Worksheet 7.5 Strategy for Integration

1) $\int x\sin^2 x\,dx$

$u = x \quad dv = \sin^2 x\,dx = \left(\frac{1-\cos 2x}{2}\right)dx$

$du = dx \quad v = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right)$

$$\int x\sin^2 x\,dx = \frac{x}{2}\left(x - \frac{1}{2}\sin 2x\right) - \int \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right)dx$$

$$= \frac{x}{2}\left(x - \frac{1}{2}\sin 2x\right) - \frac{1}{2}\left[\left(\frac{x^2}{2}\right) + \frac{1}{4}\cos 2x\right] + C$$

$$= \boxed{\frac{x^2}{4} - \frac{x}{4}\sin 2x - \frac{1}{8}\cos 2x + C}$$

2) $\int x4^x\,dx$

$u = x \quad dv = 4^x\,dx$

$du = dx \quad v = \frac{4^x}{\ln 4}$

$$\int x4^x\,dx = \frac{x4^x}{\ln 4} - \int \frac{1}{\ln 4}4^x\,dx$$

$$= \frac{x4^x}{\ln 4} - \frac{1}{(\ln 4)^2}4^x + C$$

$$= \boxed{\frac{4^x}{\ln 4}\left(x - \frac{1}{\ln 4}\right) + C}$$

3) $\int x \sin^{-1} x \, dx$

$u = \sin^{-1} x \quad dv = x\,dx$

$du = \frac{1}{\sqrt{1-x^2}}\,dx \quad v = \frac{x^2}{2}$

$$\int x \sin^{-1} x \, dx = \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \int \frac{x^2}{\sqrt{1-x^2}}\,dx$$

$x = \sin\theta \quad dx = \cos\theta \, d\theta$

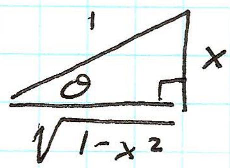
$\int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \cos\theta \, d\theta$

$\int \sin^2\theta \, d\theta$

$\int \frac{1-\cos 2\theta}{2} \, d\theta$

$\frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C$

$\sin\theta = x$



$\sin 2\theta = 2 \sin\theta \cos\theta$

$-\frac{1}{2} \Big| \quad \frac{1}{2}\left(\sin^{-1} x - \frac{1}{2}(2)(x)(\sqrt{1-x^2}) + C\right.$

$$= \boxed{\frac{x^3}{2}\sin^{-1} x - \frac{1}{4}\sin^{-1} x + \frac{x}{4}\sqrt{1-x^2} + C}$$

4) $\int \ln(9x^2-1)\,dx$

$u = \ln(9x^2-1) \quad dv = dx$

$du = \frac{1}{9x^2-1}(18x)\,dx \quad v = x$

$\int \ln(9x^2-1)\,dx = x\ln(9x^2-1) - \int \frac{18x^2}{9x^2-1}\,dx$

$$\begin{array}{r} 2 \\ 9x^2-1 \enclose{longdiv}{18x^2} \\ \underline{18x^2-2} \\ 2 \end{array}$$

$\int \left(2 + \frac{2}{9x^2-1}\right) dx$

$2x + 2\int \frac{1}{(3x+1)(3x-1)}\,dx$

$\frac{1}{(3x+1)(3x-1)} = \frac{A}{3x+1} + \frac{B}{3x-1}$

$1 = A(3x-1) + B(3x+1)$

sub $x = \frac{1}{3} \rightarrow 1 = 2B,\ B = \frac{1}{2}$

Sub $x = -\frac{1}{3} \rightarrow 1 = -2A,\ A = -\frac{1}{2}$

$\int \left[-\frac{1}{2}\left(\frac{1}{3x+1}\right) + \frac{1}{2}\left(\frac{1}{3x-1}\right)\right] dx$

$-\frac{1}{6}\ln|3x+1| + \frac{1}{6}\ln|3x-1| + C$

$= x\ln(9x^2-1) - \left[2x + 2\left(-\frac{1}{6}\ln|3x+1| + \frac{1}{6}\ln|3x-1|\right)\right] + C$

$= x\ln(9x^2-1) - 2x - 2\ln\left[\left|\frac{3x-1}{3x+1}\right|^{\frac{1}{6}}\right] + C$

