Chapter 8: Further Applications of Integration

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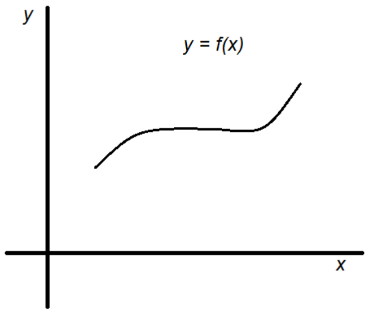
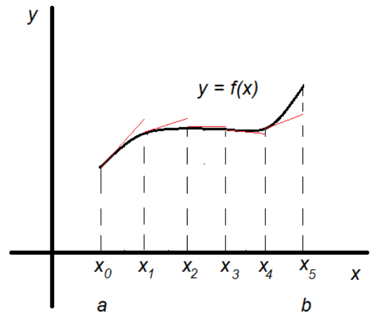
8.1-2 Arc Length and Surface Area

Estimating Arc Length

For a curve, $y = f(x)$ $a \leq x \leq b$, and $f'(x)$ is continuous on $[a, b]$:

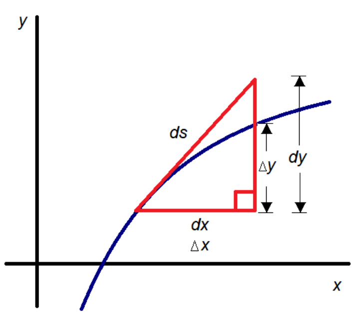




Arc Length: $L = \sum_{i=1}^{n} s_i$

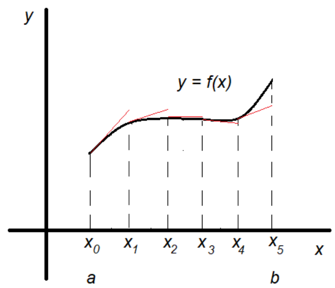
$s_i \approx ?$

Differential of Arc Length, ds



$$dy = f'(x)dx$$
$$\Delta s \approx ds = \sqrt{dx^2 + dy^2}$$
$$= \sqrt{dx^2 + (f'(x)dx)^2}$$
$$= \sqrt{dx^2(1 + (f'(x)^2)}$$
$$ds = \sqrt{1 + (f'(x))^2}dx$$

Arc Length Formula



$$s_i \approx \sqrt{1 + (f'(x_i^*)^2}\Delta x$$

$$L = \sum_{i=1}^{n} s_i$$

$$L \approx \sum_{i=1}^{n} \sqrt{1 + (f'(x_i^*))^2}\Delta x$$

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + (f'(x_i^*))^2}\Delta x$$

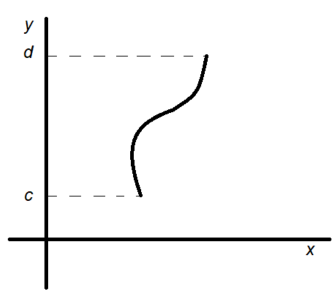
$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

$$L = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2} dx$$

$$L = \int ds$$

Alternative

For a curve, $x = g(y)$, $c \leq y \leq d$, and $g'(y)$ is continuous on $[c, d]$:



$$L = \int_c^d \sqrt{1 + (g'(y))^2} dy$$

$$L = \int_c^d \sqrt{1 + (\frac{dx}{dy})^2} dy$$

$$L = \int ds$$

Summary

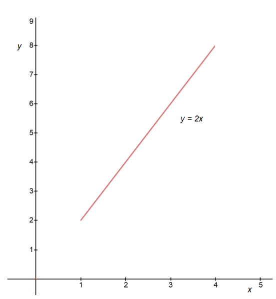
So

$ds = \sqrt{1 + (\frac{dy}{dx})^2}dx$ if $y = f(x)$

$ds = \sqrt{1 + (\frac{dx}{dy})^2}dy$ if $x = g(y)$

Simple Example

Find the length of the curve, $y = 2x$, for $1 \leq x \leq 4$.



$\frac{dy}{dx} = 2$

$$L = \int_1^4 \sqrt{1 + 2^2}dx$$

$$= \int_1^4 \sqrt{5}dx$$

$$= 3\sqrt{5}$$

OR

$x = \frac{y}{2}$

$\frac{dx}{dy} = \frac{1}{2}$

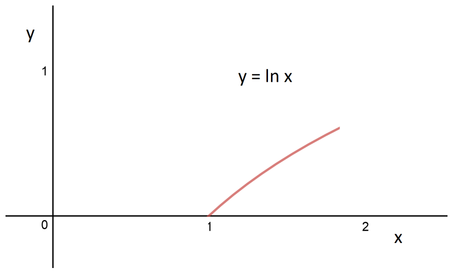
$$L = \int_2^8 \sqrt{1 + (\frac{1}{2})^2}dy$$

$$= \int_2^8 \frac{\sqrt{5}}{2}dy$$

$$= 3\sqrt{5}$$

Example Two

Find the length of $y = \ln x$, $1 \leq x \leq \sqrt{3}$



$$\frac{dy}{dx} = \frac{1}{x}$$

$$L = \int_1^{\sqrt{3}} \sqrt{1 + (\frac{1}{x})^2} dx$$

$$= \int_1^{\sqrt{3}} \sqrt{\frac{x^2+1}{x^2}} dx$$

$$= \int_1^{\sqrt{3}} \frac{\sqrt{x^2+1}}{x} dx$$

Example Two - Continued

$$L = \int_1^{\sqrt{3}} \frac{\sqrt{x^2+1}}{x} dx$$

$$x = \tan\theta$$

$$dx = \sec^2\theta d\theta$$

$$L = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sqrt{\tan^2\theta+1}}{\tan\theta} (\sec^2\theta) d\theta$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^3\theta}{\tan\theta} d\theta$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec\theta(1+\tan^2\theta)}{\tan\theta} d\theta$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\csc\theta + \tan\theta\sec\theta) d\theta$$

Example Two - Continued

$$L = \int_1^{\sqrt{3}} \sqrt{1 + (\frac{1}{x})^2} dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\csc\theta + \tan\theta\sec\theta) d\theta$$
$$= (\ln|\csc\theta - \cot\theta| + \sec\theta)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$
$$= \ln|\tfrac{2}{\sqrt{3}} - \tfrac{1}{\sqrt{3}}| + 2 - \ln|\sqrt{2} - 1| - \sqrt{2}$$
$$= \ln(\tfrac{1}{\sqrt{3}(\sqrt{2}-1)}) + 2 - \sqrt{2}$$
$$= \boxed{\ln(\tfrac{\sqrt{3}(\sqrt{2}+1)}{3}) + 2 - \sqrt{2}}$$

