Appendix H Complex Numbers

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Complex Number

Can be represented by an expression

$$a + bi$$

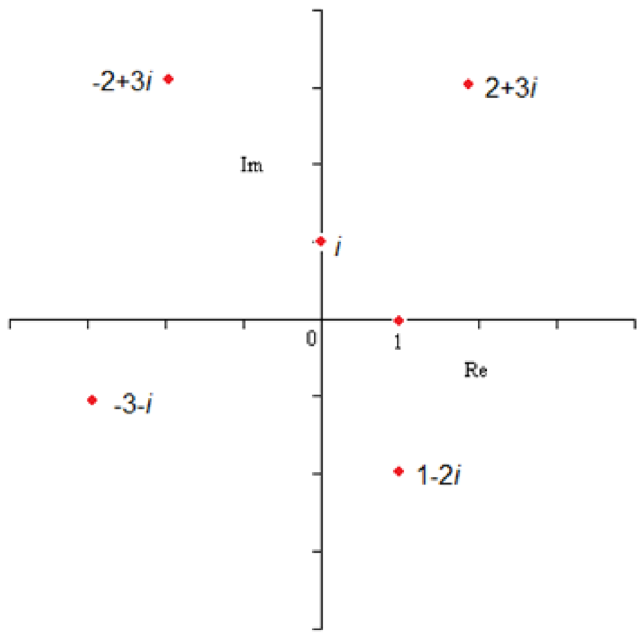
where a and b are real numbers and i has the property, $i^2 = -1$.

a is the real part

b is the imaginary part

Ordered Pair

The complex number, $a + bi$ can be represented as an ordered pair (a, b), and plotted as a point in the Argand plane, as shown.



Addition and Subtraction

Addition:

$(a + bi) + (c + di) = (a + c) + (b + d)i$

For example:

$(1 - i) + (4 + 7i) = (1 + 4) + (-1 + 7)i = 5 + 6i$

Subtraction:

$(a + bi) - (c + di) = (a - c) + (b - d)i$

For example:

$(1 - i) - (4 + 7i) = (1 - 4) + (-1 - 7)i = -3 - 8i$

Multiplication

$$\begin{aligned}(a+bi)(c+di) &= a(c+di)+bi(c+di)\\ &= ac+adi+bci+bdi^2\\ &= (ac-bd)+(ad+bc)i\end{aligned}$$

For example:

$$\begin{aligned}(1-i)(4+7i) &= 1(4+7i)-i(4+7i)\\ &= 4+7i-4i-7i^2\\ &= (4+7)+(7-4)i\\ &= 11+3i\end{aligned}$$

Complex Conjugate

If $z=a+bi$, then the complex conjugate of z is $\bar{z}=a-bi$

For example, the complex conjugate of $3+2i$ is $3-2i$.

Properties of Conjugates

Property: $\overline{z+w} = \bar{z} + \bar{w}$

$z = 2 + 3i$

$w = -1 + 2i$

$z + w = 1 + 5i$

$\overline{z+w} = 1 - 5i$

$\bar{z} = 2 - 3i$

$\bar{w} = -1 - 2i$

$\bar{z} + \bar{w} = 1 - 5i$

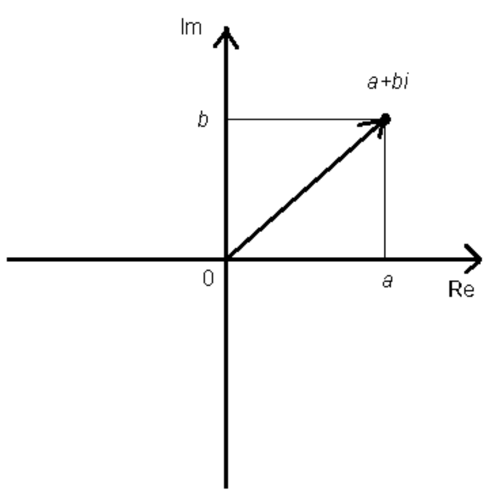
Property: $\overline{zw} = \bar{z}\bar{w}$

Prove this...

Modulus

The modulus (magnitude) of a complex number is computed as follows:

$$|a + bi| = \sqrt{a^2 + b^2}$$



For example:

$$|5 - 2i| = \sqrt{5^2 + (-2)^2} = \sqrt{29}$$

More Properties

Property: $z\bar{z} = |z|^2$

For example, if $z = 2 + 3i$ and $\bar{z} = 2 - 3i$:

$$\begin{aligned} z\bar{z} &= (2+3i)(2-3i) \\ &= 4 - 6i + 6i - 9i^2 \\ &= 4 + 9 \\ &= 13 \end{aligned}$$

$$\begin{aligned} |z| &= \sqrt{2^2 + 3^2} \\ &= \sqrt{13} \\ |z|^2 &= 13 \end{aligned}$$

Division

To simplify $\dfrac{a+bi}{c+di}$, multiply the numerator and the denominator by the complex conjugate of the denominator.

