Section 6.4: Work

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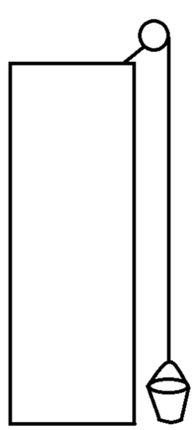
6.4 Work

Definition

Work = Force x Distance

Where force and distance are in the same direction.

Example Definition



What is the work required to lift a bucket of water to the top of a building that is 80 ft high?

The bucket weighs 4 lb and is filled with 50 lb of water.

Example Solution

$$W_{total} = W_{bucket} + W_{water}$$

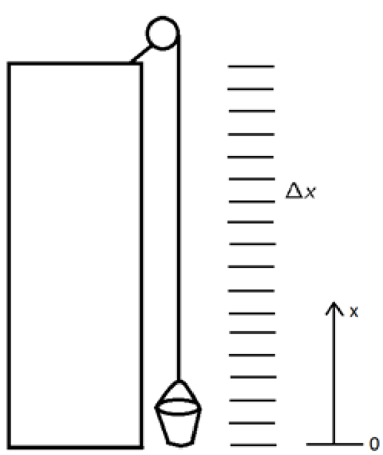
$$W_{bucket} = 4x80 = 320 \text{ ft-lb}$$
$$W_{water} = 50x80 = 4000 \text{ ft-lb}$$

$$W_{total} = 320 + 4000 = \boxed{4320 \text{ ft-lb}}$$

Heavy Chain

What if the chain used to lift the bucket is heavy enough to matter? Say 0.5lb/ft.

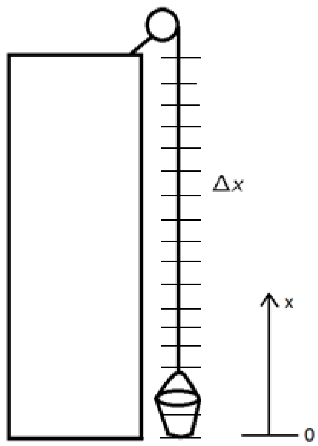
Divide distance into small equal lengths, Δx.



$$F_{chain,i} = 40 - 0.5x_i$$
$$d = \Delta x$$
$$W_{chain,i} = (40 - 0.5x_i)\Delta x$$
$$W_{chain} = \int_0^{80} (40 - 0.5x)dx$$
$$= 1600 \text{ ft-lb}$$

Alternative for Heavy Chain

Divide chain into equal lengths Δx.

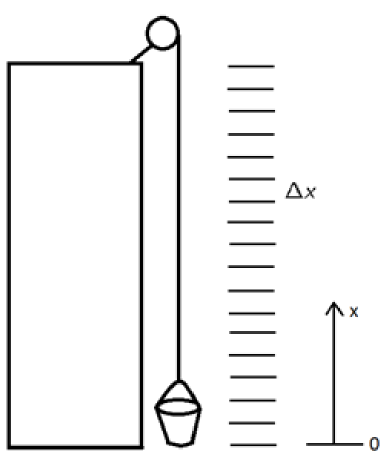


$$F_{chain,i} = 0.5\Delta x$$
$$d_i = 80 - x_i$$
$$W_{chain,i} = (80 - x_i)(0.5)\Delta x$$
$$W_{chain} = \int_0^{80} (80 - x)(0.5)dx$$
$$= 1600 \text{ ft-lb}$$

Leaky Bucket

What if the amount of water in the bucket is changing? Suppose it is leaking at 0.2lb/s as bucket is lifted at a constant rate of 2ft/s.

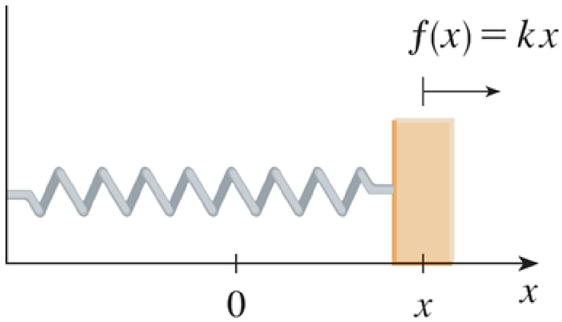
Divide distance into small equal lengths, Δx.



$$F_{water,i} = 50lb - \frac{0.2lb/sec}{2ft/sec}x_i$$
$$F_{water,i} = 50 - 0.1x_i$$
$$W_{water,i} = (50 - 0.1x_i)\Delta x$$
$$W_{water} = \int_0^{80} (50 - 0.1x)dx$$
$$= 3680 \text{ ft-lb}$$
$$W_{total} = W_{water} + W_{bucket} + W_{chain}$$
$$= 320 + 3680 + 1600$$
$$W_{total} = \boxed{5600 \text{ ft-lb}}$$

Spring - Hooke's Law

A force of 35N is required to hold a spring that has been stretched 7 cm past its natural length of 20cm. How much work is required to stretch the spring from 20cm to 30cm?



