) \cdot \cos^8 x \\
&\gg\gg \quad \sin x\,\left(\cos^8 x - 2\cos^{10} x + \cos^{12} x\right) \\
&\gg\gg \quad \sin x \cos^8 x - 2\sin x \cos^{10} x + \sin x \cos^{12} x
\end{aligned}$$

u-sub for power-one

If $m=1$ or $n=1$, *perform u-substitution* to do the integral.

The *other* trig power becomes a u power; the power-one becomes du.

For example, using $u=\cos x$ and thus $du=-\sin x\,dx$ we can do:

$$\int \sin x \cos^8 x\,dx \quad \gg\gg \quad \int -\cos^8 x(-\sin x\,dx) \quad \gg\gg \quad -\int u^8\,du$$

- By combining these tricks you can do any power product with at least one odd power!
  - Leave a power-one from the odd power when swapping an even bunch.
- Notice: $1=\sin^0 x=\cos^0 x$, even powers. So the method works for $\int \sin^3 x\,dx$ and similar.

Power-to-frequency conversion

Using these 'power-to-frequency' identities (maybe repeatedly):

$$\sin^2 x = \tfrac{1}{2}(1-\cos 2x), \qquad \cos^2 x = \tfrac{1}{2}(1+\cos 2x)$$

change an even power (either type) into an odd power of cosine.

For example, consider the power product:

$$\sin^4 x \cdot \cos^6 x$$

You can substitute appropriate powers of $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ and $\cos^2 x = \frac{1}{2}(1+\cos 2x)$:

$$\begin{aligned}
\sin^4 x \cdot \cos^6 x \quad &\gg\gg \quad \left(\sin^2 x\right)^2 \cdot \left(\cos^2 x\right)^3 \\
&\gg\gg \quad \left(\tfrac{1}{2}(1-\cos 2x)\right)^2 \cdot \left(\tfrac{1}{2}(1+\cos 2x)\right)^3
\end{aligned}$$

By doing some annoying algebra, this expression can be expanded as a sum of *smaller* powers of $\cos 2x$:

$$\left(\tfrac{1}{2}(1-\cos 2x)\right)^2 \cdot \left(\tfrac{1}{2}(1+\cos 2x)\right)^3$$

$$\gg\gg \quad \frac{1}{32}\Big(1+\cos(2x)-2\cos^2(2x)-2\cos^3(2x)+\cos^4(2x)+\cos^5(2x)\Big)$$

Each of these terms can be integrated by repeating the same techniques.

02 Illustration

Example - Trig power product with an odd power

Compute the integral:

$$\int \cos^2 x \cdot \sin^5 x \, dx$$

Solution

1. Swap over the even bunch.
   - Max even bunch leaving power-one is $\sin^4 x$:

   $$\sin^5 x \qquad \gg\gg \qquad \sin x \left(\sin^2 x\right)^2 \qquad \gg\gg \qquad \sin x \left(1-\cos^2 x\right)^2$$

   - Apply to $\sin^5 x$ in the integrand:

   $$\int \cos^2 x \cdot \sin^5 x \, dx \qquad \gg\gg \qquad \int \cos^2 x \cdot \sin x \left(1-\cos^2 x\right)^2 dx$$

2. Perform u-substitution on the power-one integrand.
   - Set $u = \cos x$.
   - Hence $du = \sin x \, dx$. Recognize this in the integrand.
   - Convert the integrand:

   $$\int \cos^2 x \cdot \sin x \left(1-\cos^2 x\right)^2 dx \qquad \gg\gg \qquad \int \cos^2 x \cdot \left(1-\cos^2 x\right)^2 \left(\sin x \, dx\right)$$

   $$\gg\gg \qquad \int u^2 \cdot (1-u^2)^2 \, du$$

3. Perform the integral.
   - Expand integrand and use power rule to obtain:

   $$\int u^2 \cdot (1-u^2)^2 \, du = \frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 + C$$

   - Insert definition $u = \cos x$:

   $$\int \cos^2 x \cdot \sin^5 x \, dx \quad \gg\gg \quad \int u^2 \cdot (1-u^2)^2 \, du$$

   $$\gg\gg \quad \frac{1}{3}\cos^3 x - \frac{2}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C$$

4. This is our final answer.

03 Theory

Trig power product: $\tan$ / $\sec$ **or** $\cot$ / $\csc$

A tan / sec power product has this form:

$$\int \tan^m x \cdot \sec^n x \, dx$$

A cot / csc power product has this form:

$$\int \cot^m x \cdot \csc^n x \, dx$$

To integrate these, **swap an even bunch** using:

- $\tan^2 x + 1 = \sec^2 x$

OR:

- $\cot^2 x + 1 = \csc^2 x$

Or do **u-substitution** using:

- $u = \tan x \quad \rightsquigarrow \quad du = \sec^2 x \, dx$
- $u = \sec x \quad \rightsquigarrow \quad du = \sec x \, \tan x \, dx$

OR:

- $u = \cot x \quad \rightsquigarrow \quad du = -\csc^2 x \, dx$
- $u = \csc x \quad \rightsquigarrow \quad du = -\csc u \, \cot u \, dx$

Note:

- ⓘ There is no simple "power-to-frequency conversion" for tan / sec !

