Chapter 11 Infinite Sequences and Series

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11.1 Sequences

Definition

A sequence is a list in a particular order.

1,2,3,4,5

5,3,1,2,4

The above two sequences are different.

Infinite Sequence

We will be concerned with infinite sequences of numbers, in which there is a pattern to the terms.

Examples:

$1, 2, 3, 4, 5, ...$

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...$

Sequence Descriptions

Here are multiple ways to describe the same infinite sequence:

$$\{1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, ...\}$$

$$\left\{\frac{1}{n}\right\}$$

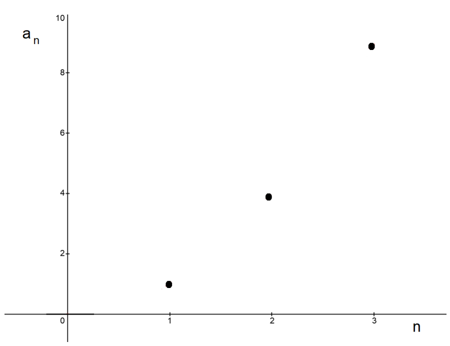
$$\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$$

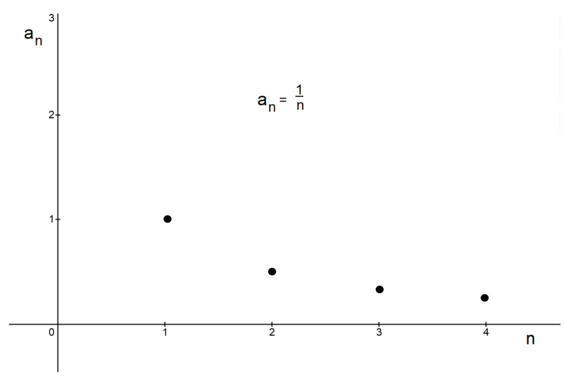
$$\{a_n\}, a_n = \frac{1}{n}$$

Graphing Sequences

Graphing a sequence involves plotting discrete points, rather than plotting curves.

Example: plot $\{n^2\}$ and $\left\{\frac{1}{n}\right\}$





Convergence or Divergence

If $\lim_{n\to\infty} a_n = L$, and L is finite,

then $\{a_n\}$ converges to L;

otherwise, the sequence diverges.

If $\lim_{n\to\infty} a_n = \infty$ (or $-\infty$),

then $\{a_n\}$ diverges to ∞ (or $-\infty$).

Examples

$\left\{\frac{1}{n}\right\}$ converges to 0

$\{n^2\}$ diverges to ∞

$\{(-1)^n\} = \{-1, 1, -1, 1, \ldots\}$ diverges (no corresponding function)

Theorem

If $a_n = f(n)$, and $\lim_{x\to\infty} f(x) = L$, where L is finite, then $\{a_n\}$ converges to L.

If L is ∞ (or $-\infty$), then the sequence diverges to ∞ (or $-\infty$).

Example

Is the sequence, $\left\{\dfrac{\ln n}{n}\right\}$, convergent or divergent?

Solution:

$f(x) = \dfrac{\ln x}{x}$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0$$

$\left\{\dfrac{\ln n}{n}\right\}$ is convergent

Note that L'Hospital's Rule cannot be applied to a sequence, but can be applied to the corresponding continuous function.

Another Example

Is the sequence, $\{\sin(2\pi n)\}$, convergent or divergent?

Solution:

$\lim_{x\to\infty} \sin(2\pi x)$ does not exist,

but

$\lim_{n\to\infty} \sin(2\pi n) = 0$, so the sequence is convergent.

Theorem 2

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$

Example 1

Does the sequence, $\{(-1)^n \frac{1}{n}\}$, converge or diverge?

Solution:

$|(-1)^n \frac{1}{n}| = \frac{1}{n}$

$\lim_{n\to\infty} \frac{1}{n} = 0$, so $\lim_{n\to\infty} (-1)^n \frac{1}{n} = 0$

$\{(-1)^n \frac{1}{n}\}$ converges to 0.