$= \boxed{x\ln(9x^2-1) - 2x - \ln\left(\left|\frac{3x-1}{3x+1}\right|^{\frac{1}{3}}\right) + C}$

5) $\displaystyle\int \frac{\sqrt{9+x^2}}{x}\,dx$

$x = 3\tan\theta \quad dx = 3\sec^2\theta\, d\theta$

$$\int \frac{\sqrt{9+x^2}}{x}\,dx = \int \frac{\sqrt{9+9\tan^2\theta}}{3\tan\theta}\,(3\sec^2\theta)\,d\theta$$

$$= \int \frac{3\sec\theta}{3\tan\theta}\,(3\sec^2\theta)\,d\theta$$

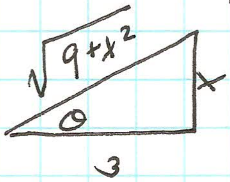
$$= \int 3\left(\frac{\sec\theta}{\tan\theta}\right)(1+\tan^2\theta)\,d\theta$$

$$= \int 3\left(\frac{\sec\theta}{\tan\theta} + \sec\theta\tan\theta\right)d\theta$$

$$= \int 3\left(\csc\theta + \sec\theta\tan\theta\right)d\theta$$

$$= 3\left[\ln|\csc\theta - \cot\theta| + \sec\theta\right) + C$$

$\tan\theta = \frac{x}{3}$



$$= 3\left[\ln\left|\frac{\sqrt{9+x^2}}{x} - \frac{3}{x}\right| + \frac{\sqrt{9+x^2}}{3}\right] + C$$

$$= \boxed{\sqrt{9+x^2} + 3\ln\left|\frac{\sqrt{9+x^2}-3}{x}\right| + C}$$

6) $\int \frac{\sqrt{1-9x^2}}{x}\,dx$

$3x = \sin\theta \quad dx = \frac{1}{3}\cos\theta\, d\theta$

$$\int \frac{\sqrt{1-9x^2}}{x}\,dx = \int 3\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}\left(\frac{1}{3}\cos\theta\right)d\theta$$

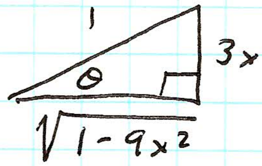
$$= \int \frac{\cos^2\theta}{\sin\theta}\,d\theta$$

$$= \int \frac{1-\sin^2\theta}{\sin\theta}\,d\theta$$

$$= \int (\csc\theta - \sin\theta)\,d\theta$$

$$= \ln|\csc\theta - \cot\theta| + \cos\theta + C$$

$\sin\theta = 3x$



$$= \ln\left|\frac{1}{3x} - \frac{\sqrt{1-9x^2}}{3x}\right| + \sqrt{1-9x^2} + C$$

$$= \boxed{\ln\left|\frac{1-\sqrt{1-9x^2}}{3x}\right| + \sqrt{1-9x^2} + C}$$

7) $$\int \frac{1}{x\sqrt{9x^2-1}}\,dx$$

$3x = \sec\theta \qquad dx = \frac{1}{3}\sec\theta\tan\theta\,d\theta$

$$\int \frac{1}{x\sqrt{9x^2-1}}\,dx = \frac{3}{\sec\theta\sqrt{\sec^2\theta-1}}\left(\frac{1}{3}\sec\theta\tan\theta\right)d\theta$$

$$= \frac{3}{\sec\theta\tan\theta}\left(\frac{1}{3}\sec\theta\tan\theta\right)d\theta$$

$$= \int d\theta$$

$$= \theta + C$$

$$= \sec^{-1}(3x) + C$$

8) $$\int \frac{x}{\sqrt{9x^2-1}}\,dx$$

$u = 9x^2 - 1$

$du = 18x\,dx$

$$\int \frac{x}{\sqrt{9x^2-1}}\,dx = \int \frac{1}{18}\frac{du}{\sqrt{u}}$$

$$= \frac{2u^{1/2}}{18} + c$$

$$= \boxed{\frac{1}{9}\sqrt{9x^2-1} + C}$$

9) $\int \frac{1}{x\sqrt{9x-1}}\,dx$

$u = \sqrt{9x-1}$

$u^2 = 9x-1 \rightarrow x = \frac{u^2+1}{9}$

$2u\,du = 9\,dx \rightarrow dx = \frac{2}{9}u\,du$

$$\int \frac{1}{x\sqrt{9x-1}}\,dx = \int \frac{1}{\left(\frac{u^2+1}{9}\right)u}\left(\frac{2}{9}u\right)du$$

