# Chapter 7: Techniques of Integration

Stacie Pisano
University of Virginia

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## 7.1 Integration by Parts

## Intro - Quiz - 1

True or False?

If $f(x) = g(x)h(x)$, then $f'(x) = g'(x)h'(x)$

A) True

B) False



## Intro - Quiz - 2

True or False?

If $f(x) = g(x)h(x)$, then $\int f(x)dx = (\int g(x)dx)(\int h(x)dx)$

A) True

B) False

## When to Use

What if we want to evaluate $\int xe^x dx$ ?

If $u$ and $v$ are functions of $x$,

$\int uvdx \neq \int udx \int vdx$

So

$\int xe^x dx \neq \int xdx \int e^x dx$

Instead, we want to reverse the product rule of differentiation.



## Reverse Product Rule

Product Rule: $(uv)' = u'v + uv'$

Reverse it: $\int (uv)' dx = \int u'vdx + \int uv'dx$

$$uv = \int v\frac{du}{dx}dx + \int u\frac{dv}{dx}dx$$

$$uv = \int vdu + \int udv$$

$$\int udv = uv - \int vdu$$

This is the master equation for integration by parts.

# Example 1

Evaluate $\int xe^x dx$

Determine the parts:

$u = x \quad dv = e^x dx$

$du = dx \quad v = e^x$

Apply the master equation:

$\int udv = uv - \int vdu$

$\int xe^x dx = xe^x - \int e^x dx$

$= xe^x - e^x + C$

$$\boxed{\int xe^x dx = xe^x - e^x + C}$$



# Example 2

Evaluate $\int x \cos x dx$

Solution:
Determine the parts:

$u = x \quad dv = \cos x dx$

$du = dx \quad v = \sin x$

Apply the master equation:

$\int udv = uv - \int vdu$

$\int x \cos x dx = x \sin x - \int \sin x dx$

$= x \sin x + \cos x + C$

$$\boxed{\int x \cos x dx = x \sin x + \cos x + C}$$

## Try It

Evaluate $\int_0^{\pi} 2\pi x \sin x dx$



## Other Applications of IBD

Sometimes, an integral/anti-derivative can be obtained by knowing the derivative of a function.

We will apply that to evaluating:

$\int \ln x dx$

$\int \sin^{-1} x dx$

## Integral of $\ln x$

Evaluate $\int \ln x dx$

Solution:

$u = \ln x \quad dv = dx$

$du = \frac{1}{x} dx \quad v = x$

$\int u dv = uv - \int v du$

$\int \ln x dx = x \ln x - \int x(\frac{1}{x}) dx$

$= x \ln x - \int dx$

$= x \ln x - x$

$\boxed{\int \ln x dx = x \ln x - x + C}$

$\int \ln x dx = x \ln x - x + C$ (Memorize.)



## Integrals of Inverse Functions

Evaluate $\int \sin^{-1} x dx$

Solution:

$u = \sin^{-1} x \quad dv = dx$

$du = \frac{1}{\sqrt{1-x^2}} dx \quad v = x$

$\int u dv = uv - \int v du$

$\int \sin^{-1} x dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx$

# Integrals of Inverse Functions - Continued

$\int \sin^{-1} x dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx$

$w = 1 - x^2$

$dw = -2x dx$

$\int -\frac{1}{2} \frac{dw}{\sqrt{w}}$

$-\sqrt{w}$

$-\sqrt{1-x^2}$

$$\boxed{\int \sin^{-1} x dx = x \sin^{-1} x + \sqrt{1-x^2} + C}$$



# Choosing U

I - Inverse

L - Log

A - Algebra (power functions)

T - Trigonometry

E - Exponential

## Circular Integrals

Evaluate $\int e^x \cos x dx$

Solution:

$u = \cos x \quad dv = e^x dx$

$du = -\sin x dx \quad v = e^x$

$\int u dv = uv - \int v du$

$\int e^x \cos x dx = e^x \cos x - \int -e^x \sin x dx$

$e^x \cos x + \int e^x \sin x dx$



## Circular Integrals - Continued

$\int e^x \cos x dx = e^x \cos x + \int e^x \sin x dx$

$u = \sin x \quad dv = e^x dx$

$du = \cos x \quad v = e^x$

$e^x \sin x - \int e^x \cos x dx$

$\int e^x \cos x dx = e^x \cos x + e^x \sin x - \int e^x \cos x dx$

$2 \int e^x \cos x dx = e^x \cos x + e^x \sin x$

$$\boxed{\int e^x \cos x dx = \frac{e^x}{2}(\cos x + \sin x) + C}$$

# Quiz

$\int x \sec^2 x dx =$

A) $\frac{x^2}{2} \tan x + C$

B) $\frac{x^2}{2} - \ln|\sec x| + C$

C) $x \tan x - \ln|\sec x| + C$

D) $x \tan x - \tan x + C$



# 7.2 Trigonometric Integrals

## Trigonometric Integrals

We will focus on integrals of these types:

$\int \sin^m x \cos^n x dx$

$\int \tan^m x \sec^n x dx$

where $m$ and $n$ are positive integers.



## Strategy

The strategy depends on relationships between $\sin x$ and $\cos x$ and between $\tan x$ and $\sec x$.

## Relationships Between $\sin x$ and $\cos x$

$\cos^2 x + \sin^2 x = 1$

$\frac{d(\sin x)}{dx} = \cos x$

$\int \cos x dx = \sin x + C$

$\frac{d(\cos x)}{dx} = -\sin x$

$\int \sin x dx = -\cos x + C$



## Strategy

$\frac{d(\sin x)}{dx} = \cos x$

$\frac{d(\cos x)}{dx} = - \; sinx$

So, for $\int \sin^m x \cos^n x dx$, choose either $du = \cos x dx$ or $du = -\sin x dx$.

If $n$ is odd, choose $du = \cos x dx$. If $m$ is odd, choose $du = -\sin x dx$.