Example Three

Find the length of the following curve:

$y = \frac{x^4}{8} + \frac{1}{4x^2} \qquad 1 \leq x \leq 2$

Solution:

$\frac{dy}{dx} = \frac{x^3}{2} - \frac{1}{2x^3}$

$L = \int_1^2 \sqrt{1 + (\frac{x^3}{2} - \frac{1}{2x^3})^2} dx$

$= \int_1^2 \sqrt{1 + \frac{x^6}{4} - 2(\frac{x^3}{2})(\frac{1}{2x^3}) + \frac{1}{4x^6}} dx$

$= \int_1^2 \sqrt{1 + \frac{x^6}{4} - \frac{1}{2} + \frac{1}{4x^6}} dx$

$= \int_1^2 \sqrt{\frac{x^6}{4} + \frac{1}{2} + \frac{1}{4x^6}} dx$

Example Three - Continued

$$\begin{aligned} L &= \int_1^2 \sqrt{\tfrac{x^6}{4} + \tfrac{1}{2} + \tfrac{1}{4x^6}}\,dx \\ &= \int_1^2 \sqrt{(\tfrac{x^3}{2} + \tfrac{1}{2x^3})^2}\,dx \\ &= \int_1^2 (\tfrac{x^3}{2} + \tfrac{1}{2x^3})\,dx \\ &= (\tfrac{x^4}{8} - \tfrac{1}{4x^2})\Big|_1^2 \\ &= \boxed{\tfrac{33}{16}} \end{aligned}$$

Quiz

Which integral will find the length of the curve,
$y = e^{2x}$ for $0 \leq x \leq 4$?

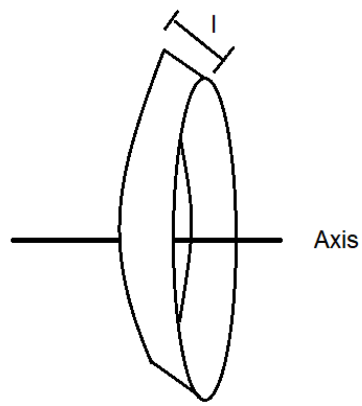
A) $\int_0^4 e^x \sqrt{1 + e^{2x}}\,dx$

B) $\int_0^4 \sqrt{1 + e^{4x}}\,dx$

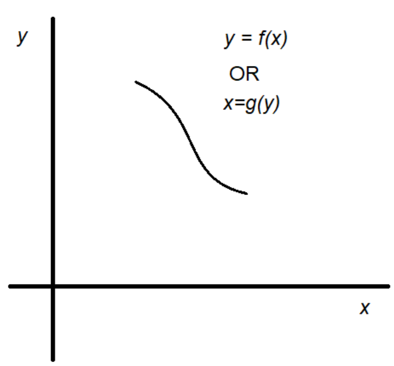
C) $\int_0^4 \sqrt{1 + 4e^{4x}}\,dx$

Revolve Line Segment About an Axis

A line segment of length, l, is revolved around an axis. The surface area of revolution is $\triangle S \approx 2\pi \bar{r} l$ where $\bar{r}$ is the average radius of revolution.



Revolve Curve About x-axis



$$S_i \approx 2\pi r_i \Delta s_i$$

$$S = \sum_{i=1}^{n} S_i$$

$$S = \int 2\pi y ds$$

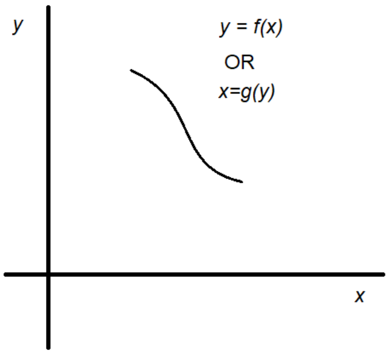
$$ds = \sqrt{1 + (\frac{dy}{dx})^2} dx$$

OR

$$ds = \sqrt{1 + (\frac{dx}{dy})^2} dy$$

Revolve Curve About y-axis

Likewise, When revolving a curve around the y-axis:



$$S = \int 2\pi x ds$$

$$ds = \sqrt{1 + (\frac{dy}{dx})^2} dx$$

OR

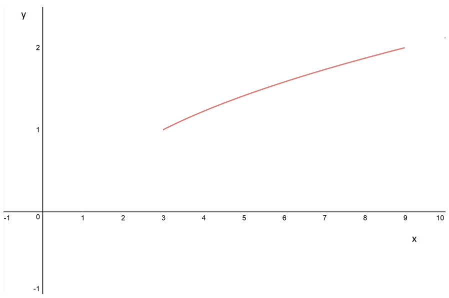
$$ds = \sqrt{1 + (\frac{dx}{dy})^2} dy$$

One Formula

$$S = \int 2\pi r ds$$

Example

Find the area of the surface obtained by revolving the curve, $y = \sqrt{\frac{x-1}{2}}$, $3 \leq x \leq 9$, around the x-axis



$$\frac{dy}{dx} = \frac{1}{4}\sqrt{\frac{2}{x-1}}$$

$$ds = \sqrt{1 + \frac{1}{8(x-1)}}dx$$

$$= \sqrt{\frac{8x-7}{8(x-1)}}dx$$

Example - Continued

$$S = \int 2\pi r ds$$

$$= \int 2\pi y ds$$

$$= \int_3^9 2\pi \sqrt{\frac{x-1}{2}}\sqrt{\frac{8x-7}{8(x-1)}}dx$$

$$= \int_3^9 2\pi \frac{\sqrt{x-1}}{\sqrt{2}\sqrt{8}}\frac{\sqrt{8x-7}}{\sqrt{x-1}}dx$$

$$= \int_3^9 \frac{\pi}{2}\sqrt{8x-7}dx$$

$$= \frac{\pi}{2}\left(\frac{2}{24}\right)(8x-7)^{\frac{3}{2}}\Big|_3^9$$

$$= \boxed{\frac{\pi}{24}(65\sqrt{65} - 17\sqrt{17})}$$

Summary

$$S = \int 2\pi r ds$$

Choose ds first. Choice depends on shape of curve ($y = f(x)$ or $x = g(y)$) and ease of integration.

r depends only on axis of revolution, but its form depends on ds.

Quiz 1

Which integral will calculate the area of the surface obtained by rotating $y = (x-5)^2$, $4 \leq x \leq 6$, around the x-axis?

