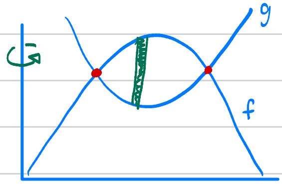
1. Using the method of shells, find the volumes of the solids given by revolving the regions:

- (a) The region enclosed between $\overset{f}{-x^2 + 5x}$ and $\overset{g}{x^2 - 5x + 8}$. Rotate about the y-axis.
- (b) The region under the curve $y = \frac{1}{\sqrt{x}}$ for $1 \le x \le 2$. Rotate about the line $x = -2$.

(a) Notice $f + g = 8$.

$-x^2 + 5x = x(5-x)$

$x^2 - 5x + 8 = 8 - x(5-x)$



Shells: $dV = 2\pi R h\,dx = 2\pi x(f-g)\,dx$

Intersections: •, • at $f = g \rightsquigarrow -x^2 + 5x = x^2 - 5x + 8$

$\rightsquigarrow 2x^2 - 10x + 8 = 0$

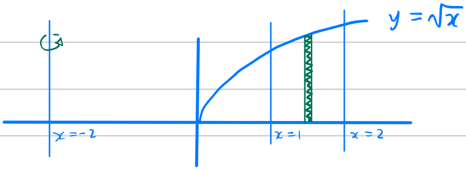
$\rightsquigarrow x^2 - 5x + 4 = 0$

$\rightsquigarrow (x-1)(x-4) = 0$

$\rightsquigarrow x = 1, 4$

$$V = \int_1^4 2\pi x(-2x^2 + 10x - 8)\,dx = \boxed{45\pi}$$

(b)



$dV = 2\pi R h\,dx = 2\pi(x+2)\sqrt{x}\,dx$

$$V = \int_1^2 2\pi(x+2)\sqrt{x}\,dx$$

$$x^{3/2} + 2x^{1/2} \to \left.\frac{x^{5/2}}{5/2} + \frac{2x^{3/2}}{3/2}\right|_1^2$$

$$= \frac{2}{5}2^{5/2} + \frac{4}{3}2^{3/2} - \left(\frac{2}{5} + \frac{4}{3}\right)$$

$$= \frac{8}{5}\sqrt{2} + \frac{8}{3}\sqrt{2} - \frac{26}{15} = \frac{64}{15}\sqrt{2} - \frac{26}{15}$$

$$\xrightarrow{\times 2\pi} \boxed{\frac{4\pi}{15}(32\sqrt{2} - 13)}$$

3. Consider the curve $y = x^2$ on $0 \le x \le 1$.

- (a) Find the arc length of this curve.
- (b) Using the method of bands, find the surface area of the revolution about the x-axis.

(a) $ds = \sqrt{1+f'^2}\,dx \ggg \sqrt{1+(2x)^2}\,dx \ggg \sqrt{1+4x^2}\,dx$

$$\text{arclength} = \int_a^b ds \ggg \int_0^1 \sqrt{1+4x^2}\,dx$$

sub $x = \frac{1}{2}\tan\theta$, $1+4x^2 = 1+\tan^2\theta = \sec^2\theta$

$dx = \frac{1}{2}\sec^2\theta\,d\theta$ $\qquad x \in [0,1] \rightsquigarrow \theta \in [0, \pi/4]$

$$\rightsquigarrow \int_0^{\pi/4} \sqrt{\sec^2\theta} \cdot \frac{1}{2}\sec^2\theta\,d\theta$$

$$\ggg \frac{1}{2}\int_0^{\pi/4} \sec^3\theta\,d\theta$$

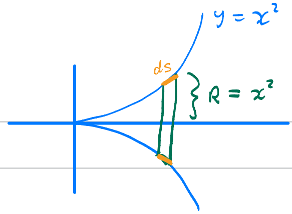
☆ Use $\int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C$

$\int \sec^3\theta\,d\theta = \frac{\tan\theta\sec\theta}{2} + \frac{1}{2}\int\sec\theta\,d\theta$

$$☆ \rightsquigarrow \frac{1}{2}\left(\frac{1}{2}\tan\theta\sec\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta|\right)\Big|_0^{\pi/4}$$

$$\ggg \frac{1}{4}\left(\sqrt{2} + \ln|\sqrt{2}+1|\right) - \frac{1}{4}\left(0 + \frac{1}{2}\ln|1+0|\right)$$

$$\ggg \boxed{\frac{\sqrt{2}}{4} + \frac{1}{4}\ln|\sqrt{2}+1|}$$



3) (b) $\quad ds = \sqrt{1+4x^2}\, dx$

$$dA = 2\pi R\, ds = \text{area of "band"}$$

$$A = \int_a^b dA \;>>>\; \int_0^1 2\pi R\, ds \;>>>\; \int_0^1 2\pi x^2 \sqrt{1+4x^2}\, dx$$