We can modify the power-one technique to solve some of these. We need to swap over an even bunch *from the odd power* so that exactly the du factor is left behind.

Considering all the possibilities, one sees that this method works when:

- $\tan^m x$ is an *odd* power
- $\sec^n x$ is an *even* power

Quite a few cases escape this method:

- Any $\int \tan^m x \, dx$ with no power of $\sec x$
- Any $\int \tan^m x \cdot \sec^n x \, dx$ for m even and n odd

These tricks don't work for $\int \tan x \, dx$ or $\int \sec x \, dx$ or $\int \tan^4 x \, \sec^5 x \, dx$, among others.

Special integrals: tan and sec

We have:

$$\int \tan x \, dx = \ln |\sec x| + C$$

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C$$

- ⓘ These integrals should be memorized individually.

Extra - Deriving special integrals - tan and sec

The first formula can be found by u-substitution, considering that $\tan x = \frac{\sin x}{\cos x}$.

The second formula can be derived by multiplying $\sec x$ by a special "1", computing instead $\int \frac{\sec x(\sec x+\tan x)}{\sec x+\tan x}\, dx$ by expanding the numerator and doing u-sub on the denominator.

04 Illustration

Example - Trig power product with tan and sec

Compute the integral:

$$\int \tan^5 x \cdot \sec^3 x \, dx$$

Solution

1. Try $du = \sec^2 x \, dx$.
   - Factor du out of the integrand:

     $$\int \tan^5 x \cdot \sec^3 x \, dx \qquad \gg\gg \qquad \int \tan^5 x \cdot \sec x \left(\sec^2 x \, dx\right)$$

   - We then must swap over remaining $\sec x$ into the $\tan x$ type.
   - Cannot do this because $\sec x$ has odd power. Need even to swap.
2. Try $du = \sec x \, \tan x \, dx$.
   - Factor du out of the integrand:

     $$\int \tan^5 x \cdot \sec^3 x \, dx \qquad \gg\gg \qquad \int \tan^4 x \cdot \sec^2 x \left(\sec x \, \tan x \, dx\right)$$

   - Swap remaining $\tan x$ into $\sec x$ type:

     $$\int (\tan^2 x)^2 \cdot \sec^2 x \left(\sec x \, \tan x \, dx\right)$$

     $$\gg\gg \qquad \int \left(\sec^2 x - 1\right)^2 \cdot \sec^2 x \left(\sec x \, \tan x \, dx\right)$$

   - Substitute $u = \sec x$ and $du = \sec x \, \tan x \, dx$:

     $$\gg\gg \qquad \int (u^2 - 1)^2 \cdot u^2 \, du$$

3. Compute the integral in u and convert back to x.
   - Expand the integrand:

     $$\gg\gg \qquad \int u^6 - 2u^4 + u^2 \, du$$

   - Apply power rule:

     $$\gg\gg \qquad \frac{u^7}{7} - 2\frac{u^5}{5} + \frac{u^3}{3} + C$$

   - Plug back in, $u = \sec x$:

     $$\gg\gg \qquad \frac{\sec^7 x}{7} - 2\frac{\sec^5 x}{5} + \frac{\sec^3 x}{3} + C$$

4. ≡ This is our final answer.

Trig substitution

Videos, Math Dr. Bob:

- Trig sub 1: Basics and $\int \frac{1}{\sqrt{36-x^2}}\,dx$ and $\int \frac{x}{36+x^2}\,dx$ and $\int \frac{1}{\sqrt{x^2-36}}\,dx$
- Trig sub 2: $\int \frac{dx}{(1+x^2)^{5/2}}$
- Trig sub 3: $\int \frac{x^2}{\sqrt{1-4x^2}}\,dx$
- Trig sub 4: $\int \sqrt{e^{2x}-1}\,dx$
- Trig sub 5: $\int \frac{\sqrt{4-36x^2}}{x^2}\,dx$

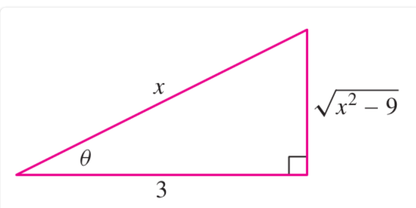
05 Theory

Certain algebraic expressions have a secret meaning that comes from the Pythagorean Theorem. This meaning has a very simple expression in terms of trig functions of a certain angle.

For example, consider the integral:

$$\int \frac{1}{x^2\sqrt{x^2-9}}\,dx$$

Now consider this triangle:



The triangle determines the relation $x = 3\sec\theta$, and it implies $\sqrt{x^2-9} = 3\tan\theta$.

