Section 7.8: Improper Integrals

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7.8 Improper Integrals

Two Types

Infinite Interval

Ex.

$$\int_1^{\infty} \frac{1}{x^2}\,dx$$

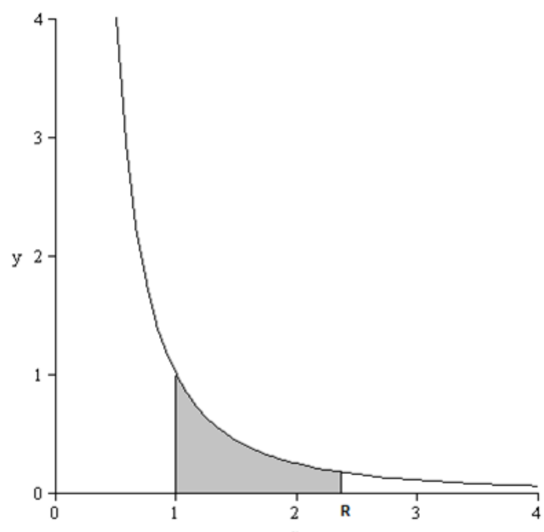
Discontinuous Integrands

Ex.

$$\int_0^{2} \frac{1}{x}\,dx$$

Example

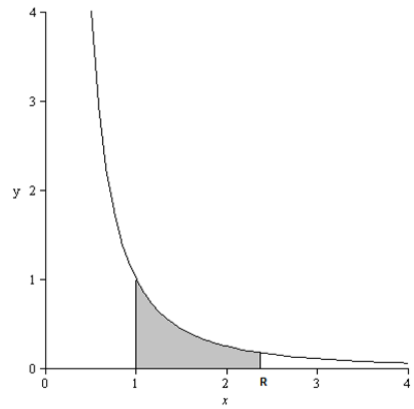
$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{R\to\infty} \int_1^{R} \frac{1}{x^2}\,dx$$

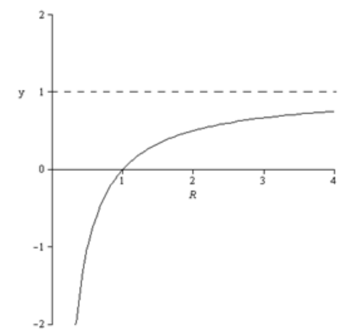


Example Cont

$$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{R\to\infty} \int_1^{R} \frac{1}{x^2} dx$$
$$= \lim_{R\to\infty} \left(-\frac{1}{x}\right)\Big|_1^R$$
$$= \lim_{R\to\infty} \left(-\frac{1}{R}+1\right)$$
$$= 1$$

So
$$\int_1^{\infty} \frac{1}{x^2} dx = 1$$
and $\int_1^{\infty} \frac{1}{x^2} dx$ is convergent.





Definition

If $\int_a^R f(x)dx$ exists for every $R \geq a$, then
$$\int_a^{\infty} f(x)dx = \lim_{R\to\infty} \int_a^R f(x)dx, \text{ if this limit exists.}$$

If $\int_R^b f(x)dx$ exists for every $R \leq b$, then
$$\int_{-\infty}^{b} f(x)dx = \lim_{R\to-\infty} \int_R^b f(x)dx, \text{ if this limit exists.}$$

In both cases, the integral is convergent if a finite limit exists, and the integral is divergent otherwise. Thus, an improper integral may diverge to ∞ or to $-\infty$, or it may diverge with no limit.

Example

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{R\to\infty} \int_1^R \frac{1}{x}\,dx$$

$$= \lim_{R\to\infty} \ln|x| \Big|_1^R$$

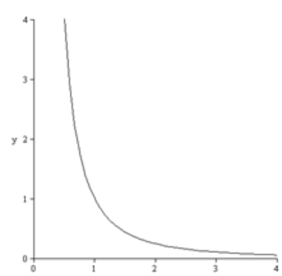
$$= \lim_{R\to\infty} (\ln|R| - \ln 1)$$

$$= \infty$$

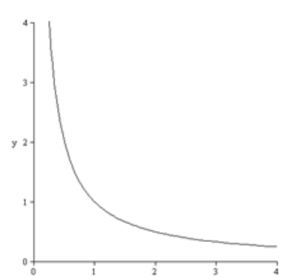
Integral is divergent. (Diverges to ∞.)

Recap

$\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is convergent.



$\displaystyle\int_1^\infty \frac{1}{x}\,dx$ is divergent.



For what values of p is $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ convergent?

p

$$\int_1^\infty \frac{1}{x^p} dx = \lim_{R\to\infty} \int_1^R x^{-p} dx$$
$$= \lim_{R\to\infty} \left. \frac{x^{-p+1}}{-p+1} \right|_1^R \text{ if } p \neq 1$$
$$= \lim_{R\to\infty} \frac{R^{1-p} - 1}{1-p}$$
$$= \begin{cases} \infty & p < 1 \\ \frac{-1}{1-p} & p > 1 \end{cases}$$

$\int_1^\infty \frac{1}{x^p} dx$ is convergent for $p > 1$ and divergent for $p \leq 1$.