A) $\int_4^6 2\pi x \sqrt{1 + 4(x-5)^2} dx$

B) $\int_4^6 2\pi y \sqrt{1 + 4(x-5)^2} dx$

C) $\int_4^6 2\pi (x-5)^2 \sqrt{1 + 4(x-5)^2} dx$

D) $\int_4^6 2\pi (x-5)^2 \sqrt{1 - 4(x-5)^2} dx$

Quiz 2

Which integral will calculate the surface area generated by revolving the same curve around the y-axis?

A) $\int_4^6 2\pi x\sqrt{1+4(x-5)^2}dx$

B) $\int_5^6 2\pi x\sqrt{1+4(x-5)^2}dx$

C) $\int_0^1 2\pi y\sqrt{1+\frac{1}{4y}}dy$

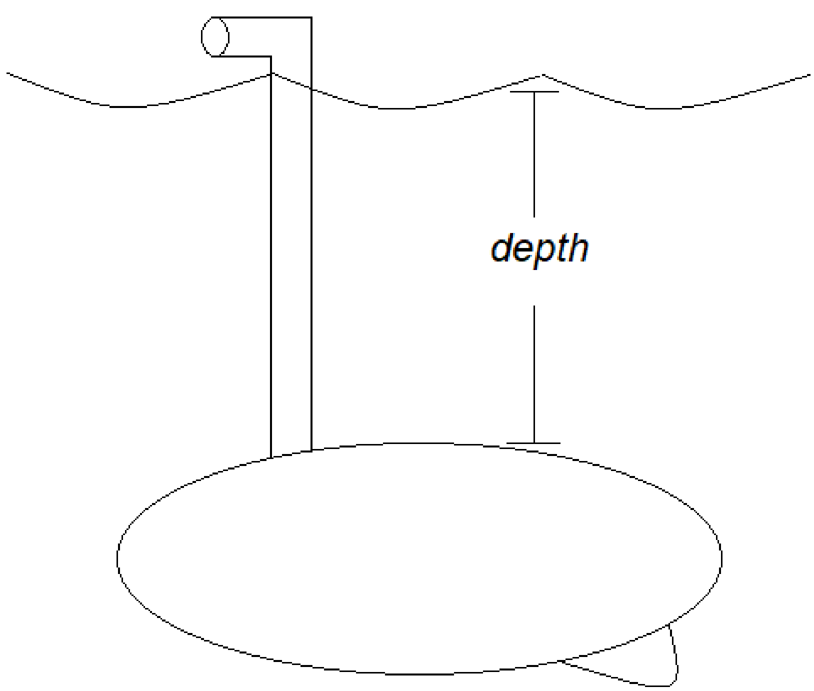
8.3a Hydrostatic Pressure and Force

Review

$\rho = 1000 kg/m^3$	mass density of water
$g = 9.8 m/s^2$	gravitational acceleration
$\delta = \rho g = 9800 N/m^3$	weight density of water
OR	
$\delta = \rho g = 62.5 lb/ft^3$	weight density of water

Definition

Depth - distance below the surface



Quiz

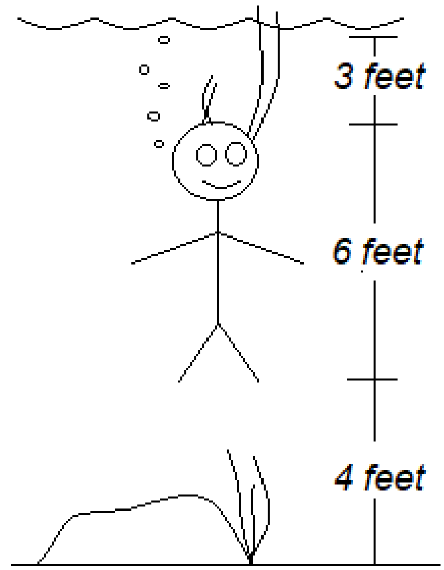
What is the depth of the diver's feet?

A) 3 feet

B) 4 feet

C) 6 feet

D) 9 feet



Hydrostatic Pressure

Fluid pressure is equal in all directions and is exerted on a surface in the perpendicular direction. It is dependent only on depth.

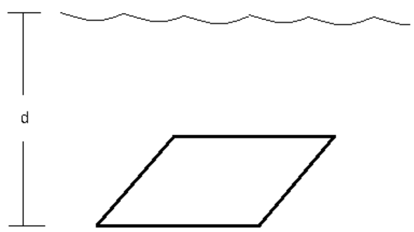
$$P = \rho g d$$

$$P = \delta d$$

If the pressure on a surface is constant, then

$$F = PA$$

Horizontal Plate



$F = PA$

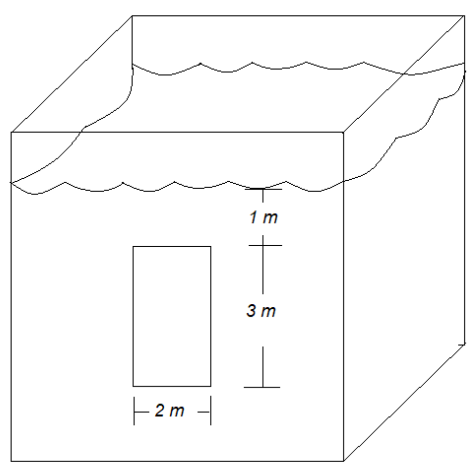
Pressure is constant.

$P = \rho g d$

$F = \rho g d A$

Vertical Plate

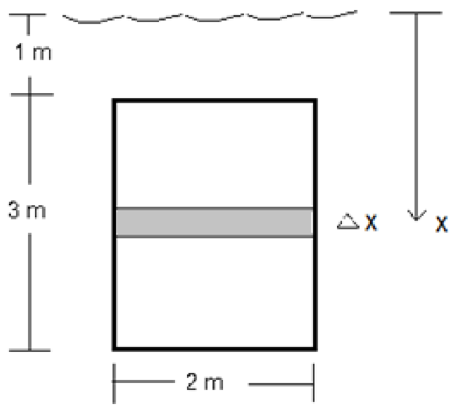
What if the plate is vertical? What is the force on one face of the plate?



$F = PA$, but the pressure varies with depth so we cannot simply multiply P times A.