Example 2

For what values of r does $\{r^n\}$ converge?

Solution:

$$\lim_{x\to\infty} r^x = \begin{cases} 0, & \text{for } 0 < r < 1 \\ 1, & \text{for } r = 1 \\ \infty, & \text{for } r > 1 \end{cases}$$

so

$$\lim_{n\to\infty} r^n = \begin{cases} 0, & \text{for } 0 < r < 1 \\ 1, & \text{for } r = 1 \\ \infty, & \text{for } r > 1 \end{cases}$$

Note that r^x is defined only for $r > 0$. What about $r \leq 0$?

Example 2 Cont

$$\lim_{n\to\infty} r^n = 0 \text{ if } |r| < 1$$

$$\lim_{n\to\infty} r^n \text{ does not exist if } r \leq -1$$

For example:

$$\left\{\left(-\frac{1}{2}\right)^n\right\} = \left\{-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \cdots\right\} \text{ convergent}$$

$$\{(-1)^n\} = \{-1, 1, -1, 1, \cdots\} \text{ divergent}$$

$$\{(-2)^n\} = \{-2, 4, -8, 16, \cdots\} \text{ divergent}$$

Summary for $\{r^n\}$

$\{r^n\}$ converges to 0 for $-1 < r < 1$

$\{r^n\}$ converges to 1 for $r = 1$

$\{r^n\}$ diverges otherwise

Try It

Show whether the following sequences converge or diverge.

$\{-3(\frac{2^{n+1}}{5^n})\}$

$\{-3(\frac{2^{n+1}}{5})\}$

$\{-3(\frac{2}{5^n})\}$

Try It

For what value of p does $\{\frac{1}{n^p}\}$ converge?

Try It

Show whether the following sequences converge or diverge.

$\{(-1)^n \frac{1}{\sqrt{n}}\}$

$\{\frac{n}{n+1}\}$

$\{(-1)^n \sqrt{n}\}$

$\{(-1)^n \frac{n}{n+1}\}$

Try It

Show whether the following sequences converge or diverge.

$$\left\{\frac{n\ln(e^4)}{3^n}\right\}$$

$$\{(-1)^{2n+1}\}$$

$$\left\{\frac{(-1)^n + n}{(-1)^n - n}\right\}$$

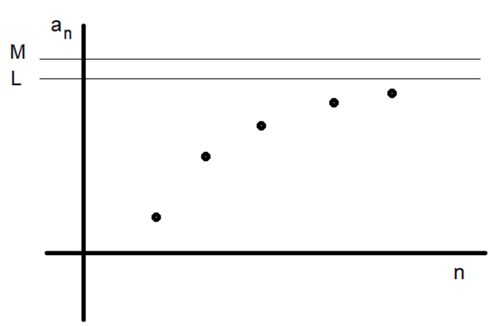
Monotonic Theorem

A monotonic bounded sequence converges.

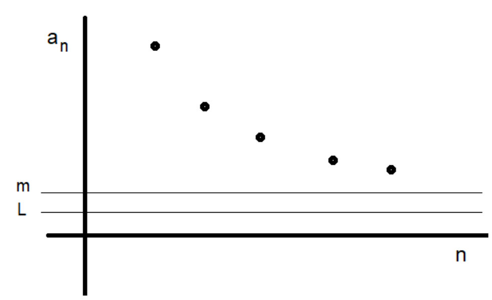
monotonically increasing: $a_{n+1} \geq a_n$ for all n

monotonically decreasing: $a_{n+1} \leq a_n$ for all n

Explanation



Monotonically increasing sequence is always bounded below.



Monotonically decreasing sequence is always bounded above.

Example

Does the sequence, $0.2, 0.22, 0.222, 0.2222, \ldots$ converge?