For example:

$$\begin{aligned} \tfrac{-1+3i}{2+5i} &= \tfrac{-1+3i}{2+5i}\left(\tfrac{2-5i}{2-5i}\right) \\ &= \tfrac{-2+5i+6i-15i^2}{4-25i^2} \\ &= \tfrac{13+11i}{29} \\ &= \tfrac{13}{29} + \tfrac{11}{29}i \end{aligned}$$

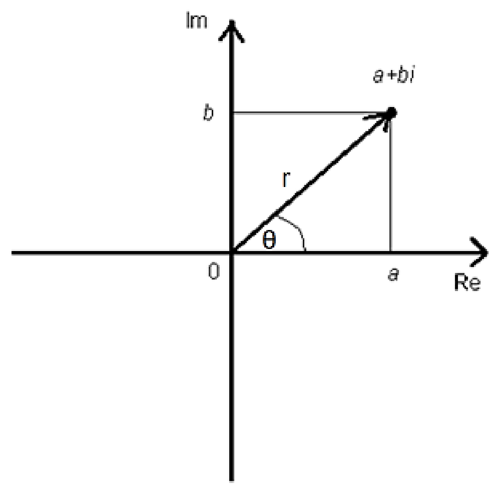
Polar Form

It is often convenient to write complex numbers in polar form:

$z = r(\cos\theta + i\sin\theta)$

r is the modulus (magnitude).

θ is the argument (angle).



The real part is $a = r\cos\theta$, and the imaginary part is $b = r\sin\theta$.

Multiplication and Division in Polar Form

It is convenient to multiply and divide complex numbers in polar form:

$z = r_z(\cos\theta_z + i\sin\theta_z)$

$w = r_w(\cos\theta_w + i\sin\theta_w)$

$zw = r_z r_w(\cos\theta_z\cos\theta_w + i\cos\theta_z\sin\theta_w + i\sin\theta_z\cos\theta_w + i^2\sin\theta_z\sin\theta_w)$

$zw = r_z r_w(\cos\theta_z\cos\theta_w - \sin\theta_z\sin\theta_w + i(\cos\theta_z\sin\theta_w + i\sin\theta_z\cos\theta_w))$

$zw = r_z r_w(\cos(\theta_z + \theta_w) + i\sin(\theta_z + \theta_w))$

Multiplication and Division in Polar Form

So, to multiply two complex numbers, multiply the moduli and add the arguments:

$z = r_z(\cos\theta_z + isin\theta_z)$
$w = r_w(\cos\theta_w + i\sin\theta_w)$

$zw = r_z r_w(\cos(\theta_z + \theta_w) + i\sin(\theta_z + \theta_w)$

To divide two complex numbers, divide the moduli and subtract the arguments:

$\frac{z}{w} = \frac{r_z}{r_w}(\cos(\theta_z - \theta_w) + i\sin(\theta_z - \theta_w))$

Quiz

$4\sqrt{3} - 4i =$

A. $4(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3})$

B. $4(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3})$

C. $8(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6})$

D. $8(\cos\frac{11\pi}{3} + i\sin\frac{11\pi}{3})$

Quiz

$(4\sqrt{3} - 4i)(1 + i) =$

A. $16(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12})$

B. $16(\cos(-\frac{\pi}{12}) + i\sin(-\frac{\pi}{12}))$

C. $8\sqrt{2}(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12})$

D. $8\sqrt{2}(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$

De Moivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then:

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

Quiz

$(3+3i)^4 =$

A. $324(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$

B. $81(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$

C. -324

D. $81i$

Roots of a Complex Number

If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then z has n distinct roots:

$$w_k = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right]$$

for $k = 0, 1, 2, \ldots, n-1$

Example

Find the fourth roots of $1 + i$:

Solution:

First write in polar form:
$1 + i = \sqrt{2}(cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$

Include equivalent angles:
$1 + i = \sqrt{2}(cos(\frac{\pi}{4} + 2k\pi) + i \sin(\frac{\pi}{4} + 2k\pi))$

Take fourth root of modulus and divide argument by four:
$(1 + i)^{\frac{1}{4}} = (\sqrt{2})^{\frac{1}{4}}(cos(\frac{1}{4}(\frac{\pi}{4} + 2k\pi)) + i \sin(\frac{1}{4}(\frac{\pi}{4} + 2k\pi)))$ for $k = 0, 1, 2, 3$

$(1 + i)^{\frac{1}{4}} = 2^{\frac{1}{8}}(cos(\frac{\pi}{16} + \frac{2k\pi}{4}) + i \sin(\frac{\pi}{16} + \frac{2k\pi}{4}))$ for $k = 0, 1, 2, 3$

Example - Continued

$w_k = 2^{\frac{1}{8}}(cos(\frac{\pi}{16} + \frac{2k\pi}{4}) + i \sin(\frac{\pi}{16} + \frac{2k\pi}{4}))$ for $k = 0, 1, 2, 3$

$$\begin{aligned} w_0 &= 2^{\frac{1}{8}}(cos(\frac{\pi}{16} + \frac{0\pi}{4}) + i \sin(\frac{\pi}{16} + \frac{0\pi}{4})) \\ &= 2^{\frac{1}{8}}(cos(\frac{\pi}{16}) + i \sin(\frac{\pi}{16})) \end{aligned}$$