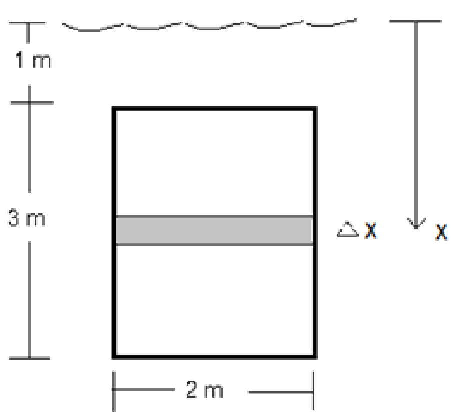
Vertical Plate - Continued

Divide plate into thin strips that are parallel to the surface of the water. Then, the depth (and pressure) within each strip will be approximately constant. Let F_i be the approximate force on one rectangular strip.



$$F_i = P_i A_i$$
$$P_i = \rho g d_i = \rho g x_i$$
$$A_i = w\Delta x = 2\Delta x$$
$$F_i = 2\rho g x_i \Delta x$$

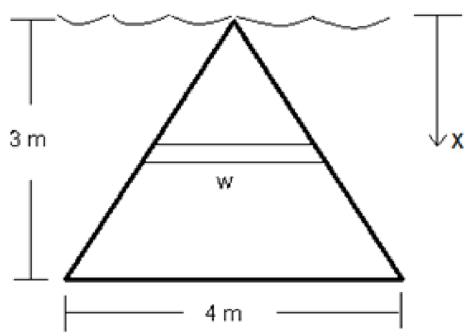
Vertical Plate - Continued



$$F \approx \sum_{i=1}^{n} F_i$$
$$\approx \sum_{i=1}^{n} 2\rho g x_i \Delta x$$
$$F = \lim_{n\to\infty} \sum_{i=1}^{n} 2\rho g x_i \Delta x$$
$$= \int_1^4 2\rho g x dx$$
$$= 15\rho g$$
$$= 147000 \text{ N}$$

Vertical Triangle

Find the hydrostatic force on one face of the plate shown.



(x, w)

$(0, 0)$

$(3, 4)$

$w - 0 = \frac{4}{3}(x - 0)$

$w = \frac{4}{3}x$

$d = x$

$F_i = \rho g d_i A_i$

$A_i = w_i \Delta x$

$F_i = \rho g x_i (\frac{4}{3}x_i)\Delta x$

Vertical Triangle - Continued

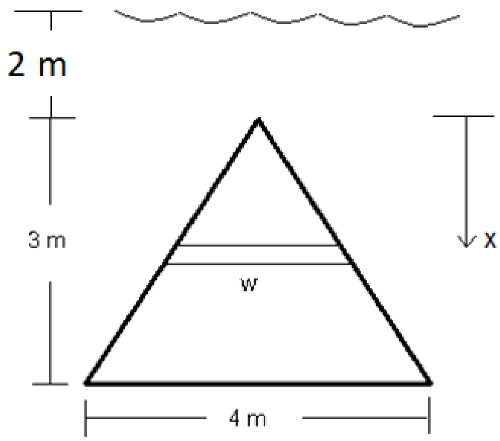
$$F \approx \sum_{i=1}^{n} F_i$$

$$\approx \sum_{i=1}^{n} \rho g x_i (\frac{4}{3}x_i)\Delta x$$

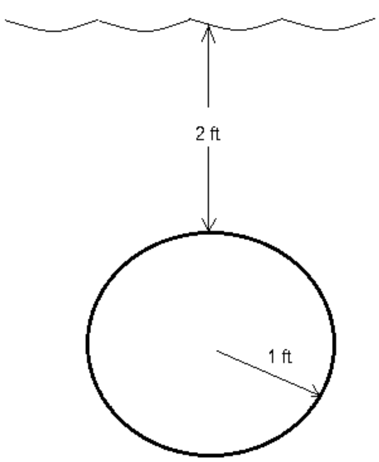
$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \rho g x_i (\frac{4}{3}x_i)\Delta x$$

$$= \int_0^3 \rho g x (\tfrac{4}{3}) x dx$$

What Changes?



Circular Plate

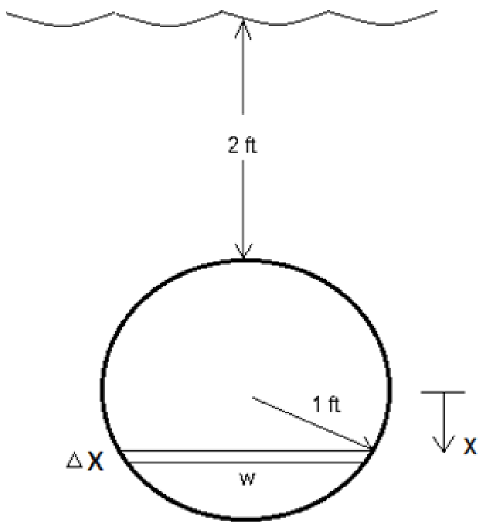


$F_i = \delta d_i A_i$

Note: Here we use δ instead of ρg because the units are conventional units.

In the case of a circle (or a semi-circle or any part of a circle) put the origin at the center of the circle.

Circular Plate



$x^2 + (\frac{w}{2})^2 = 1$

$(\frac{w}{2})^2 = 1 - x^2$

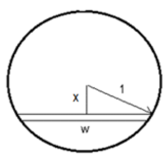
$w = 2\sqrt{1 - x^2}$

$d = x + 3$

$F_i = \delta d_i A_i$

$F_i = \delta(x_i + 3)(2\sqrt{1 - x_i^2})\Delta x$

$F = \int_{-1}^{1} \delta(x + 3)(2\sqrt{1 - x^2})dx$



Circular Plate - Continued

$F = \int_{-1}^{1} \delta(x + 3)(2\sqrt{1 - x^2})dx$

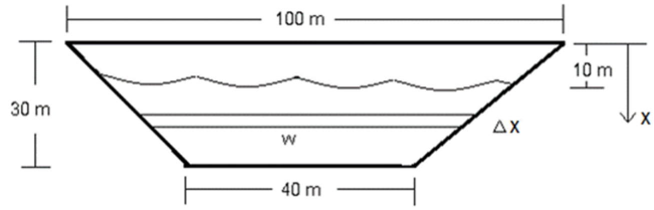
$= 2\delta \int_{-1}^{1}(x + 3)(\sqrt{1 - x^2})dx$

$2\delta[\int_{-1}^{1} x(\sqrt{1 - x^2})dx + 3\int_{-1}^{1} \sqrt{1 - x^2}dx]$

$= 2\delta[0 + 3(\frac{\pi}{2})]$

$F = \boxed{3\pi\delta}$ lb

Dam



$F_i = P_i A_i$
$P_i = \rho g d_i$
$A_i = w_i \Delta x$

(x, w)
(0, 100)
(30, 40)