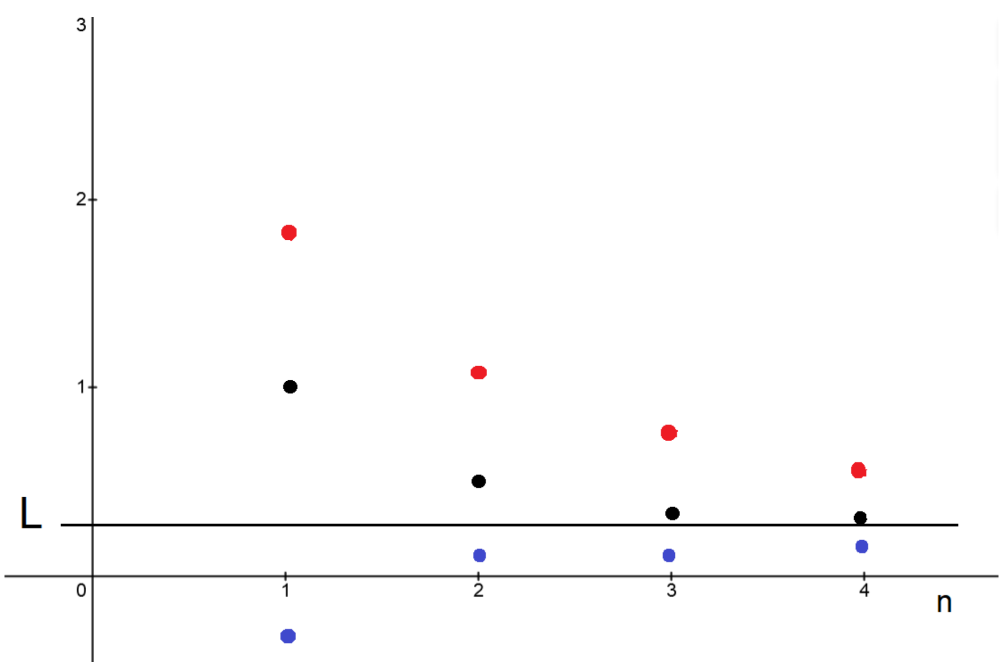
Solution:

Sequence is monotonically increasing. It is bounded by 0.2 (below) and 0.3 (above). The sequence is convergent.

Squeeze Theorem

If $a_n \leq b_n \leq c_n$ for all $n \geq n_0$, and

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.



Review

Factorial

$0! = 1$

$n! = n(n-1)!$

$5! = 5 \times 4 \times 3 \times 2 \times 1$

$67! = 67 \times 66!$

$(n+2)! = (n+2)(n+1)n!$

Example

Does the sequence, $\{\frac{4^n}{n!}\}$, converge?

Solution:

n^{th} term:

$$\frac{4 \times 4 \times 4 \times 4 \times 4 \times 4 \cdot \times 4 \times 4}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times \cdot \times (n-1) \times n} \leq \frac{4 \times 4 \times 4 \times 4}{1 \times 2 \times 3 \times 4}\left(\frac{4}{n}\right)$$

for $n \geq 5$

$0 \leq \frac{4^n}{n!} \leq \frac{1024}{24n}$ for $n \geq 5$

AND

$\lim_{n\to\infty} \frac{1024}{24n} = 0$

So $\frac{4^n}{n!} = 0$ by the squeeze theorem

11.2 Series

Definition

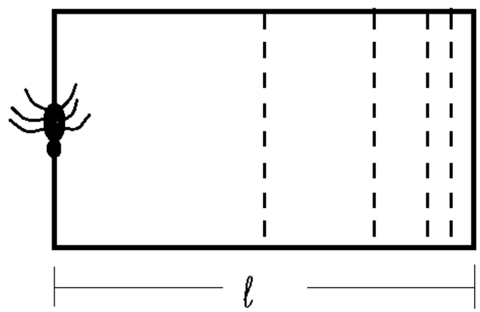
A series is the sum of the terms of an infinite sequence.

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = \sum a_n$$

A series can have a finite sum.