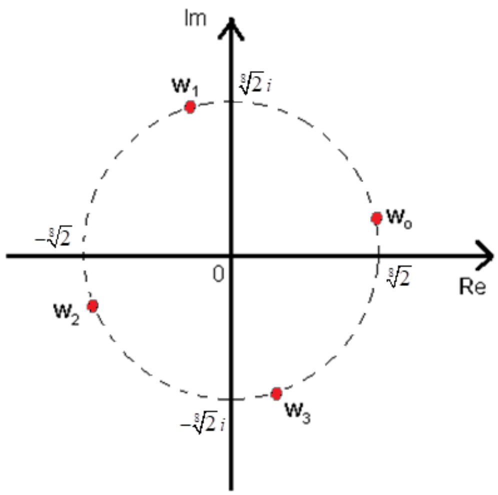
$$\begin{aligned} w_1 &= 2^{\frac{1}{8}}(cos(\frac{\pi}{16} + \frac{2\pi}{4}) + i \sin(\frac{\pi}{16} + \frac{2\pi}{4})) \\ &= 2^{\frac{1}{8}}(cos(\frac{9\pi}{16}) + i \sin(\frac{9\pi}{16})) \end{aligned}$$

$$\begin{aligned} w_2 &= 2^{\frac{1}{8}}(cos(\frac{\pi}{16} + \frac{4\pi}{4}) + i \sin(\frac{\pi}{16} + \frac{4\pi}{4})) \\ &= 2^{\frac{1}{8}}(cos(\frac{17\pi}{16}) + i \sin(\frac{17\pi}{16})) \end{aligned}$$

$$\begin{aligned} w_3 &= 2^{\frac{1}{8}}(cos(\frac{\pi}{16} + \frac{6\pi}{4}) + i \sin(\frac{\pi}{16} + \frac{6\pi}{4})) \\ &= 2^{\frac{1}{8}}(cos(\frac{25\pi}{16}) + i \sin(\frac{25\pi}{16})) \end{aligned}$$

Example - Continued

Plot these roots:



Notice that the roots are distributed evenly around the pole, on a circle of radius $\sqrt[8]{2}$.

Try It 1

Find the fifth roots of 32.

Complex Exponentials

Recall:

$$e^x = \sum_{n=0}^{\infty} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Thus:

$$\begin{aligned} e^{iy} &= 1 + iy + \tfrac{(iy)^2}{2!} + \tfrac{(iy)^3}{3!} + \tfrac{(iy)^4}{4!} + \tfrac{(iy)^5}{5!} + \cdots \\ &= 1 + iy - \tfrac{y^2}{2!} - i\tfrac{y^3}{3!} + \tfrac{y^4}{4!} + i\tfrac{y^5}{5!} + \cdots \\ &= \left(1 - \tfrac{y^2}{2!} + \tfrac{y^4}{4!} + \cdots\right) + i\left(y - \tfrac{y^3}{3!} + \tfrac{y^5}{5!} + \cdots\right) \end{aligned}$$

$$e^{iy} = \cos y + i \sin y$$

Euler's Formula

$$e^{iy} = \cos y + i \sin y$$

General Complex Exponentials

$$e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$$

Euler $\rightarrow$ DeMoivre

$$[r(cos\theta + i \sin\theta)]^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n(\cos n\theta + i \sin n\theta)$$

Try It 2

Write $e^{2+i\pi}$ in the form $a + bi$.

Proof Solution

Let $z = a + bi$ and $w = c + di$

Then,

$zw = (ac - bd) + i(ad + bc)$

$\overline{zw} = (ac - bd) - i(ad + bc)$

and

$\bar{z} = a - bi$ and $\bar{w} = c - di$

$\bar{z}\bar{w} = (ac - bd) - i(ad + bc)$

so

$\overline{zw} = \bar{z}\bar{w}$

Try It 1 Solution

$32 = 32(\cos 0 + i \sin 0)$

$32^{\frac{1}{5}} = 32^{\frac{1}{5}}(\cos(\frac{0+2k\pi}{5}) + i \sin(\frac{0+2k\pi}{5}))$ for $k = 0, 1, 2, 3, 4$

$w_k = 2(\cos(\frac{2k\pi}{5}) + i \sin(\frac{2k\pi}{5}))$ for $k = 0, 1, 2, 3, 4$

5 Solutions:

$w_0 = 2(\cos(0) + i \sin(0)) = 2$

$w_1 = 2(\cos(\frac{2\pi}{5}) + i \sin(\frac{2\pi}{5}))$

$w_2 = 2(\cos(\frac{4\pi}{5}) + i \sin(\frac{4\pi}{5}))$

$w_3 = 2(\cos(\frac{6\pi}{5}) + i \sin(\frac{6\pi}{5}))$

$w_4 = 2(\cos(\frac{8\pi}{5}) + i \sin(\frac{8\pi}{5}))$

Try It 2 Solution

$$\begin{aligned} e^{2+i\pi} &= e^2(\cos \pi + i \sin \pi) \\ &= e^2(-1 + 0i) \\ &= \boxed{-e^2} \end{aligned}$$