$w - 100 = -\frac{60}{30}(x - 0)$
$w = 100 - 2x$
$d = x - 10$

$$F_i = \rho g (x_i - 10)(100 - 2x_i) \triangle x$$
$$F = \rho g \int_{10}^{30} (x - 10)(100 - 2x) dx$$
$$F = \boxed{1.04 \times 10^8 \text{ N}}$$

Quiz

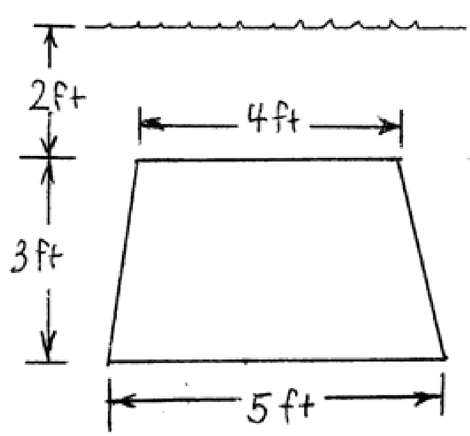
Which integral will calculate the hydrostatic force on one face of the vertical plate shown?

A) $F = \int_2^5 \delta x (4 + \frac{x}{3}) dx$

B) $F = \int_0^3 \delta x (4 + \frac{x}{3}) dx$

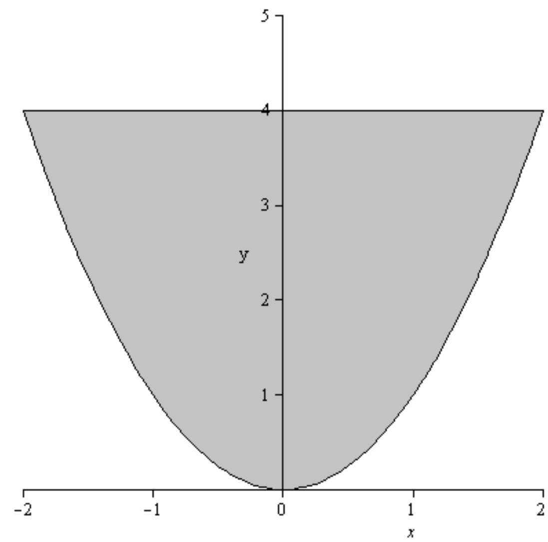
C) $F = \int_0^3 \delta (x + 2)(4 + \frac{x}{3}) dx$

D) $F = \int_2^5 \delta (x + 2)(\frac{10}{3} + \frac{x}{3}) dx$

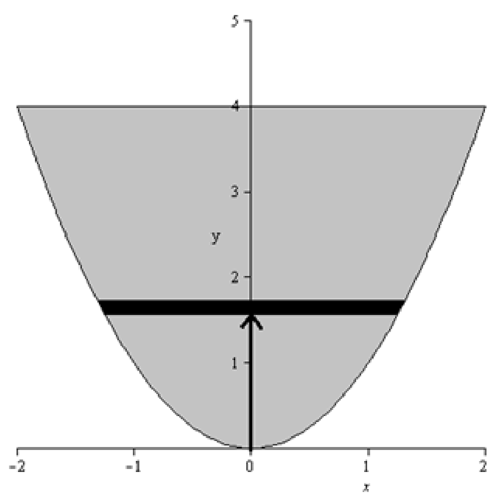


Try

A plate shaped as the region bounded by the curves, $y = x^2$ and $y = 4$, is submerged vertically in water so that the top is even with the surface of the water, as shown. Compute the force on one face of the plate. All lengths are in meters.



Solution



$$F_i = P_i A_i$$

$$P_i = \rho g d_i$$

$$A_i = w_i \Delta y$$

$$d_i = 4 - y_i$$

$$w_i = 2\sqrt{y_i}$$

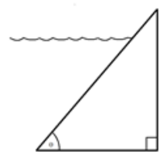
$$F_i = \rho g (4 - y_i)(2\sqrt{y_i})\Delta y$$

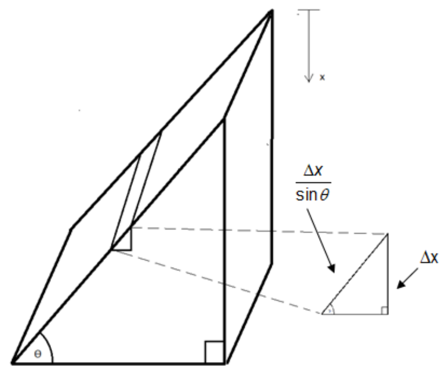
$$F = \int_0^4 \rho g (4 - y)(2\sqrt{y}) dy$$

$$F = \boxed{\frac{256}{16}\rho g}$$

Inclined Plate

If the plate is inclined, then the area of the strip must be adjusted.





$A_i = w_i h$ where h is the slant height of the strip.

$F_i = \rho g d_i w_i h$, where $h = \frac{1}{\sin\theta}\Delta x$

$F = \int_a^b \rho g d(x) \frac{w(x)}{\sin\theta} dx$

$F = \frac{\rho g}{\sin\theta} \int_a^b d(x) w(x) dx$

where $d(x)$ is the depth as a function of x and $w(x)$ is the width of the strip as a function of x.

8.3b Moments and Center of Mass

Concepts

Average or Mean

Point Mass

Moment

Lamina

Center of Mass

Centroid

Average

If 4 people have a score of 7 on a quiz, 3 people have a score of 6, and 2 people have a score of 5, what is the average score?

$$Average = \frac{4(7) + 3(6) + 2(5)}{4 + 3 + 2}$$

$$Average \approx 6.2$$

Point Mass

When all of the mass of a object can be considered to be concentrated at a single point, we can use point masses to model the situation.

Point masses have mass, but no volume.

Moment

Moment is a measure of the tendency to rotate.