## Example 1

Evaluate $\int \sin^2 x \cos x dx$

Solution:

$$du = \cos x dx$$

$$u = \sin x$$

$$\int \sin^2 x \cos x dx = \int u^2 du$$

$$= \frac{u^3}{3} + C$$

$$= \boxed{\frac{\sin^3 x}{3} + C}$$



## Example 2

Evaluate $\int \sin^3 x dx$

Solution:

$$\int \sin^3 x dx = \int \sin^2 x \sin x dx$$

$$= \int - \sin^2 x(- \sin x) dx$$

$$du = - \sin x dx$$

$$u = \cos x$$

$$= \int -(1 - \cos^2 x)(- \sin x) dx$$

$$= \int -(1 - u^2) du$$

$$= -(u - \frac{u^3}{3}) + C$$

$$= \boxed{-(\cos x - \frac{\cos^3 x}{3}) + C}$$

## Example 3

Evaluate $\int \sin^4 x \cos^3 x dx$

Solution:

$$\int \sin^4 x \cos^3 x dx = \int \sin^4 x \cos^2 x \cos x dx$$
$$du = \cos x dx$$
$$u = \sin x$$
$$= \int \sin^4 x(1 - \sin^2 x) \cos x dx$$
$$= \int u^4(1 - u^2) du$$
$$= \int (u^4 - u^6) du$$
$$= \frac{u^5}{5} - \frac{u^7}{7} + C$$
$$\int \sin^4 x \cos^3 x dx = \boxed{\frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C}$$



## Even Powers

So, for $\int \sin^m x dx \cos^n x dx$, it is clear what to do if either $m$ or $n$ is odd. What if both "m" and "n" are even?

If both "m" and "n" are even, then reduce the order using these helpful identities:

$\cos^2 x = \frac{1+\cos 2x}{2}$

$\sin^2 x = \frac{1-\cos 2x}{2}$

## Example 4

Evaluate $\int \cos^2 x dx$

Solution:

$$\int \cos^2 x dx = \int \frac{1+\cos 2x}{2} dx$$
$$= \int \frac{1}{2}(1 + \cos 2x) dx$$
$$\int \cos^2 x dx = \boxed{\frac{1}{2}(x + \frac{1}{2}\sin 2x) + C}$$



## Example 5

Evaluate $\int \cos^2 x \sin^2 x dx$

Solution:

$$\int \cos^2 x \sin^2 x dx = \int \frac{(1+\cos 2x)}{2} \frac{(1-\cos 2x)}{2} dx$$
$$= \int \frac{1}{4}(1 - \cos^2 2x) dx$$
$$= \int \frac{1}{4}(1 - \frac{1}{2}(1 + \cos 4x)) dx$$
$$= \int \frac{1}{4}(1 - \frac{1}{2} - \frac{1}{2}\cos 4x) dx$$
$$= \int (\frac{1}{8} - \frac{1}{8}\cos 4x) dx$$
$$\int \cos^2 x \sin^2 x dx = \boxed{\frac{1}{8}x - \frac{1}{32}\sin 4x + C}$$

# Quiz

After an appropriate substitution, which integral is equivalent to

$\int \sin^2 3x dx$ ?

A) $\int \frac{(1-\cos^2 x)}{2} dx$

B) $\int \frac{(1+\cos 6x)}{2} dx$

C) $\int \frac{(1+\cos 6x)}{6} dx$

D) $\int \frac{(1-\cos 6x)}{2} dx$



# Relationships Between $\tan x$ and $\sec x$

$1 + \tan^2 x = \sec^2 x$

$\frac{d(\tan x)}{dx} = \sec^2 x$ $\qquad$ $\int \sec^2 x dx = \tan x + C$

$\frac{d(\sec x)}{dx} = \tan x \sec x$ $\qquad$ $\int \tan x \sec x dx = \sec x + C$

## Strategy

$$\frac{d(\tan x)}{dx} = \sec^2 x$$

$$\frac{d(\sec x)}{dx} = \ tanx \sec x$$

So, for $\int \tan^m x \sec^n x dx$, choose either $du = \sec^2 x dx$ or $du = \tan x \sec x dx$.

If $n$ is even, choose $du = \sec^2 x dx$. If $m$ and $n$ are both odd, choose $du = \tan x \sec x dx$.



## Example 6

Evaluate $\int \tan^4 x \sec^4 x dx$

Solution:

$$\int \tan^4 x \sec^4 x dx = \int \tan^4 x \sec^2 x \sec^2 x dx$$

$$du = \sec^2 x dx$$

$$u = \tan x$$

$$= \int \tan^4 x(1 + \tan^2 x) \sec^2 x dx$$

$$= \int u^4(1 + u^2) du$$

$$= \int (u^4 + u^6) du$$

$$= \frac{u^5}{5} + \frac{u^7}{7} + C$$

$$\int \tan^4 x \sec^4 x dx = \boxed{\frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C}$$

## Summary

$$\int \sin^m x \cos^n x dx$$

| $m$ | $n$ | Strategy |
|---|---|---|
| even | odd | $u = \sin x$<br>$du = \cos x dx$ |
| odd | even | $u = \cos x$<br>$du = -\sin x dx$ |
| odd | odd | $u = \sin x$<br>$du = \cos x dx$ |
| even | even | Reduce order. |



## Summary

$$\int \tan^m x \sec^n x dx$$

| $m$ | $n$ | Strategy |
|---|---|---|
| even | odd | No explicit strategy. |
| odd | even | $u = \tan x$<br>$du = \sec^2 x dx$ |
| odd | odd | $u = \sec x$<br>$du = \tan x \sec x dx$ |
| even | even | $u = \tan x$<br>$du = \sec^2 x dx$ |

# Memorize

You are expected to know the derivatives and antiderivatives of all of the trigonometric functions:

$\frac{d(sin(x))}{dx} = \cos x$

$\frac{d(cos(x))}{dx} = -\sin x$

$\frac{d(tan(x))}{dx} = \sec^2 x$

$\frac{d(sec(x))}{dx} = \tan x \sec x$

$\frac{d(csc(x))}{dx} = -\cot x \csc x$

$\frac{d(cot(x))}{dx} = -\csc^2 x$

$\int \sin x dx = -\cos x + C$

$\int \cos x dx = \sin x + C$

$\int \tan x dx = ln|\sec x| + C$

$\int \sec x dx = ln|\sec x + \tan x| + C$

$\int \csc x dx = ln|\csc x - \cot x| + C$

$\int \cot x dx = ln|\sin x| + C$



# Quiz

After an appropriate substitution, which integral is equivalent to $\int \tan^3 x \sec^5 x dx$ ?