Try

$$\int_{-\infty}^0 \frac{1}{x-1} dx$$

Another Kind of Infinite Integral

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx$$

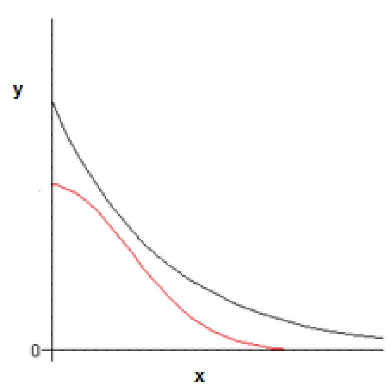
The equality holds if both integrals on the right are convergent. The integral on the left is divergent if either integral on the right is divergent.

Comparison Theorem

For infinite integrals, the comparison theorem applies:
Suppose f and g are continuous functions and $f(x) \geq g(x) \geq 0$ for $x \geq a$,

If $\int_{a}^{\infty} f(x)dx$ is convergent, then $\int_{a}^{\infty} g(x)dx$ is convergent.

If $\int_{a}^{\infty} g(x)dx$ is divergent, then $\int_{a}^{\infty} f(x)dx$ is divergent.



$y = f(x)$
$y = g(x)$

Example

$\int_1^\infty \frac{1}{x^2+x+1} dx$ is convergent because

$0 \leq \frac{1}{x^2+x+1} \leq \frac{1}{x^2}$ and $\int_1^\infty \frac{1}{x^2} dx$ is convergent.

$\int_2^\infty \frac{x}{x^2-1} dx$ is divergent because

$\frac{x}{x^2-1} \geq \frac{x}{x^2} = \frac{1}{x} > 0$ and $\int_2^\infty \frac{1}{x} dx$ is divergent.

Quiz

$$\int_1^\infty \frac{sin^2 x}{x^2} dx$$

A) is convergent.

B) diverges to ∞.

C) diverges to $-\infty$.

D) diverges with no limit.

Discontinuous Integrand

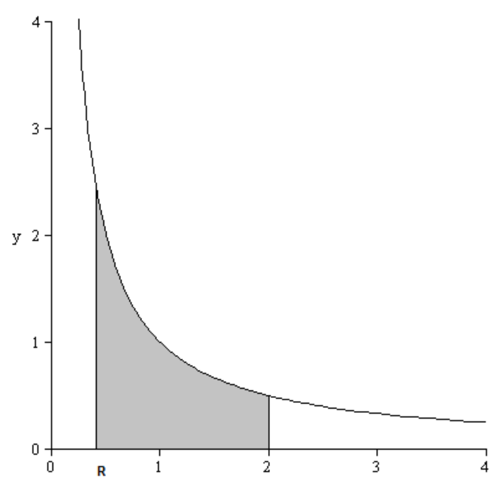
If $f(x)$ is continuous on $[a, b)$ and discontinuous at b,
then $\int_a^b f(x)dx = \lim_{R \to b^-} \int_a^R f(x)dx$

If $f(x)$ is continuous on $(a, b]$ and discontinuous at a,
then $\int_a^b f(x)dx = \lim_{R \to a^+} \int_R^b f(x)dx$

Example

$$\begin{aligned}\int_0^2 \frac{1}{x}dx &= \lim_{R \to 0^+} \int_R^2 \frac{1}{x}dx \\ &= \lim_{R \to 0^+} \ln|x| \Big|_R^2 \\ &= \lim_{R \to 0^+} (\ln 2 - \ln R) \\ &= \infty\end{aligned}$$

$\int_0^2 \frac{1}{x}dx$ diverges to ∞



Quiz

$$\int_0^1 \frac{1}{x^2} dx$$

A) is convergent.

B) diverges to ∞.

C) diverges to $-\infty$.

D) diverges with no limit.

Note

$\displaystyle\int_a^\infty \frac{1}{x^p} dx$ is convergent for $p > 1$ and divergent for $p \leq 1$.

$\displaystyle\int_0^a \frac{1}{x^p} dx$ is convergent for $p < 1$ and divergent for $p \geq 1$.

Inside Discontinuity

If $f(x)$ is discontinuous at c where $a < c < b$, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

The equality holds if both integrals on the right are convergent.
The integral on the left is divergent if either integral on the right is divergent.

Example

$$\int_{-2}^2 \frac{1}{x}dx = \int_{-2}^0 \frac{1}{x}dx + \int_0^2 \frac{1}{x}dx$$

$\uparrow$ Also divergent.

$\uparrow$ Already shown to be divergent.

So $\int_{-2}^2 \frac{1}{x}dx$ is divergent.

Example Cont.

If we ignore the discontinuity:

$$\int_{-2}^{2} \frac{1}{x}\,dx = \ln|x|\Big|_{-2}^{2}$$
$$= \ln 2 - \ln|-2|$$
$$= \ln 2 - \ln 2$$
$$= 0$$

Wrong!

Try

$$\int_0^{\pi} \sec x dx$$