Moments of Point Masses

Moment about an axis:

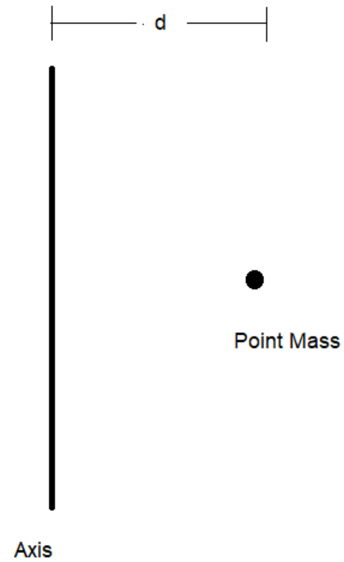
$M = md$

where

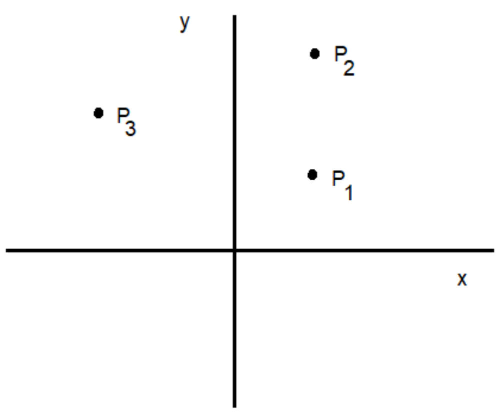
M is the moment

m is the mass

d is the distance



System of Point Masses

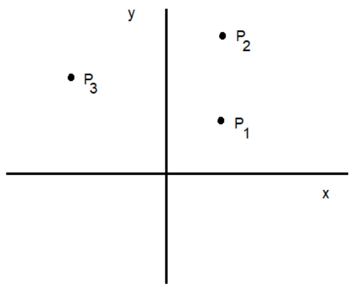


$m = m_1 + m_2 + m_3$

$M_x = m_1 y_1 + m_2 y_2 + m_3 y_3$

$M_y = m_1 x_1 + m_2 x_2 + m_3 x_3$

Center of Mass



$$m = m_1 + m_2 + m_3$$

$$M_x = m_1 y_1 + m_2 y_2 + m_3 y_3$$

$$M_y = m_1 x_1 + m_2 x_2 + m_3 x_3$$

Definition:
Center of Mass - $(\bar{x}, \bar{y})$, location where total mass of entire system can be placed and point mass will have same moments as the system.

$$m\bar{x} = M_y$$

$$m\bar{y} = M_x$$

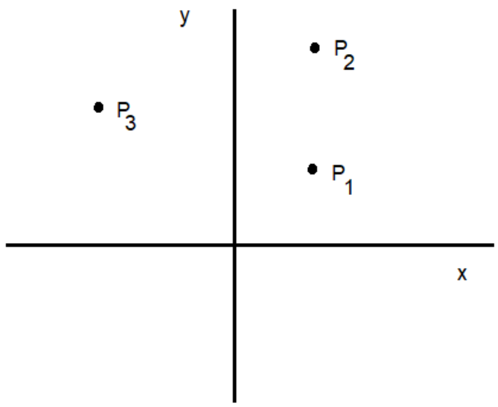
$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$$

Compare

$$Average = \frac{4(7) + 3(6) + 2(5)}{4 + 3 + 2}$$

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} = \frac{M_y}{m}$$

Example



$P1 : m_1 = 5$ at $(1, 1)$

$P2 : m_2 = 4$ at $(1, 3)$

$P3 : m_3 = 2$ at $(-2, 2)$

$m = 5 + 4 + 2 = 11$

$M_x = 5(3) + 4(1) + 2(2) = 23$

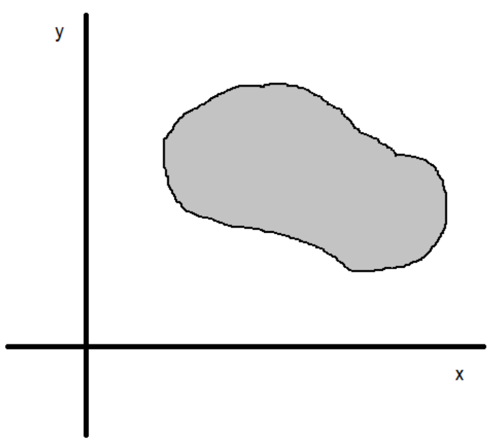
$M_y = 5(1) + 4(1) + 2(-2) = 5$

$\bar{x} = \frac{5}{11}$

$\bar{y} = \frac{23}{11}$

Lamina

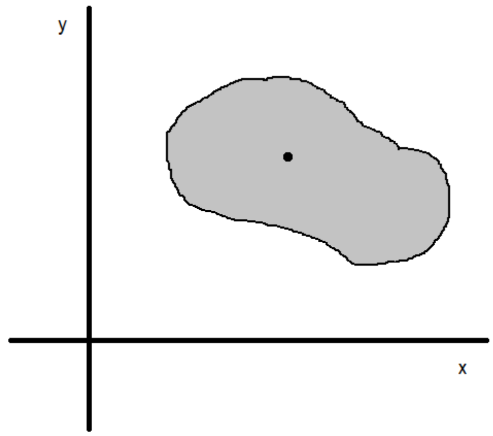
Lamina – Model. Plane region with an area and a mass, but with no thickness (no volume).



For APMA 1110 we will study laminas with constant mass density, $\rho = mass/area$.

Centroid

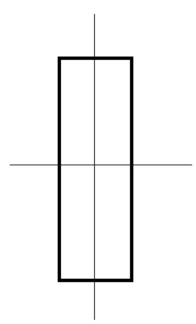
Centroid – center of mass of an object if the mass density is constant.



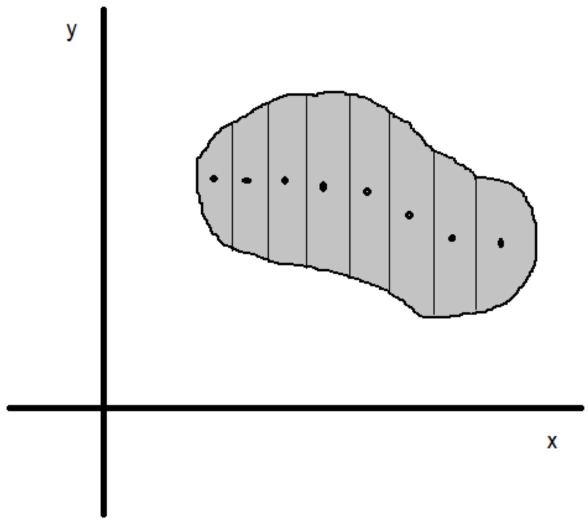
For APMA 1110, centroid = center of mass

Symmetry Principle

If R is a region that is symmetric about a line, l, then the centroid of R lies on l.