A) $\int \tan^3 x dx \int \sec^5 x dx$

B) $\int u^5 du$

C) $\int (u^6 + u^4) du$

D) $\int (u^2 - 1) u^4 du$

# 7.3 Trigonometric Substitution



# When To Use?

$\int \sqrt{9-x^2}dx$ $\int \frac{1}{\sqrt{x^2-25}}dx$ $\int_{\frac{2\sqrt{3}}{3}}^{2} \frac{1}{x^2\sqrt{x^2+4}}dx$

The method of trig substitution recommends a substitution that rationalizes the integrand in cases like these.

# Example 1

Evaluate $\int \sqrt{9 - x^2}dx$

Solution:

Substitute: $x = 3\sin\theta$

$dx = 3\cos\theta d\theta$

$$\begin{aligned}\int \sqrt{9 - x^2}dx &= \int \sqrt{9 - 9\sin^2\theta}(3\cos\theta)d\theta \\ &= \int \sqrt{9(1 - \sin^2\theta)}(3\cos\theta)d\theta \\ &= \int \sqrt{9\cos^2\theta}(3\cos\theta)d\theta \\ &= \int (3\cos\theta)(3\cos\theta)d\theta \\ &= \int 9\cos^2\theta d\theta\end{aligned}$$



# Example 1 - continued

$$\begin{aligned}\int \sqrt{9 - x^2}dx &= \int 9\cos^2\theta d\theta \\ &= \int \tfrac{9}{2}(1 + \cos 2\theta)d\theta\end{aligned}$$

Now integrate:

$$\int \sqrt{9 - x^2}dx = \tfrac{9}{2}(\theta + \tfrac{1}{2}\sin 2\theta) + C$$

Now resubstitute:

We need to write $\theta$ and $\sin 2\theta$ in terms of $x$.

We know: $x = 3\sin\theta$

$\sin\theta = \frac{x}{3}$

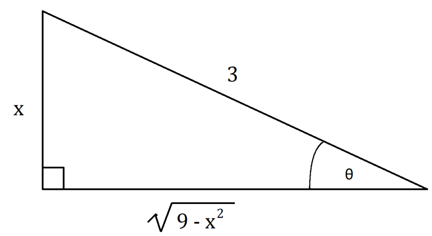
$\theta = \sin^{-1}\frac{x}{3}$ ✓

Now what about $\sin 2\theta$?

## Example 1 - conclusion

We know that $\sin 2\theta = 2\sin\theta\cos\theta$, and we know that $\sin\theta = \frac{x}{3}$ so all we need is $\cos\theta$ in terms of x.

Draw a right triangle with an angle $\theta$ such that $\sin\theta = \frac{x}{3}$. Then determine the third side of the triangle using the Pythagorean Theorem.




So $\cos\theta = \frac{\sqrt{9-x^2}}{3}$ and $\sin 2\theta = 2(\frac{x}{3})(\frac{\sqrt{9-x^2}}{3})$ ✓ .



## Example 1 - conclusion

$$\int \sqrt{9-x^2}dx = \int 9\cos^2\theta d\theta$$

$$= \frac{9}{2}(\theta + \frac{1}{2}\sin 2\theta) + C$$

$$= \boxed{\frac{9}{2}\left(\sin^{-1}\frac{x}{3} + \frac{x}{9}\sqrt{9-x^2}\right) + C}$$

## Generalize

When the factor, $\sqrt{a^2 - x^2}$, appears in the integrand, use the substitution, $x = a\sin\theta$. This is equivalent to substituting $\theta = \sin^{-1}\frac{x}{a}$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

So,

$x = a\sin\theta$

$dx = a\cos\theta d\theta$

$$\begin{aligned}\sqrt{a^2 - x^2} &= \sqrt{a^2 - a^2\sin\theta} \\ &= \sqrt{a^2(1-\sin^2\theta)} \\ &= \sqrt{a^2\cos^{\theta}} \\ &= a|\cos\theta| \\ &= a\cos\theta\end{aligned}$$



## Rationalizing Substitution

| Factor | Rationalizing Substitution | Result |
|---|---|---|
| $\sqrt{a^2 - x^2}$ | $x = a\sin\theta$<br>$dx = a\cos\theta d\theta$<br>$\theta = \sin^{-1}\frac{x}{a}$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ | $\sqrt{a^2 - x^2} = a\cos\theta$ |
| $\sqrt{a^2 + x^2}$ | $x = a\tan\theta$<br>$dx = a\sec^2\theta d\theta$<br>$\theta = \tan^{-1}\frac{x}{a}$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ | $\sqrt{a^2 + x^2} = a\sec\theta$ |
| $\sqrt{x^2 - a^2}$ | $x = a\sec\theta$<br>$dx = a\tan\theta\sec\theta d\theta$<br>$\theta = \sec^{-1}\frac{x}{a}$, $0 < \theta \leq \frac{\pi}{2}$<br>or $\pi < \theta < \frac{3\pi}{2}$ | $\sqrt{x^2 - a^2} = a\tan\theta$ |

# Quiz

For the integral, $\int \frac{dx}{x^2\sqrt{x^2-9}}$, use the substitution:

A) $x = 3\sin\theta,\ dx = 3\cos\theta d\theta$

B) $x = 3\tan\theta,\ dx = 3\sec^2\theta d\theta$

C) $x = 9\sec\theta,\ dx = 9\tan\theta\sec\theta d\theta$

D) $x = 3\sec\theta,\ dx = 3\tan\theta\sec\theta d\theta$



# Quiz

Using the substitution, $x = 3\sec\theta,\ dx = 3\tan\theta\sec\theta d\theta$,

$\int \frac{dx}{x^2\sqrt{x^2-9}} =$

A) $\int \frac{d\theta}{9\sec^2\theta\tan\theta}$

B) $\int \frac{d\theta}{27\sec^2\theta\tan\theta}$

C) $\int \frac{1}{9}\cos\theta d\theta$

# Quiz

$$\int \frac{dx}{x^2\sqrt{x^2-9}} = \int \frac{1}{9}\cos\theta d\theta = \frac{1}{9}\sin\theta + C =$$