Again set $\quad x = \frac{1}{2}\tan\theta, \quad dx = \frac{1}{2}\sec^2\theta$

$$\leadsto \int_0^{\pi/4} 2\pi \, \frac{1}{4}\tan^2\theta \sqrt{\sec^2\theta}\, \frac{1}{2}\sec^2\theta \, d\theta$$

$$>>> \frac{\pi}{4}\int_0^{\pi/4} \tan^2\theta \sec^3\theta \, d\theta \;>>>\; \frac{\pi}{4}\int_0^{\pi/4} \sec^3\theta \, d\theta + \frac{\pi}{4}\int_0^{\pi/4} \sec^5\theta \, d\theta$$

$$= \frac{\pi}{2}\overbrace{\left(\frac{\sqrt{2}}{4} + \frac{1}{4}\ln\left|\sqrt{2}+1\right|\right)}^{\text{(from (a) above)}} + \frac{\pi}{4}\int_0^{\pi/4} \sec^5\theta \, d\theta$$

$$\int_0^{\pi/4} \sec^5\theta \, d\theta \overset{\star}{=} \left.\frac{\tan\theta\sec^3\theta}{4}\right|_0^{\pi/4} + \frac{4}{3}\int_0^{\pi/4} \sec^3\theta \, d\theta$$

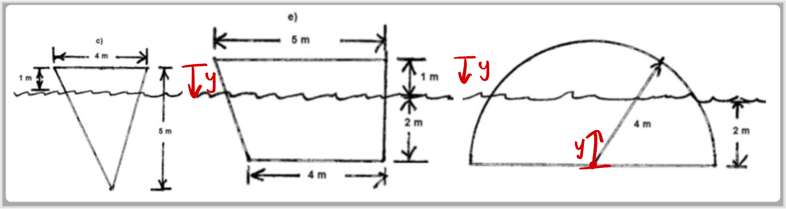
$\star$ Use $\displaystyle\int \sec^5\theta \, d\theta = \frac{\tan\theta \sec^3\theta}{4} + \frac{4}{3}\int \sec^3\theta \, d\theta$

$$>>> \frac{1\cdot\sqrt{2}^3}{4} + \frac{4}{3}\left(\frac{\sqrt{2}}{4} + \frac{1}{4}\ln\left|\sqrt{2}+1\right|\right)$$

So, get

$$\boxed{\left(\frac{\pi}{2} + \frac{4}{3}\right)\left(\frac{\sqrt{2}}{4} + \frac{1}{4}\ln\left|\sqrt{2}+1\right|\right) + \frac{\sqrt{2}}{2}}$$

4. Set up the integrals that give the hydrostatic force on these shapes:



(a) depth (below water line) $= y-1$

width $w(y) = 4 - \frac{4-0}{5}y$

bounds (below water line only) $1 \le y \le 5$

$$\boxed{\int_1^5 \rho g (y-1)\left(4-\frac{4}{5}y\right)dy}$$

(b) depth $= y-1$

$w(y) = 5 - \frac{1}{3}y$

bounds : $1 \le y \le 3$

$$\boxed{\int_1^3 \rho g (y-1)\left(5-\frac{1}{3}y\right)dy}$$

(c) depth $= 2-y$

$w(y) = 2\cdot$("x-coord on circle")

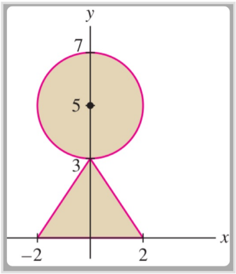
$= 2\sqrt{4^2-y^2}$

bounds : $0 \le y \le 2$

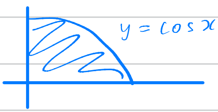
$$\boxed{\int_0^2 \rho g (2-y)\cdot 2\sqrt{16-y^2}\,dy}$$

5. Find the CoMs of the regions:

(a) Area under the curve $y = \cos x$ for $0 \le x \le \frac{\pi}{2}$. (b) See figure:



(a)



"distance to y-axis"

height of slice

width

$$\bar{x} = \frac{M_y}{M}, \quad M_y = \int_0^{\pi/2} \rho\, x\, dA \;\;>>> \;\; \rho \int_0^{\pi/2} x \underbrace{\cos x}\, \underbrace{dx}$$

$u = x \mid dv = \cos x$

$du = 1 \mid v = \sin x$

$$>>> \;\; \rho\, x \sin x \Big|_0^{\pi/2} - \rho \int_0^{\pi/2} \sin x\, dx$$

$$>>> \;\; \rho \frac{\pi}{2} \cdot 1 - \rho(-\cos x)\Big|_0^{\pi/2} \;\; >>> \;\; \frac{\pi}{2}\rho - \rho$$

$$\leadsto \quad M_y = \left(\frac{\pi}{2} - 1\right)\rho$$

$$M = ? \quad = \int_0^{\pi/2} \rho \cos x\, dx \quad = \text{"area under the curve"}$$

$$= \rho \sin x \Big|_0^{\pi/2} = \rho$$

Therefore $\bar{x} = \frac{M_y}{M} = \frac{\pi}{2} - 1.$

....

$$\bar{y} = \frac{M_x}{M} = \frac{M_x}{\rho} = \int_0^{\pi/2} \frac{1}{2}\left(f_1^2 - f_0^2\right)dx$$