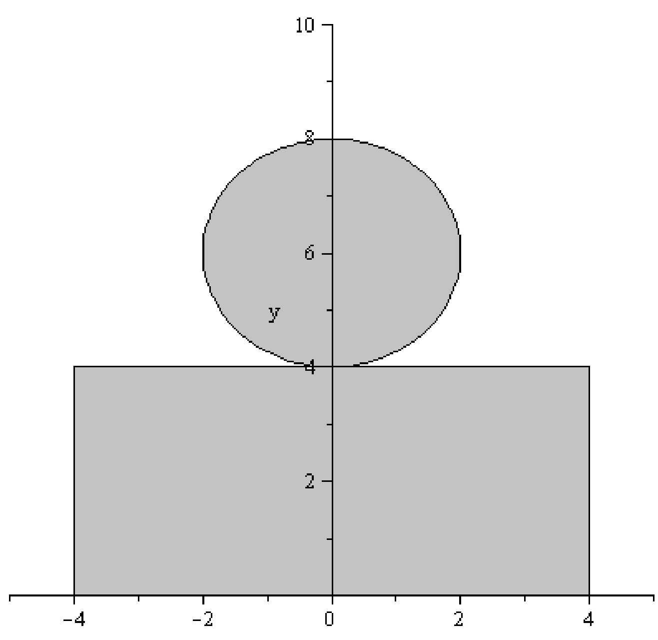


Divide Lamina into Manageable Shapes



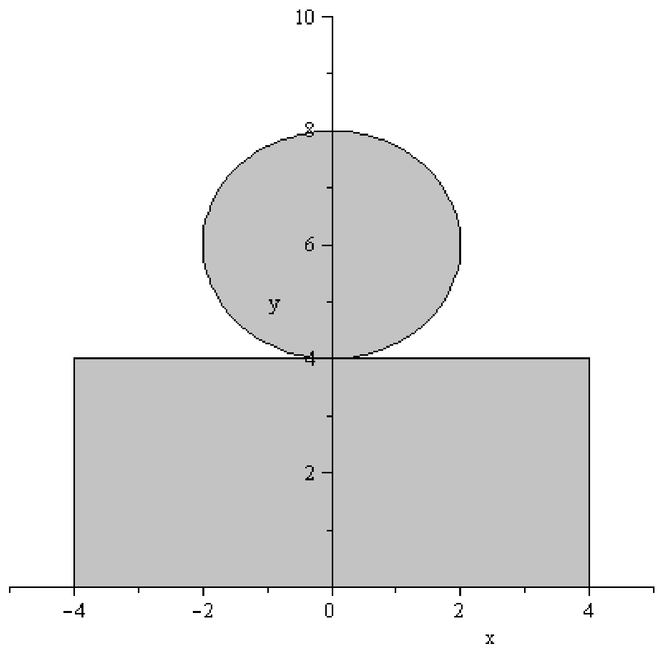
Try

Find the mass and find the moments, M_x and M_y , of the shaded region below. Use ρ for the mass density.



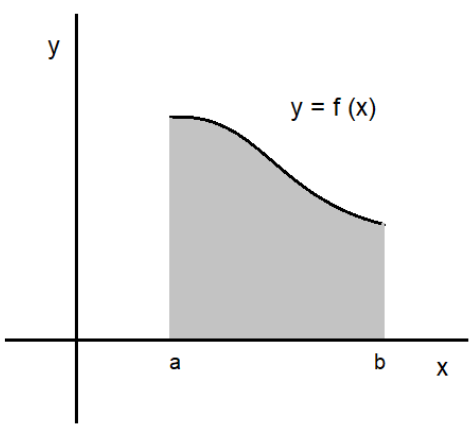
Try

Find the centroid of the shaded region below:



General Region

Assume $f(x) \geq 0$



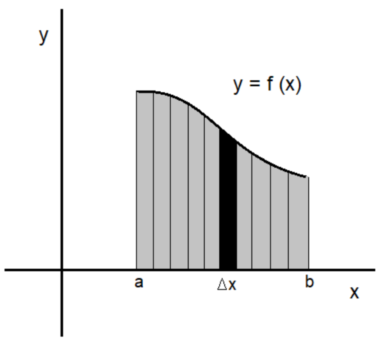
$m\bar{x} = M_y$

$m\bar{y} = M_x$

$(\bar{x}, \bar{y}) = (\frac{M_y}{m}, \frac{M_x}{m})$

Now, how are the mass (m) and the moments (M_x and M_y) computed?

Mass of General Region



Let m_i be the approximate mass of one "rectangle".

Let A_i be the approximate area of one "rectangle".

$m_i = \rho A_i$

$A_i = f(x_i^*)\Delta x$

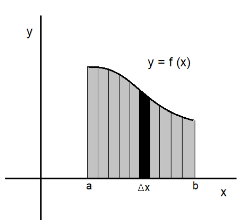
$m_i = \rho f(x_i^*)\Delta x$

$m = \int_a^b \rho dA$ where $dA = f(x)dx$

$m = \rho \int_a^b f(x)dx$

$m = \rho A$ where A is the area of the region.

Moments of General Region



$M_{yi} = \bar{x}_i m_i$

$\quad = \bar{x}_i \rho A_i$

$M_y = \int \bar{x} \rho dA$

where $\bar{x} = x$ and $dA = f(x)dx$

$M_y = \int_a^b \rho x f(x) dx$

$M_{xi} = \bar{y} m_i$

$\quad = \bar{y}_i \rho A_i$

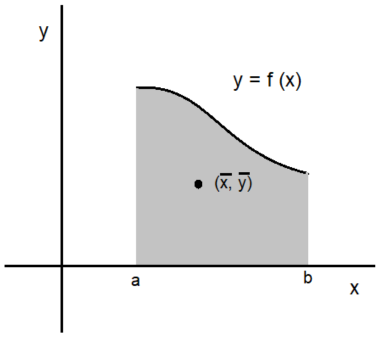
$M_x = \int \bar{y} \rho dA$

where $\bar{y} = \frac{1}{2} f(x)$ and $dA = f(x)dx$

$M_x = \int_a^b \frac{1}{2} f(x) \rho f(x) dx$

$M_x = \int_a^b \rho \frac{1}{2} (f(x))^2 dx$

Center of Mass of General Region

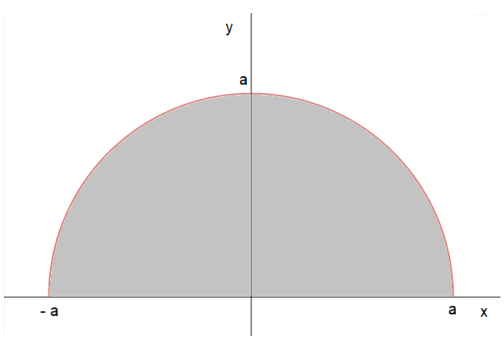


$\bar{x} = \frac{M_y}{m}$

$\bar{y} = \frac{M_x}{m}$

Example

If R is the region between the x-axis and the curve, $y = \sqrt{a^2 - x^2}$, find the coordinates of the centroid of R.