A) $\frac{1}{9}\frac{\sqrt{x^2-9}}{x} + C$

B) $\frac{1}{3}\frac{1}{x} + C$

C) $\frac{1}{9}\frac{x}{\sqrt{x^2-9}} + C$



# Example 2

Evaluate $\int \frac{1}{\sqrt{x^2-25}}dx$

Solution:

$x = 5\sec\theta$

$dx = 5\tan\theta\sec\theta d\theta$

$$\begin{aligned}\int \frac{1}{\sqrt{x^2-25}}dx &= \int \frac{1}{\sqrt{25\sec^2\theta-25}}(5\tan\theta\sec\theta)d\theta \\ &= \int \frac{5\tan\theta\sec\theta}{\sqrt{25(\sec^2\theta-1)}}d\theta \\ &= \int \frac{5\tan\theta\sec\theta}{5\tan\theta}d\theta \\ &= \int \sec\theta d\theta \\ &= ln|\sec\theta + \tan\theta| + C\end{aligned}$$

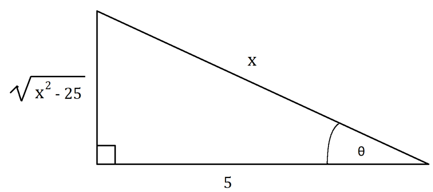
Now resubstitute.

## Example 2 - continued

$\int \frac{1}{\sqrt{x^2-25}} dx = ln|\sec\theta + \tan\theta| + C$

We need to write $\sec\theta$ and $\tan\theta$ in terms of $x$. We know that $\sec\theta = \frac{x}{5}$. ✓

Draw a right triangle with an angle $\theta$ such that $\sec\theta = \frac{x}{5}$. Then determine the third side of the triangle using the Pythagorean Theorem.




From the triangle, $\tan\theta = \frac{\sqrt{x^2-25}}{5}$. ✓



## Example 2 - continued

$$\begin{aligned}\int \frac{1}{\sqrt{x^2-25}} dx &= \int \sec\theta d\theta \\ &= ln|\sec\theta + \tan\theta| + C \\ &= \boxed{ln|\tfrac{x}{5} + \tfrac{\sqrt{x^2-25}}{5}| + C}\end{aligned}$$

This can be simplified further:

$$\begin{aligned}\int \frac{1}{\sqrt{x^2-25}} dx &= \ln\left|\frac{x+\sqrt{x^2-25}}{5}\right| + C \\ &= \ln|x + \sqrt{x^2-25}| - \ln 5 + C \\ &= \boxed{\ln|x + \sqrt{x^2-25}| + C_2}\end{aligned}$$

## Example 3

Evaluate $\int_{\frac{2\sqrt{3}}{3}}^{2} \frac{1}{x^2\sqrt{x^2+4}}dx$

Solution:

$x = 2\tan\theta$

$dx = 2\sec^2\theta d\theta$

$$\int_{\frac{2\sqrt{3}}{3}}^{2} \frac{dx}{x^2\sqrt{x^2+4}} = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{2\sec^2\theta}{4\tan^2\theta\sqrt{4\tan^2\theta+4}}d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{2\sec^2\theta}{4\tan^2\theta\sqrt{4\sec^2\theta}}d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sec\theta}{4\tan^2\theta}d\theta$$



## Example 3 - continued

$$\int_{\frac{2\sqrt{3}}{3}}^{2} \frac{dx}{x^2\sqrt{x^2+4}} = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sec\theta}{4\tan^2\theta}d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos\theta}{4\sin^2\theta}d\theta$$

$$u = \sin\theta, du = \cos\theta d\theta$$

$$= \int_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} \frac{1}{4}u^{-2}du$$

$$= -\frac{1}{4}\frac{1}{u}\Big|_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}}$$

$$= -\frac{1}{2\sqrt{2}} + \frac{1}{2}$$

$$= \boxed{\frac{1}{2}(1 - \frac{\sqrt{2}}{2})}$$

# Try It

Evaluate: $\int \sqrt{1-9x^2}dx$



# 7.4 Integration of Rational Functions by Partial Fractions

## Which do you prefer?

$$\int \frac{x^2 - x + 4}{x^3 - 4x^2 + 4x} dx$$

OR

$$\int \left( \frac{1}{x} + \frac{3}{(x-2)^2} \right) dx$$



## Warm-Up Examples

$$\int \frac{3}{x-4} dx = 3 \ln|x-4| + C$$

$$\int \frac{5}{(x-4)^3} dx = \frac{5}{-2(x-4)^2} + C$$

$$\int \frac{3x}{x^2+4} dx = \tfrac{3}{2} \ln(x^2+4) + C$$

$$\int \frac{3}{x^2+4} dx = \tfrac{3}{2} \tan^{-1} \tfrac{x}{2} + C$$

## Warm-Up Generalized

| Integral | Criteria | Result |
|---|---|---|
| $\int \frac{A}{x+a}dx$ | $A \neq 0, a \in R$ | $A \ln\|x+a\| + C$ |
| $\int \frac{A}{(x+a)^n}dx$ | $A \neq 0, a \in R, n \neq 1$ | $\frac{A}{-(n-1)(x+a)^{n-1}} + C$ |
| $\int \frac{Ax}{x^2+a}dx$ | $A \neq 0, a > 0$ | $\frac{A}{2}\ln(x^2+a) + C$ |
| $\int \frac{A}{x^2+a^2}dx$ | $A \neq 0, a > 0$ | $\frac{A}{a}\tan^{-1}\frac{x}{a} + C$ |



## Quiz

$$\int \left[\frac{1}{x} + \frac{3}{(x-2)^3} + \frac{5}{x^2+4} + \frac{x}{x^2+9}\right] dx =$$

A) $\frac{-1}{x^2} + \frac{3}{(x-2)^4} + 5\tan^{-1}\frac{x}{2} + \ln(x^2+9) + C$

B) $\ln|x| - \frac{3}{2(x-2)^2} + \frac{5}{2}\tan^{-1}\frac{x}{2} + \frac{1}{2}\ln(x^2+9) + C$

C) $\ln|x| - \frac{6}{(x-2)^3} + \frac{5}{2}\tan^{-1}\frac{x}{2} + \ln(x^2+9) + C$

## Integrating Rational Functions

1) Make sure the degree of the numerator is less than the degree of the denominator. (If not, then divide.)