$(M = \rho)$

upper function

lower

$$f_1 = \cos x, \quad f_0 = 0 = x\text{-axis}$$

$$\bar{y} = \int_0^{\pi/2} \frac{1}{2}\cos^2 x \, dx$$

$$\ggg \int_0^{\pi/2} \frac{1}{4}\left(1+\cos(2x)\right)dx \ggg \left.\frac{x}{4} + \frac{1}{2}\sin(2x)\right|_0^{\pi/2}$$

$$\rightsquigarrow \bar{y} = \frac{\pi}{8}$$

$$\rightsquigarrow \boxed{(\bar{x}, \bar{y}) = \left(\frac{\pi}{2} - 1, \frac{\pi}{8}\right)}$$

(b) $\boxed{\bar{x} = 0}$ by symmetry.

$$\bar{y} = \frac{M_x}{M}.$$

$$M = \rho \cdot \text{Area} = \rho\left(\tfrac{1}{2}\cdot b\cdot h + \pi r^2\right) = \rho\left(\tfrac{1}{2}\cdot 4\cdot 3 + \pi\cdot 2^2\right) = \rho(6 + 4\pi)$$

$$M_x = ? \qquad \bar{y}^{circ} = 5, \; M^{circ} = 4\pi\rho \rightsquigarrow M_x^{circ} = 20\pi\rho$$

$$M_x^{tri} = ?$$

$$= \int_{triangle} \rho y \, dA = \int_0^3 \rho y \, w(y) dy \ggg \int_0^3 \rho y \left(4 - \tfrac{4}{3}y\right)dy$$

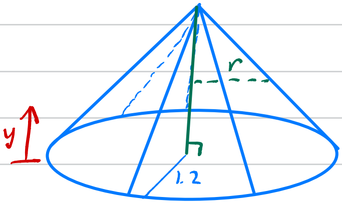
width $= 4 - \frac{4}{3}y$

$$\ggg \rho\left(4\cdot\frac{3}{2} - \frac{4}{3}\cdot\frac{3^3}{3}\right) = 6\rho = M_x^{tri}$$

$$M_x = M_x^{circ} + M_x^{tri} = (20\pi + 6)\rho$$

$$\rightsquigarrow \boxed{\bar{y} = \frac{20\pi + 6}{4\pi + 6} \approx 3.71}$$

7. Set up an integral that computes the work done (against gravity) to build a circular cone-shaped tower of height 4m and base radius 1.2m out of a material with mass density 600 $\frac{\text{kg}}{\text{m}^3}$.



horizontal plane slices

$dm = \rho \cdot dA = \rho \cdot \pi r^2 dy$

$r = 1.2 - \frac{1.2}{4} \cdot y$

$dm = \rho \pi \left(\frac{12}{10} - \frac{3}{10} y\right)^2 dy$

work to raise slice to "y" is $y \cdot g \cdot dm$

$>>> \frac{9 \rho g \pi}{100} y(4-y)^2 dy$

$y^3 - 8y^2 + 16y$

Thus total work is $\int_0^4 \frac{9\rho g \pi}{100} y(4-y)^2 dy$

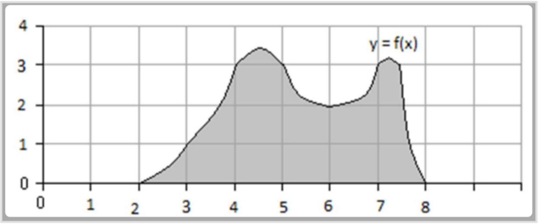
$>>> \frac{9 \rho g \pi}{100} \left(\frac{y^4}{4} - 8\frac{y^3}{3} + 16\frac{y^2}{2}\right)\Big|_0^4$

$4 - \frac{32}{3} + 8 = 12 - \frac{32}{3} = \frac{36-32}{3} = \frac{4}{3}$

$>>> \rho g \pi \frac{9}{100} \cdot 16 \left(\frac{16}{4} - \frac{8 \cdot 4}{3} + \frac{16}{2}\right)$

$>>> \frac{3 \cdot 64}{100} \pi \rho g \quad >>> \boxed{\frac{3 \cdot 64}{100} \pi \cdot \left(600 \frac{\text{kg}}{\text{m}^3}\right) \cdot \left(9.8 \frac{\text{m}}{\text{s}^2}\right)}$

8. Use Simpson's Rule with $n = 6$ to approximate the area of the pictured region:



Simpson's:

$$\text{Area} \approx \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \dots\right)$$

$\Delta x = 1, \quad x_0 = 2, \quad x_1 = 3, \quad \dots, \quad x_6 = 8$

$$\text{Area} \approx \frac{1}{3}\left(0 + 4\cdot 1 + 2\cdot 3 + 4\cdot 3 + 2\cdot 2 + 4\cdot 3 + 1\cdot 0\right)$$

$>>> \frac{1}{3}(4 + 6 + 12 + 4 + 12)$

$>>> \boxed{\frac{38}{3}}$