Solution:

$\bar{x} = 0$ by the Symmetry Principle.

(Also, $M_y = 0$.)

$\bar{y} = \frac{M_x}{m}$

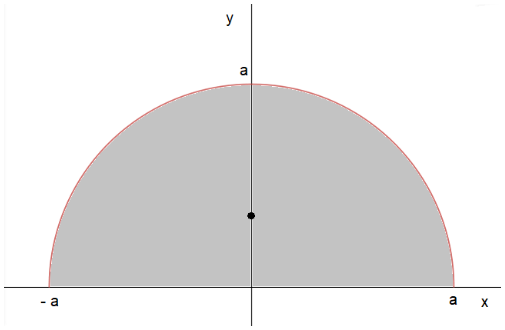
$m = \frac{\pi a^2}{2}\rho$

Example - Continued

$$\begin{aligned} M_x &= \int_{-a}^{a} \rho(\tfrac{1}{2})(\sqrt{a^2 - x^2})^2 dx \\ &= \tfrac{\rho}{2} \int_{-a}^{a} (a^2 - x^2) dx \\ &= \tfrac{\rho}{2}(a^2 x - \tfrac{x^3}{3})|_{-a}^{a} \\ &= \tfrac{\rho}{2}(a^3 - \tfrac{a^3}{3} + a^3 - \tfrac{a^3}{3}) \\ &= \tfrac{2a^3\rho}{3} \end{aligned}$$

$\bar{y} = \frac{M_x}{m} = \frac{4a}{3\pi}$

$(\bar{x}, \bar{y}) = (0, \frac{4a}{3\pi})$



Quiz

Which of the following pairs of coordinates gives the centroid of the quarter circle, $y = \sqrt{36 - x^2}$, $0 \leq x \leq 6$?

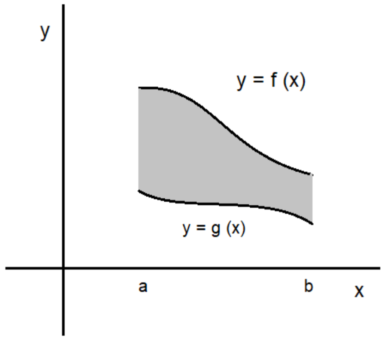
A) $(\frac{8}{\pi}, \frac{8}{\pi})$

B) $(0, \frac{8}{\pi})$

C) $(\frac{8}{\pi}, 0)$

D) $(\frac{4}{3\pi}, \frac{4}{3\pi})$

Region Between Two Curves



$$m = m_f - m_g$$
$$M_y = M_{fy} - M_{gy}$$
$$M_x = M_{fx} - M_{gx}$$

$$m = \rho \int_a^b (f(x) - g(x))dx$$
$$M_y = \rho \int_a^b x[f(x) - g(x)]dx$$
$$M_x = \rho \int_a^b \frac{1}{2}[(f(x))^2 - (g(x))^2]dx$$

$$\bar{x} = \frac{M_y}{m}$$
$$\bar{y} = \frac{M_x}{m}$$

Try

Find the centroid of the region enclosed by the curves, $y = \sqrt{x}$, and $y = x$.

Quiz

For the problem on the previous slide, which set of equations is correct?

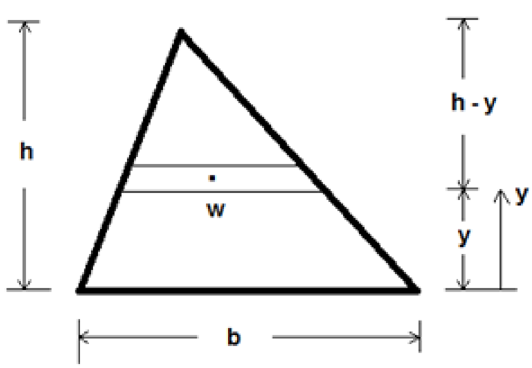
A) $M_y = \rho \int_0^1 (x^2 - x\sqrt{x})dx$, $M_x = \rho \int_0^1 \frac{1}{2}(x^2 - x)dx$,
$m = \rho \int_0^1 (x - \sqrt{x})dx$

B) $M_y = \rho \int_0^1 (x\sqrt{x} - x^2)dx$, $M_x = \rho \int_0^1 \frac{1}{2}(\sqrt{x} - x)^2 dx$,
$m = \rho \int_0^1 (\sqrt{x} - x)dx$

C) $M_y = \rho \int_0^1 (x\sqrt{x} - x^2)dx$, $M_x = \rho \int_0^1 \frac{1}{2}(x - x^2)dx$,
$m = \rho \int_0^1 (\sqrt{x} - x)dx$

Triangle

Find the moment of a general triangle about its base:

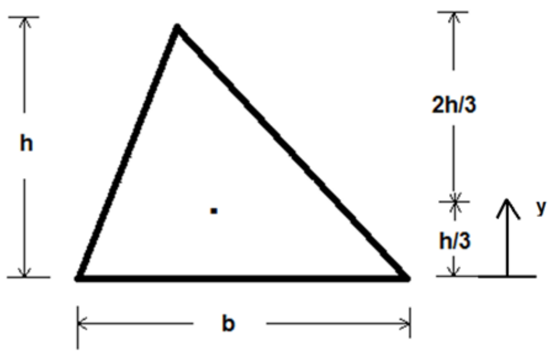


$$
\begin{aligned}
M_{bi} &= \rho \bar{y}_i A_i \\
&= \rho y_i w_i \Delta y \\
&= \rho y_i (b - \tfrac{b}{h} y_i)\Delta y
\end{aligned}
$$

$$
\begin{aligned}
M_b &= \int_0^h \rho y (b - \tfrac{b}{h} y) dy \\
&= \rho \left(\tfrac{by^2}{2} - \tfrac{by^3}{3h}\right)\Big|_0^h \\
&= \rho \tfrac{bh^2}{6}
\end{aligned}
$$

Triangle

How far is the centroid of a triangle from its base?



$$M_b = \rho \frac{bh^2}{6}$$

$$\bar{y} = \frac{M_b}{m}$$

$$= \frac{(\rho \frac{bh^2}{6})}{(\rho \frac{bh}{2})}$$

$$\bar{y} = \frac{h}{3}$$