2) Fully factor the denominator.

3) Rewrite the integrand as a sum of partial fractions. (Method of Partial Fractions.)



## Example 1

Evaluate $\int \frac{2x^3 + 3x^2 + 7x + 4}{x + 1} dx$

The degree of the numerator (3) is not less than the degree of the denominator (1), so we must divide first...

$$\int \frac{2x^3 + 3x^2 + 7x + 4}{x + 1} dx = \int (2x^2 + x + 6 - \tfrac{2}{x+1}) dx$$

(There is only one factor in the denominator so proceed to integration.)

$$\int \frac{2x^3 + 3x^2 + 7x + 4}{x + 1} dx = \boxed{\tfrac{2x^3}{3} + \tfrac{x^2}{2} + 6x - 2\ln|x + 1| + C}$$

## Example 2

Evaluate $\int \frac{dx}{x^2+5x+6}$

The degree of the numerator (0) is less than the degree of the denominator (2).

Now fully factor the denominator...

$\int \frac{dx}{x^2+5x+6} = \int \frac{dx}{(x+2)(x+3)}$

Now rewrite the integrand as a sum of partial fractions...

First decompose the integrand into partial fractions with undetermined coefficients...

$\frac{1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$



## Example 2 - continued

Now determine the coefficients, $A$ and $B$...

$\frac{(x+2)(x+3)}{(x+2)(x+3)} = \frac{A(x+2)(x+3)}{x+2} + \frac{B(x+2)(x+3)}{x+3}$

$1 = A(x+3) + B(x+2)$

$1 = Ax + 3A + Bx + 2B$

$1 = x(A+B) + (3A+2B)$

Equalize the coefficients above...

$A + B = 0$

$3A + 2B = 1$

Solve this system of equations...

$A = 1$ , $B = -1$

So $\frac{1}{(x+2)(x+3)} = \frac{1}{x+2} - \frac{1}{x+3}$

## Example 2 - continued

Now integrate...

$$\int \frac{1}{(x+2)(x+3)}dx = \int (\frac{1}{x+2} - \frac{1}{x+3})dx$$

$$= \boxed{\ln|x+2| - \ln|x+3| + C}$$



## Alternative

Here is an alternative way to determine the coefficients...

$$\frac{1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

$$1 = A(x+3) + B(x+2)$$

The polynomial equation above must be true for all x.

Substitute $x = -3 \quad \rightarrow \quad 1 = 0A - B \quad \rightarrow \quad B = -1$

Substitute $x = -2 \quad \rightarrow \quad 1 = 1A - 0B \quad \rightarrow \quad A = 1$

$$\frac{1}{(x+2)(x+3)} = \frac{1}{x+2} - \frac{1}{x+3}$$

# Quiz

Decompose into partial fractions with determined coefficients.

$\frac{6}{(x-2)(x+1)} =$

A) $\frac{1}{x-2} - \frac{2}{x+1}$

B) $\frac{2}{x-2} - \frac{2}{x+1}$

C) $\frac{1}{x-2} - \frac{1}{x+1}$



# Try It

Evaluate $\int \frac{x^4+2}{x^2-1} dx$

# Example 3

Evaluate $\int \frac{3}{(x^2+2)(x-1)} dx$

The degree of the numerator is less than the degree of the denominator.

The denominator is fully factored.

Now rewrite the integrand as a sum of partial fractions...



# Example 3 - continued

Decompose the integrand into partial fractions with undetermined coefficients...

$\frac{3}{(x^2+2)(x-1)} = \frac{Ax+B}{x^2+2} + \frac{C}{x-1}$

Now determine the coefficients...

$3 = (Ax + B)(x - 1) + C(x^2 + 2)$

Substitute $x = 1 \rightarrow 3 = 3C \rightarrow C = 1$

$3 = Ax^2 + Bx - Ax - B + x^2 + 2$

$3 = x^2(A + 1) + x(B - A_)  + (2 - B)$

$A + 1 = 0 \rightarrow A = -1$

$B - A = 0 \rightarrow B = A \rightarrow B = -1$

$2 - B = 3$ ? ✓

## Example 3 - continued

$$\frac{3}{(x^2+2)(x-1)} = \frac{-x-1}{x^2+2} + \frac{1}{x-1}$$

$$\int \frac{3}{(x^2+2)(x-1)} dx = \int \left( \frac{-x-1}{x^2+2} + \frac{1}{x-1} \right) dx$$

$$= \int \left( \frac{-x}{x^2+2} + \frac{-1}{x^2+2} + \frac{1}{x-1} \right) dx$$

$$= \boxed{-\frac{1}{2}\ln(x^2+2) - \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + \ln|x-1| + C}$$



## Example 4

Evaluate $\int \frac{1}{x^3(x^2+1)} dx$

Solution:

$\frac{1}{x^3(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+1}$

$1 = Ax^2(x^2+1) + Bx(x^2+1) + C(x^2+1) + (Dx+E)x^3$

Substitute $x = 0 \to 1 = C$

$1 = x^4(A+D) + x^3(B+E) + x^2(A+1) + x(B) + 1$

$B = 0$

$A + 1 = 0 \to A = -1$

$B + E = 0 \to E = 0$

$A + D = 0 \to D = -A = 1$

$\frac{1}{x^3(x^2+1)} = \frac{-1}{x} + \frac{1}{x^3} + \frac{x}{x^2+1}$

# Example 4 - continued

$$\int \frac{1}{x^3(x^2+1)}\,dx = \int \Big( -\frac{1}{x} + \frac{1}{x^3} + \frac{x}{x^2+1} \Big)\,dx$$

$$= \boxed{-\ln|x| - \frac{1}{2}x^{-2} + \frac{1}{2}\ln(x^2+1) + C}$$



# Example 5

Decompose $\frac{x}{(x-2)^3(x^2+4)}$ as a sum of partial fractions with undetermined coefficients.

