In-Class Practice Problems Solutions

Show whether the series is absolutely convergent (AC), conditionally convergent (CC), or Divergent (D).

1) $\sum (-1)^n \frac{1}{n^4}$

$a_n = (-1)^n \frac{1}{n^4}$ $\quad b_n = |a_n| = \frac{1}{n^4}$

$\sum b_n$ is a convergent p-series, $p = 4 > 1$

$\sum a_n$ is $\boxed{\text{absolutely convergent}}$

2) $\sum (-1)^n \frac{n}{n^2+2}$

$a_n = (-1)^n \frac{n}{n^2+2}$ $\quad b_n = |a_n| = \frac{n}{n^2+2} \geq 0$ $\quad c_n = \frac{1}{n} \geq 0$

$\lim_{n\to\infty} \frac{b_n}{c_n} = \lim_{n\to\infty} \frac{n}{n^2+2} \cdot \frac{n}{1} = 1 \neq 0$; finite

$\sum c_n$ is a divergent p-series $p = 1 \not> 1$

$\sum b_n$ is divergent by LCT

$\sum a_n$ is Not AC

$b_{n+1} \leq b_n$

$\lim_{n\to\infty} b_n = 0$

$\sum a_n$ is convergent by AST

$\sum a_n$ is $\boxed{CC}$

3) $\sum (-1)^n \frac{n^2+2}{n}$

$\lim_{n\to\infty} (-1)^n \frac{n^2+2}{n} \neq 0$ (DNE)

$\sum (-1)^n \frac{n^2+2}{n}$ is $\boxed{\text{divergent}}$ by DT

What is the Maclaurin series for $f(x)$?

4) $f(x) = 5x^2 \cos(3x^2)$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos(3x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (3x^2)^{2n}}{(2n)!}$$

$$5x^2 \cos(3x^2) = 5x^2 \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n}}{(2n)!} x^{4n}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n}(5)}{(2n)!} x^{4n+2}}$$

5) $f(x) = 6e^{5x^3}$

$$6e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$6e^{5x^3} = 6\sum_{n=0}^{\infty} \frac{(5x^3)^n}{n!}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{6(5^n)}{n!} x^{3n}}$$

6) $f(x) = -\ln(1+4x)$

$$\ln(1-x) = \sum_{n=0}^{\infty} -\frac{x^{n+1}}{n+1}$$

$$\ln(1+4x) = \sum_{n=0}^{\infty} \frac{-(-4x)^{n+1}}{n+1}$$

$$-\ln(1+4x) = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(4^{n+1})}{n+1} x^{n+1}}$$

What is the function for the Maclaurin series?

7) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{2n+1}$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$\tan^{-1} x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{2n+1}$$

$$\tan^{-1} x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{2n+1}$$

$$\boxed{x \tan^{-1} x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{2n+1}$$

8) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(2n)!}$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$$

$$\boxed{x^3 \cos x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n)!}$$

9) $\sum_{n=0}^{\infty} (-5)^n \frac{x^{2n}}{n!}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-5x^2} = \sum_{n=0}^{\infty} \frac{(-5x^2)^n}{n!}$$

$$\boxed{e^{-5x^2}} = \sum_{n=0}^{\infty} \frac{(-1)^n 5^n x^{2n}}{n!}$$

What is the sum of the series?

10) $\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n+2}}{4^n(2n+1)!}$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin \frac{\pi}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{2^{2n+1}(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{2(4^n)(2n+1)!}$$

$$2\pi \sin \frac{\pi}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+2}}{4^n(2n+1)!} = \boxed{2\pi}$$

11) $\sum_{n=0}^{\infty}(-1)^n \frac{5^{2n}}{n!}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-5^2} = \sum_{n=0}^{\infty} \frac{(-5^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n}}{n!}$$

$$\boxed{e^{-25}}$$

12) If $f(x) = x^3\cos(2x^2)$, then $f^{(83)}(0) = ?$

$$x^3\cos(2x^2) = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n (2x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{4n+3}}{(2n)!}$$

$$C_{83} = \frac{f^{(83)}(0)}{83!} = \frac{(-1)^{20} 2^{40}}{40!} \longrightarrow f^{(83)}(0) = \boxed{\frac{2^{40}(83!)}{40!}}$$

13) If $f(x) = 3xe^{-x^2}$, then $f^{(44)}(0) = ?$

$$3xe^{-x^2} = 3x \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (3) x^{2n+1}}{n!}$$

$$C_{44} = 0 \quad \text{so} \quad \boxed{f^{(44)}(0) = 0}$$

14) Estimate $\int_0^{0.1} 3e^{-x^2}\,dx$ with |error| < 0.000001.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$3e^{-x^2} = 3\sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n(3)}{n!}x^{2n}$$

$$\int_0^{0.1} 3e^{-x^2}dx = \int_0^{0.1}\sum_{n=0}^{\infty}\frac{(-1)^n(3)}{n!}x^{2n}\,dx$$

$$= \left[\sum_{n=0}^{\infty}\frac{(-1)^n(3)}{n!}\frac{x^{2n+1}}{2n+1}\right]_0^{0.1}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n(3)}{n!(2n+1)}(0.1)^{2n+1} - \cancel{\sum_{n=0}^{\infty}\frac{(-1)^n(3)}{n!(2n+1)}(0)^{2n+1}} \to 0$$

$$= 3(0.1) - \frac{3}{3}(0.1)^3 + \frac{3}{2(5)}(0.1)^5 - \frac{3}{6(7)}(0.1)^7 + \cdots$$

$$\approx 0.3 - 0.1^3 + 3(0.1)^6 \qquad |\text{error}| < \frac{3}{42}(0.1)^7 < 1\times10^{-6}$$

$$\approx \boxed{0.299003}$$

15) If $f(x)$ is equal to its power series, $\sum_{n=0}^{\infty}(-1)^n\frac{(x-1)^{2n}}{5^n}$, what is the power series representation for $f'(x)$, centered at a = 1?

$$f'(x) = \boxed{\sum_{n=1}^{\infty}\frac{(-1)^n}{5^n}(2n)(x-1)^{2n-1}}$$

16) Find the radius and interval of convergence for the power series, $\sum_{n=0}^{\infty}(-1)^n\frac{(x-3)^n}{n^2+1}$.