Now plug these into the integrand above:

$$\frac{1}{x^2\sqrt{x^2-9^2}} \quad \gg\gg \quad \frac{1}{9\sec^2\theta\cdot 3\tan\theta}$$

Considering that $dx = 3\sec\theta\,\tan\theta\,d\theta$, we obtain a very reasonable trig integral:

$$\int \frac{1}{x^2\sqrt{x^2-9^2}}\,dx \quad \gg\gg \quad \int \frac{3\sec\theta\,\tan\theta}{27\sec^2\theta\,\tan\theta}\,d\theta$$

$$\gg\gg \quad \frac{1}{9}\int \cos\theta\,d\theta \quad \gg\gg \quad \frac{1}{9}\sin\theta + C$$

We must rewrite this in terms of x using $x = 3\sec\theta$ to finish the problem. We need to find $\sin\theta$ assuming that $\sec\theta = \frac{x}{3}$. To do this, refer back to the triangle to see that $\sin\theta = \frac{\sqrt{x^2-9}}{x}$. Plug this in for our final value of the integral:

$$\frac{1}{9}\sin\theta + C \quad \gg\gg \quad \frac{\sqrt{x^2-9}}{9x} + C$$

Here is the moral of the story:

- Re-express the Pythagorean expression *using a triangle and a trig substitution*.
  - In this way, square roots of quadratic polynomials can be eliminated.

There are always three steps for these trig sub problems:

- (1) Identify the trig sub: find the sides of a triangle and relevant angle θ.
- (2) Solve a trig integral (often a power product).
- (3) Refer back to the triangle to convert the answer back to x.

To speed up your solution process for these problems, *memorize* these three transformations:

(1)

$$\sqrt{a^2-x^2} \quad \overset{x=a\sin\theta}{\gg\gg} \quad \sqrt{a^2-a^2\sin^2\theta}=a\cos\theta \qquad \text{from} \quad 1-\sin^2\theta=\cos^2\theta$$

(2)

$$\sqrt{a^2+x^2} \quad \overset{x=a\tan\theta}{\gg\gg} \quad \sqrt{a^2+a^2\tan^2\theta}=a\sec\theta \qquad \text{from} \quad 1+\tan^2\theta=\sec^2\theta$$

(3)

$$\sqrt{x^2-a^2} \quad \overset{x=a\sec\theta}{\gg\gg} \quad \sqrt{a^2\sec^2\theta-a^2}=a\tan\theta \qquad \text{from} \quad \sec^2\theta-1=\tan^2\theta$$

For a more complex quadratic with linear and constant terms, you will need to first *complete the square* for the quadratic and then do the trig substitution.

06 Illustration

Example - Trig sub in quadratic: completing the square

Compute the integral:

$$\int \frac{dx}{\sqrt{x^2-6x+11}}$$

Solution

1. Notice square root of a quadratic.
2. Complete the square to obtain Pythagorean form.
   - Find constant term for a complete square:

   $$x^2-6x+\left(\frac{-6}{2}\right)^2=x^2-6x+9=(x-3)^2$$

   - Add and subtract desired constant term:

   $$x^2-6x+11 \quad \gg\gg \quad x^2-6x+9-9+11$$

   - Simplify:

   $$x^2-6x+9-9+11 \quad \gg\gg \quad (x-3)^2+2$$

3. Perform shift substitution.
   - Set $u=x-3$ as inside the square:

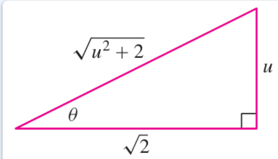
   $$(x-3)^2+2=u^2+2$$

   - Infer $du=dx$.
   - Plug into integrand:

   $$\int \frac{dx}{\sqrt{x^2-6x+11}} \quad \gg\gg \quad \int \frac{du}{\sqrt{u^2+2}}$$

4. ⚠ Trig sub with $\tan\theta$.

- Identify triangle:



- Use substitution $u = \sqrt{2}\tan\theta$. (From triangle or memorized tip.)
- Infer $du = \sqrt{2}\sec^2\theta\, d\theta$.
- Plug in data:

$$\int \frac{du}{\sqrt{u^2+2}} \qquad \gg\gg \qquad \int \frac{\sec^2\theta}{\sec\theta}\, d\theta = \int \sec\theta\, d\theta$$

5. Compute trig integral.

- Use ad hoc formula:

$$\int \sec\theta\, d\theta = \ln|\tan\theta + \sec\theta| + C$$

6. Convert trig back to x.

- First in terms of u, referring to the triangle:

$$\tan\theta = \frac{u}{\sqrt{2}}, \qquad \sec\theta = \frac{\sqrt{u^2+2}}{\sqrt{2}}$$

- Then in terms of x using $u = x - 3$.
- Plug everything in:

$$\ln|\tan\theta + \sec\theta| + C \qquad \gg\gg \qquad \ln\left|\frac{x-3}{\sqrt{2}} + \frac{\sqrt{(x-3)^2+2}}{\sqrt{2}}\right| + C$$

7. Simplify using log rules.

- Log rule for division gives us:

$$\ln\frac{f(x)}{a} = \ln f(x) - \ln a$$

- The common denominator $\frac{1}{\sqrt{2}}$ can be pulled outside as $-\ln\sqrt{2}$.
- The new term $-\ln\sqrt{2}$ can be "absorbed into the constant" (redefine C).
- So we write our final answer thus:

$$\ln\left|x - 3 + \sqrt{(x-3)^2+2}\right| + C$$