Solution:

$$\frac{x}{(x-2)^3(x^2+4)} = \boxed{\frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3} + \frac{Dx+E}{x^2+4}}$$

# Quiz

Rewrite $\frac{x^2}{(x^2-4)(x-4)^2(x^2+4)}$ as a sum of partial fractions with undetermined coefficients.

A) $\frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x-4} + \frac{Dx+E}{(x-4)^2} + \frac{Fx+G}{x^2+4}$

B) $\frac{Ax+B}{x^2-4} + \frac{C}{x-4} + \frac{Dx+E}{(x-4)^2} + \frac{Fx+G}{x^2+4}$

C) $\frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x-4} + \frac{D}{(x-4)^2} + \frac{Fx+G}{x^2+4}$



# Rationalizing Substitution

Evaluate $\int \frac{1}{x-\sqrt{x+2}} dx$

Solution:

We want to rationalize the integrand, but trig substitution is not appropriate. Try this...

$u = \sqrt{x+2}$

$u^2 = x + 2$

$x = u^2 - 2$

$dx = 2udu$

$$\begin{aligned}\int \frac{1}{x-\sqrt{x+2}} dx &= \int \frac{1}{(u^2-2)-u}(2u)du \\ &= \int \frac{2u}{(u-2)(u+1)} du \\ &= \int \frac{2u}{(u-2)(u+1)} du\end{aligned}$$

## Rationalizing Substitution - continued

$\frac{2u}{(u-2)(u+1)} = \frac{A}{u-2} + \frac{B}{u+1}$

$2u = A(u+1) + B(u-2)$

Substitute $u = -1 \to -2 = -3B \to B = \frac{2}{3}$

Substitute $u = 2 \to 4 = 3A \to A = \frac{4}{3}$

$$\begin{aligned}\int \frac{1}{x-\sqrt{x+2}}dx &= \int \left(\frac{4}{3}\left(\frac{1}{u-2}\right) + \frac{2}{3}\left(\frac{1}{u+1}\right)\right)du \\ &= \frac{4}{3}\ln|u-2| + \frac{2}{3}\ln|u+1| + C \\ &= \boxed{\frac{4}{3}\ln|\sqrt{x+2}-2| + \frac{2}{3}\ln(\sqrt{x+2}+1) + C}\end{aligned}$$



## 7.5 Strategy for Integration

# First Step

What should the first step be?

A) Integrate

B) U-substitution

C) Trig Substitution

D) Divide

E) Decompose into partial fractions



1) $\int \frac{1}{\sqrt{x^2+4}} dx$

2) $\int \frac{1}{\sqrt{x^2-4}} dx$

3) $\int \frac{1}{x^2+4} dx$

4) $\int \frac{1}{x^2-4} dx$

5) $\int \frac{1}{\sqrt{4-x^2}} dx$

6) $\int \frac{x}{\sqrt{x^2+4}} dx$

7) $\int \frac{x}{\sqrt{x^2-4}} dx$

8) $\int \frac{x}{x^2+4} dx$

9) $\int \frac{x}{x^2-4} dx$

10) $\int \frac{x}{\sqrt{4-x^2}} dx$



11) $\int \frac{x^2}{\sqrt{x^2+4}} dx$

12) $\int \frac{x^2}{\sqrt{x^2-4}} dx$

13) $\int \frac{x^2}{x^2+4} dx$

14) $\int \frac{x^2}{x^2-4} dx$

15) $\int \frac{x^2}{\sqrt{4-x^2}} dx$

16) $\int \frac{1}{\sqrt{x+4}} dx$

17) $\int \frac{1}{\sqrt{x-4}} dx$

18) $\int \frac{1}{x+4} dx$

19) $\int \frac{1}{x-4} dx$

20) $\int \frac{1}{\sqrt{4-x}} dx$



21) $\int \frac{x}{\sqrt{x+4}} dx$

22) $\int \frac{x}{\sqrt{x-4}} dx$

23) $\int \frac{x}{x+4} dx$

24) $\int \frac{x}{x-4} dx$

25) $\int \frac{x}{\sqrt{4-x}} dx$

## 7.7 Approximate Integration



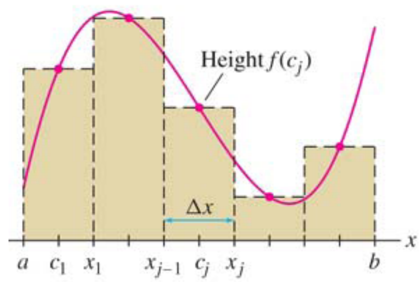
## Why Approximate?

If an anti-derivative cannot be found in terms of elemental functions, then approximation techniques can be used to approximate definite integrals.

E.g. Integrals of tabular functions, or integrals like:

$$\int_a^b \frac{\sin x}{x} dx \qquad \text{OR} \qquad \int_a^b e^{x^2} dx$$

## Midpoint Rule




$M_N$ is the sum of the areas of the midpoint rectangles.

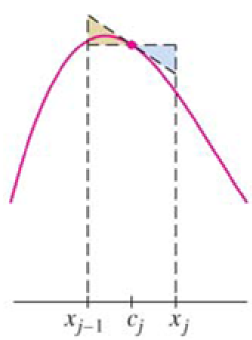
$c_i = \frac{x_{i-1}+x_i}{2}$

$$M_N = \sum_{i=1}^{N} f(c_i)\Delta x$$

$$M_N = \Delta x[f(c_1) + f(c_2) + f(c_3) + \cdots + f(c_N)]$$

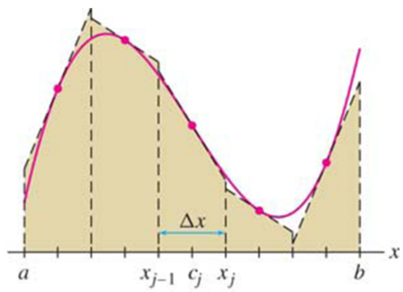


## Estimation Error - Midpoint Rule




The rectangle and the tangential trapezoid have the same area.