Hooke's Law: $F = kx$

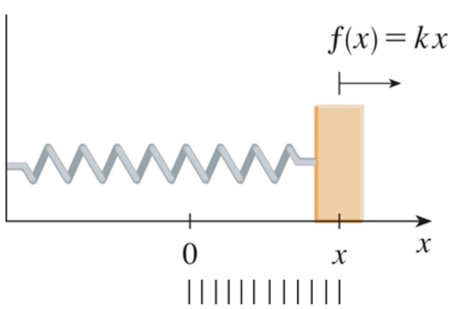
In this case:

$35 = k(0.7)$

$k = 500$ N/m

Example Solution

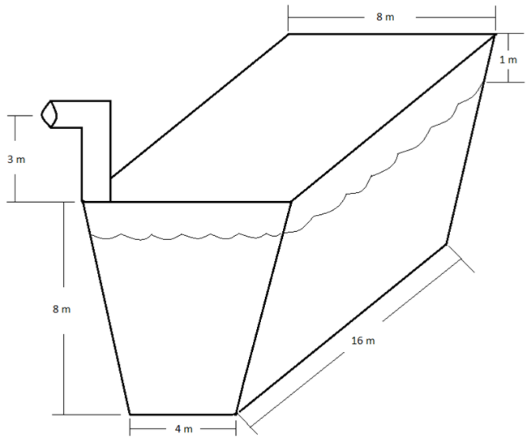
Divide distance into equal sublengths, such that the force is approximately the same as the spring is stretched through one sublength.



$$F_i = kx_i$$
$$d_i = \Delta x$$
$$W_i = kx_i \Delta x$$
$$W = \int_0^{0.1} kx\,dx$$
$$= \int_0^{0.1} 500x\,dx$$
$$= \boxed{2.5 \text{ Nm}}$$

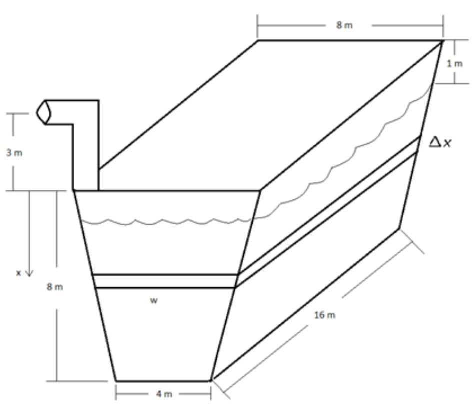
Trapezoidal Tank

How much work is required to pump the water out of the tank shown?



Divide water into thin "slices" of equal thickness so that the distance traveled by all of the water within each slice is the same.

Example Solution

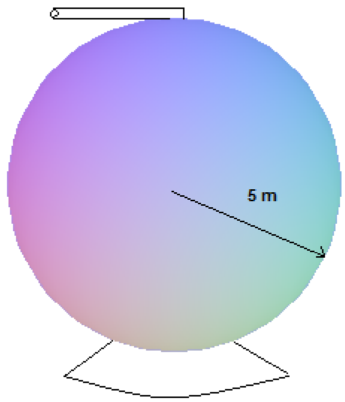


(x, w)
$(0, 8)$
$(8, 4)$
$w - 8 = \frac{4-8}{8-0}(x - 0)$
$w = 8 - \frac{4}{8}x$

$$F_i = \rho g V_i$$
$$V_i = 16 w_i \Delta x$$
$$w_i = 8 - 0.5x_i$$
$$F_i = \rho g(16)(8 - 0.5x_i)\Delta x$$
$$d_i = x_i + 3$$
$$W_i = \rho g(16)(8 - 0.5x_i)(x_i + 3)\Delta x$$
$$W = \int_1^8 \rho g(16)(8 - 0.5x)(x + 3)dx$$
$$= \boxed{4.5 \times 10^7 \text{ Nm}}$$

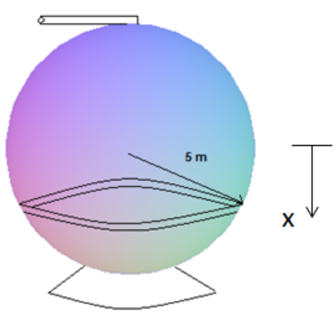
Spherical Tank

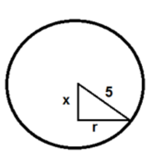
The spherical tank (radius = 5m) is full of water. Find the work required to pump the water out of the outlet.



Example Solution

Slices are disks.





$x^2 + r^2 = 25$
$r^2 = 25 - x^2$

$$
\begin{aligned}
F_i &= \rho g V_i \\
&= \rho g \pi {r_i}^2 \Delta x \\
&= \rho g \pi (25 - x_i^2) \Delta x \\
d_i &= x_i + 5 \\
W_i &= F_i d_i \\
&= \rho g \pi (25 - {x_i}^2)(x_i + 5) \Delta x \\
W &= \int_{-5}^{5} \rho g \pi (25 - x^2)(x + 5) dx \\
&= \boxed{2.6 \times 10^7 \text{ Nm}}
\end{aligned}
$$

What if?

What if the spigot is 2 m above the tank?

$$W = \int_{-5}^{5} \rho g \pi (25 - x^2)(x+7) dx$$

What if the tank is filled to a depth of only 3 m and the spigot is 2 m above the tank?

$$W = \int_{2}^{5} \rho g \pi (25 - x^2)(x+7) dx$$

Try It

What is the work required to build a solid structure with the shape shown below out of cement with a density of $1200 kg/m^3$? The front and back vertical faces of the structure are bounded by a parabola with the equation $y = x^2$ for $-2 \le x \le 2$ and the line $y = 4$.