$$= \int \frac{2}{u^2+1}\,du$$

$$= 2\tan^{-1}u + C$$

$$= \boxed{2\tan^{-1}\sqrt{9x-1} + C}$$

10) $\int \frac{1}{x^2(x^2+1)}\,dx$

$$\frac{1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

$$1 = Ax(x^2+1) + B(x^2+1) + (Cx+D)(x^2)$$

substitute $x=0$: $1 = B$

$$1 = Ax^3 + Ax + Bx^2 + B + Cx^3 + Dx^2$$

$$1 = x^3(A+C) + x^2(B+D) + Ax + B$$

$B + D = 0 \rightarrow D = -1$

$A = 0$

$A + C = 0$

$C = 0$

$$\int \frac{1}{x^2(x^2+1)}\,dx = \int \left(\frac{1}{x^2} - \frac{1}{x^2+1}\right)dx$$

$$= \boxed{-\frac{1}{x} - \tan^{-1}x + C}$$

11) $\int \frac{1}{x^2(x^2-1)}\,dx$

$$\frac{1}{x^2(x^2-1)} = \frac{1}{x^2(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{x+1}$$

$$1 = Ax(x-1)(x+1) + B(x-1)(x+1) + Cx^2(x+1) + Dx^2(x-1)$$

substitute $x=0$: $1 = -B \rightarrow B = -1$

substitute $x=1$: $1 = 2C \rightarrow C = \frac{1}{2}$

substitute $x=-1$: $1 = -2D \rightarrow D = -\frac{1}{2}$

$0x^3 = (A+C+D)$

$A+C+D=0$

$A = -C-D = -\frac{1}{2}+\frac{1}{2} = 0$

$A=0$

$$\int \frac{1}{x^2(x^2-1)}\,dx = \int \left(-\frac{1}{x^2} + \frac{1}{2}\left(\frac{1}{x-1}\right) - \frac{1}{2}\left(\frac{1}{x+1}\right)\right) dx$$

$$= \frac{1}{x} + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$$

$$= \boxed{\frac{1}{x} + \ln\sqrt{\frac{x-1}{x+1}} + C}$$

12) $\int \tan^5 x \sec^4 x \, dx$

$= \int \tan^5 x \sec^2 x \sec^2 x \, dx$

$= \int \tan^5 x (\tan^2 x + 1) \sec^2 x \, dx$

$u = \tan x \quad du = \sec^2 x \, dx$

$= \int u^5 (u^2 + 1) \, du$

$= \int (u^7 + u^5) \, du$

$= \frac{u^8}{8} + \frac{u^6}{6} + C$

$= \boxed{\frac{\tan^8 x}{8} + \frac{\tan^6 x}{6} + C}$

13) $\int_0^{\frac{\pi}{2}} \cos^3 x \sin 2x \, dx$

$= \int_0^{\frac{\pi}{2}} \cos^3 x (2 \sin x \cos x) \, dx$

$= 2 \int_0^{\frac{\pi}{2}} \cos^4 x \sin x \, dx$

$u = \cos x \quad du = -\sin x \, dx$

$= -2 \int_1^0 u^4 \, du$

$= -2 \left(\frac{u^5}{5} \right) \Big|_1^0$

$= -\frac{2}{5} (0 - 1)$

$= \frac{2}{5}$

14) $\int \frac{dx}{\sqrt{e^x-1}}$

$$u = \sqrt{e^x-1}$$

$$u^2 = e^x - 1$$

$$2u\,du = e^x\,dx$$

$$\frac{2u\,du}{e^x} = dx$$

$$\frac{2u}{u^2+1}du = dx$$

$$\int \frac{dx}{\sqrt{e^x-1}} = \int \frac{1}{u}\left(\frac{2u}{u^2+1}\right)du$$

$$= \int \frac{2}{u^2+1}\,du$$

$$= 2\tan^{-1}u + C$$

$$= \boxed{2\tan^{-1}\sqrt{e^x-1} + C}$$