## Estimation Error - Midpoint Rule

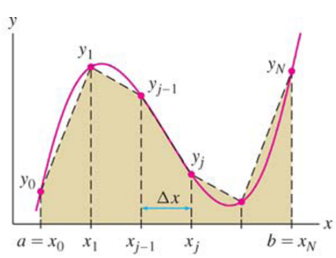



$M_N$ is the sum of the areas of the tangential trapezoids.

Notice that $M_N$ underestimates when the curve is concave up and overestimates when the curve is concave down.



## Trapezoid Rule




$T_N$ approximates the area under the graph by trapezoids.

$A_1 = \frac{1}{2}\Delta x(f(x_0) + f(x_1))$ $\quad A_2 = \frac{1}{2}\Delta x(f(x_1) + f(x_2))$

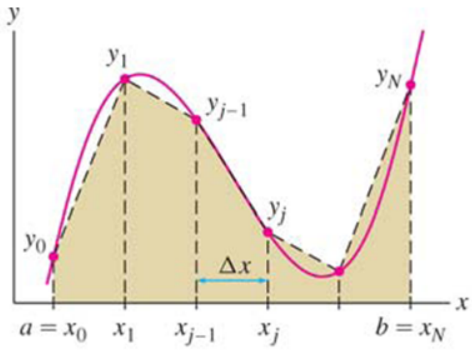
$A_3 = \frac{1}{2}\Delta x(f(x_2) + f(x_3))$ $\quad A_4 = \frac{1}{2}\Delta x(f(x_3) + f(x_4))$

$A_5 = \frac{1}{2}\Delta x(f(x_4) + f(x_5))$

$$A = \frac{\Delta x}{2}((f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + f(x_5))$$

$$T_N = \frac{\Delta x}{2}((f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{N-1}) + f(x_N))$$

# Estimation Error - Trapezoid Rule




$T_N$ overestimates when the graph of $y = f(x)$ is concave up and underestimates when the graph of $y = f(x)$ is concave down.



Which is more accurate, $M_N$ or $T_N$?

## Error Bounds

Error = Actual - Approximation

$$E_M = \int_a^b f(x)dx - M_N \qquad E_T = \int_a^b f(x)dx - T_N$$

$|E_M| \leq \frac{K_2(b-a)^3}{24N^2}$ $\qquad |E_T| \leq \frac{K_2(b-a)^3}{12N^2}$

where$|f''(x)| \leq K_2$ for $a \leq x \leq b$

Which rule is more accurate?



## Example

Consider: $\displaystyle\int_1^3 \frac{1}{x^2} dx$.

How large must $N$ be to guarantee that $T_N$ is accurate to within 0.0001? Repeat for $M_N$.

Solution:

First find $K_2$ ...

$f(x) = \frac{1}{x^2}$

$f'(x) = \frac{-2}{x^3}$

$f''(x) = \frac{6}{x^4}$

$K_2 = 6$ (because 6 is the upper bound of $f''(x)$)

## Example - continued

Using Trapezoid Rule:

$|E_T| \leq \frac{K_2(b-a)^3}{12N^2} \leq 0.0001$

$N^2 \geq \frac{K_2(b-a)^3}{12(0.0001)}$

$N^2 \geq \frac{6(2)^3}{12(0.0001)}$

$N^2 \geq 40000$

$N \geq 200$

Using Midpoint Rule:

$|E_M| \leq \frac{K_2(b-a)^3}{24N^2} \leq 0.0001$

$N^2 \geq \frac{K_2(b-a)^3}{24(0.0001)}$

$N^2 \geq \frac{6(2)^3}{24(0.0001)}$

$N^2 \geq 20000$

$N \geq 141.2$

$N \geq 142$



## Quiz

When estimating $\int_0^1 (1 - x^2)dx$, which of the following is correct? ($R_{10}, L_{10}, M_{10}$, and $T_{10}$ are, respectively, the right-hand rule, left-hand rule, midpoint rule, and trapezoid rule approximations, each with 10 subintervals.)

A) $R_{10} < L_{10} < M_{10} < T_{10} <$ Actual

B) $R_{10} < M_{10} <$ Actual $< T_{10} < L_{10}$

C) $R_{10} < T_{10} <$ Actual $< M_{10} < L_{10}$

D) $L_{10} < T_{10} <$ Actual $< M_{10} < R_{10}$

## Simpson's Rule - Introduction

Can we do better?

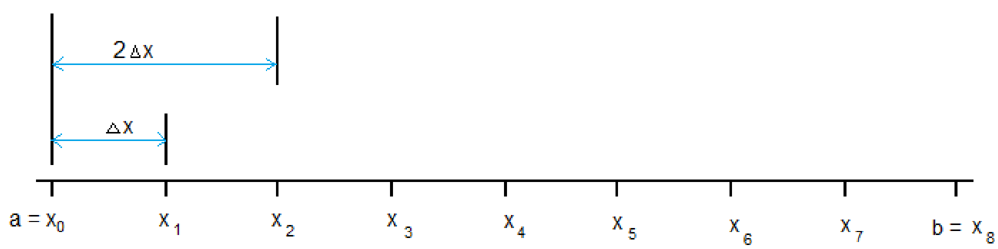
Yes. Use Simpson's Rule.

Simpson's Rule uses a weighted average of the midpoint rule approximation (2/3) and the trapezoid rule approximation (1/3).

The interval of integration must be divided into an even number of subintervals. Thus, to compute $S_N$, $N$ must be even.



## Simpson's Rule




The even-numbered endpoints divide $[a, b]$ into $\frac{N}{2}$ subintervals of length $2\Delta x$. These endpoints are used to compute $T_{\frac{N}{2}}$.

The midpoints of the subintervals, which are the odd-numbered points $x_n$, are used to compute $M_{\frac{N}{2}}$.

## Simpson's Rule - Formula

$$T_{\frac{N}{2}} = \tfrac{1}{2}(2\Delta x)(y_0 + 2y_2 + 2y_4 + \cdots + 2y_{N-2} + y_N)$$

$$M_{\frac{N}{2}} = 2\Delta x(y_1 + y_3 + y_5 + \cdots + y_{N-1})$$

$$S_N = \tfrac{1}{3}T_{\frac{N}{2}} + \tfrac{2}{3}M_{\frac{N}{2}}$$

$$= \tfrac{1}{3}\Delta x(y_0 + 2y_2 + 2y_4 + \cdots + 2y_{N-2} + y_N) + \tfrac{1}{3}\Delta x(4y_1 + 4y_3 + 4y_5 + \cdots + 4y_{N-1})$$

$$S_N = \tfrac{1}{3}\Delta x(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{N-2} + 4y_{N-1} + y_N)$$

where $\Delta x = \frac{b-a}{N}$



## Example

For $\displaystyle\int_1^3 \frac{1}{x^2}\,dx$, we wish to compute $S_{10}$.