Example

Consider: A bug crosses a room by jumping half ($\frac{1}{2}$) of the remaining distance with each jump.

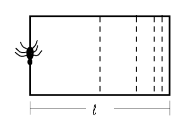


Example Cont.

Distance covered:

$d = \frac{1}{2}\ell + \frac{1}{2}(\frac{1}{2})\ell + \frac{1}{2}(\frac{1}{2})(\frac{1}{2})\ell + \cdots + (\frac{1}{2})^n\ell + \cdots = ?$

$d = \sum_{n=1}^{\infty} (\frac{1}{2})^n \ell = \ell$ (intuitively)



So a series, an infinite sum, can be finite.

Partial Sum

Definition: $s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$

$s_1 = a_1$

$s_2 = a_1 + a_2$

$s_3 = a_1 + a_2 + a_3$

Partial sums form a sequence, $\{s_n\}$.

Convergence of Series

If $\{s_n\}$ is convergent, then

$\lim_{n\to\infty} s_n = s$ is a real number,

series $\sum a_n$ is convergent, and

$\sum_{n=1}^{\infty} a_n = s$ (sum of series).

Otherwise, the series is divergent.

(Thus, series have two associated sequences. These are the sequence of terms, $\{a_n\}$, and the sequence of partial sums, $\{s_n\}$.)

Geometric Series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots$$

Geometric series are distinguished by having a common ratio, r, of subsequent terms.

Example:

$$\sum_{n=1}^{\infty} 3(\tfrac{1}{4})^{n-1} = 3 + 3(\tfrac{1}{4}) + 3(\tfrac{1}{4})^2 + 3(\tfrac{1}{4})^3 + \cdots + 3(\tfrac{1}{4})^{n-1} + \cdots$$

Convergence of Geometric Series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots$$

For what values of r is the series convergent?

$$s_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$

If $r = 1$, then $s_n = na$. Clearly, $\lim_{n\to\infty} s_n = \pm\infty$, and series is divergent.

Now check other values of r...

Convergence of Geometric Series Cont.

$s_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$ Equation 1

$rs_n = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n$ Equation 2

$s_n - rs_n = a - ar^n$ Equation 1− Equation 2

$s_n(1 - r) = a(1 - r^n)$

$s_n = \frac{a}{1-r}(1 - r^n)$ if $r \neq 1$

Convergence of Geometric Series Cont.

$s_n = \frac{a}{1-r}(1 - r^n)$ if $r \neq 1$

If $|r| < 1$ then $\lim\limits_{n\to\infty} s_n = \dfrac{a}{1-r}$ (Convergent)

If $r > 1$ then $\lim\limits_{n\to\infty} s_n = \pm\infty$ (Divergent)

If $r \leq -1$ then $\lim\limits_{n\to\infty} s_n$ does not exist. (Divergent)

Summary

The geometric series,

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots$$

is convergent if $|r| < 1$.

Otherwise, the series is divergent.

If convergent, the sum is $\frac{a}{1-r}$.

Bug Example Conclusion

$$d = \sum_{n=1}^{\infty} \ell(\tfrac{1}{2})^n = \sum_{n=1}^{\infty} (\tfrac{1}{2}\ell)(\tfrac{1}{2})^{n-1}$$

$r = \frac{1}{2}$, $a = (\frac{1}{2})\ell$, r is the common ratio, and a is the first term.

$d = s = \frac{(\frac{1}{2})\ell}{1-\frac{1}{2}} = \ell$, as expected.

Try It

Determine whether the following series are convergent or divergent. Find the sum of convergent geometric series.

$$\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{3}\right)^{n-1} \qquad \sum_{n=0}^{\infty} \frac{1}{(\sqrt{2})^n} \qquad \sum (0.0001n)^2$$

$$\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}} \qquad \sum (-1)^n \left(\frac{4}{\pi}\right)^{n-1} \qquad \sum \frac{n}{n+2}$$

Example

$$\begin{aligned}
0.22\bar{2} &= 0.2 + 0.02 + 0.002 + 0.0002 + \cdots \\
&= \frac{2}{10} + \frac{2}{100} + \frac{2}{1000} + \frac{2}{10000} + \cdots \\
&= \frac{2}{10} + \frac{2}{10}\left(\frac{1}{10}\right) + \frac{2}{10}\left(\frac{1}{10}\right)^2 + \frac{2}{10}\left(\frac{1}{10}\right)^3 + \cdots \\
&= \frac{\frac{2}{10}}{1 - \frac{1}{10}} \\
&= \frac{2}{10-1} \\
0.22\bar{2} &= \frac{2}{9}
\end{aligned}$$

Theorem

If the series, $\sum_{n=1}^{\infty} a_n$, is convergent, then $\lim_{n\to\infty} a_n = 0$.

This means that, in order for a series to be convergent, the terms must have a limit of 0.

Example

$a_n = \frac{n}{n+1}$

$\lim_{n\to\infty} \frac{n}{n+1} = 1$

$\{a_n\}$ is convergent.

$\sum a_n$ is divergent.

Example

$a_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{1}{n} = 0$

$\{a_n\}$ is convergent.

$\sum a_n$??? Not necessarily convergent. Stay tuned for the answer.

Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$, then the series, $\sum_{n=1}^{\infty} a_n$, is divergent.

(This test is a consequence of the theorem.)

Limit Laws

If $\sum a_n$ and $\sum b_n$ are convergent, and $\sum a_n = a$ and $\sum b_n = b$, then $\sum ca_n$, $\sum(a_n + b_n)$, and $\sum(a_n - b_n)$ are convergent, and

i) $\sum ca_n = ca$

ii) $\sum(a_n + b_n) = a + b$

iii) $\sum(a_n - b_n) = a - b$

Summary

If $\lim_{n\to\infty} a_n = L$, and L is finite, then $\{a_n\}$ is convergent.

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ is divergent.

A geometric series is convergent if the magnitude of the common ratio is less than 1.

If a geometric series is convergent, the sum is $s = \frac{a}{1-r}$, where r is the common ratio and a is the first term.

11.3 The Integral Test and Estimates of Sums

Quiz

True (A) or False (B)?

1) If $\lim_{n\to\infty} a_n = 0$, then $\{a_n\}$ is convergent.

2) If $\lim_{n\to\infty} a_n = 2$, then $\{a_n\}$ is divergent.

3) If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ is convergent.

4) If $\lim_{n\to\infty} a_n = 2$, then $\sum a_n$ is divergent.

Series With Positive Terms

If the terms of a series are all positive, then $\{s_n\}$ is an increasing sequence. So it is either bounded, and therefore converges to a finite positive number, or it is unbounded, and therefore diverges to ∞.

Therefore, a series with only positive terms is either convergent, or diverges to ∞. This simplifies the study of positive series.

The Integral Test

Let $a_n = f(n)$, where $f(x)$ is a continuous, positive, and decreasing function on $[1, \infty)$.

i) If $\int_1^\infty f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

ii) If $\int_1^\infty f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

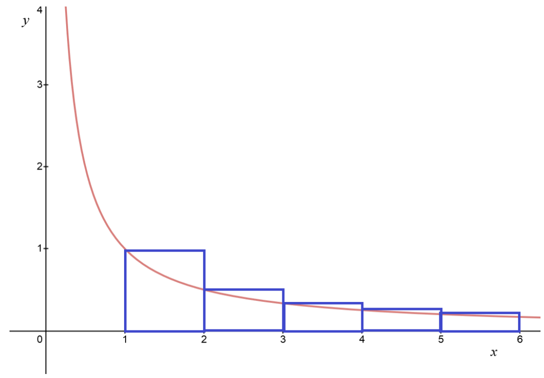
Graphical Explanation

$\sum_{n=1}^{\infty} a_n \geq \int_1^{\infty} f(x)dx$

If $\int_1^{\infty} f(x)dx$ diverges, then so does the series.