Solution:

$$\Delta x = \tfrac{b-a}{N} = \tfrac{3-1}{10} = 0.2$$

$$S_{10} = \tfrac{0.2}{3}\left(\tfrac{1}{1.0^2} + \tfrac{4}{1.2^2} + \tfrac{2}{1.4^2} + \tfrac{4}{1.6^2} + \tfrac{2}{1.8^2} + \tfrac{4}{2.0^2} + \tfrac{2}{2.2^2} + \tfrac{4}{2.4^2} + \tfrac{2}{2.6^2} + \tfrac{4}{2.8^2} + \tfrac{1}{3.0^2}\right)$$

$$S_{10} = \boxed{0.66685}$$

$$\int_1^3 \frac{1}{x^2}\,dx = 0.6666\overline{6} \qquad \text{(Actual)}$$

So, $E_S = 0.6666\overline{6} - 0.66685 = -0.00018$.

## Error Bound - Simpson's Rule

$|E_S| \leq \frac{K_4(b-a)^5}{180N^4}$

where $|f^{(4)}(x)| \leq K_4$ for $a \leq x \leq b$.



## Example

What does the error bound formula guarantee when estimating $\displaystyle\int_1^3 \frac{1}{x^2}\,dx$ with $S_{10}$?

Solution:

Find $K_4$...

$f''(x) = 6x^{-4}$

$f'''(x) = -24x^{-5}$

$f^{(4)}(x) = 120x^{-6}$

$K_4 = 120$

$|E_S| \leq \frac{120(2)^5}{180(10)^4} = \boxed{0.0021\overline{3}}$

Recall that $E_S = 0.6666\overline{6} - 0.66685 = -0.00018$

So $|E_S| = 0.00018 \leq 0.0021\overline{3}$

# Compare

$$\int_1^3 \frac{1}{x^2}\,dx = \frac{2}{3} = 0.6666\overline{6}$$

$S_{10} = 0.66685$

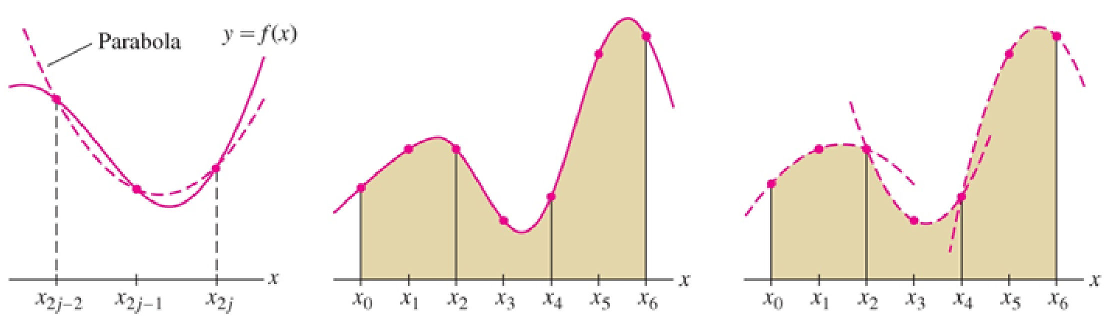
$M_{10} = 0.66350$

$T_{10} = 0.67303$



# Another View

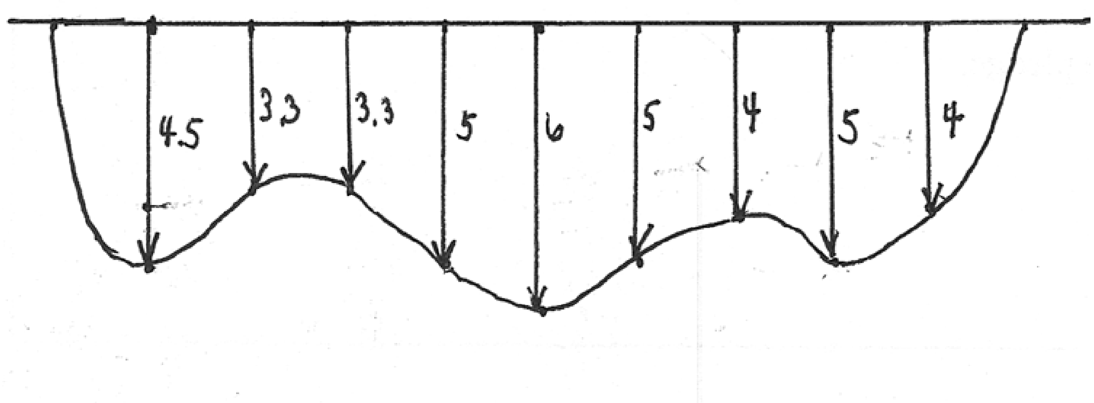
Simpson's Rule can be interpreted as approximating an integral as the sum of the areas under the unique parabolas that pass through the points, $x_{2j-2}, x_{2j-1}, x_{2j}$.




(A) Unique parabola through three points. (B) Graph of $y = f(x)$ (C) Parabolic arcs used in Simpson's Rule.

## Try It

The width of a garden bed is measured every 2 feet as shown. How much mulch (in cubic yards) should I buy to cover this garden bed with a 6-inch layer of mulch?






## Try It

The speed of a vehicle is measured every 2 seconds as shown. Estimate the distance traveled by the vehicle in 12 seconds.

| Time (s) | Speed (m/s) |
|---|---|
| 0 | 0 |
| 2 | 8 |
| 4 | 15 |
| 6 | 21 |
| 8 | 23 |
| 10 | 16 |
| 12 | 13 |

# Try It

Suppose that the region shown is rotated about the $y-$ axis to form a solid. Use Simpson's Rule with $N = 6$ to estimate the volume of the solid.