$\int_1^{\infty} \frac{1}{x} dx$ diverges,

so $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.



Graphical Explanation - 2

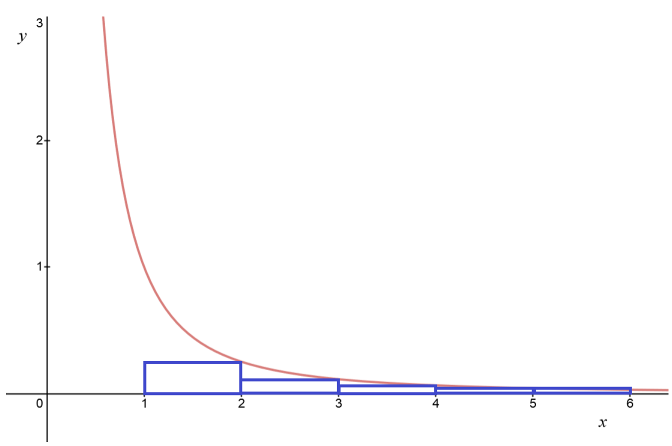
$\sum_{n=2}^{\infty} a_n \leq \int_1^{\infty} f(x)dx$

$\sum_{n=1}^{\infty} a_n \leq a_1 + \int_1^{\infty} f(x)dx$

If $\int_1^{\infty} f(x)dx$ converges, then so does the series.

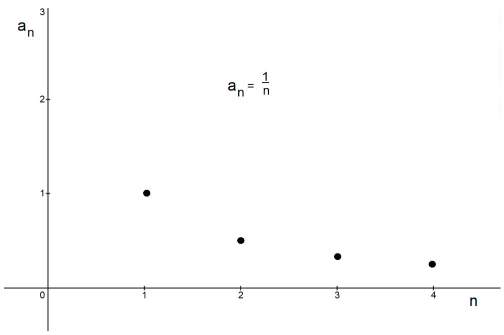
$\int_1^{\infty} \frac{1}{x^2} dx$ converges,

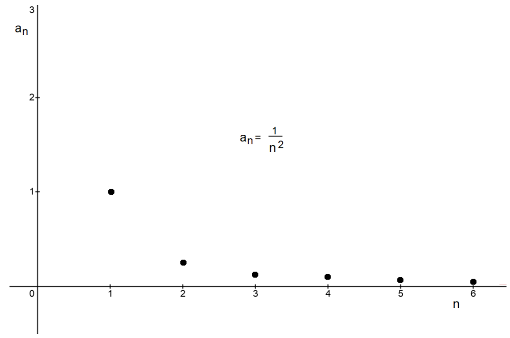
so $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.



What's the Difference?

$\frac{1}{n^2}$ decreases faster than $\frac{1}{n}$





Example

For what values of p does the series, $\sum_{n=1}^{\infty} \frac{1}{n^p}$, converge?

$f(x) = \frac{1}{x^p}$ is continuous, positive, and decreasing if $p > 0$;

so $\displaystyle\int_1^{\infty} \frac{1}{x^p} dx$ converges if $p > 1$; diverges otherwise.

Therefore,

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ by Integral Test;

diverges for $0 < p \leq 1$ by Integral Test;

diverges for $p \leq 0$ by the Divergence Test.

P-Series Test

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$; diverges otherwise.

Note

As with all series, convergence depends only on behavior of the "tail".

Examples

Determine if the following series converge or diverge.

$$\sum_{n=3}^{\infty} \frac{1}{n^4}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{-4}}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{4}}}$$

Try It

Show whether the following series converge or diverge. Pay careful attention to notation. Justify all steps.

$$\sum_{n=1}^{\infty} ne^{-n} \qquad \sum \frac{e^n}{n} \qquad \sum \frac{1}{3n}$$

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} \qquad \sum \frac{n^2}{n^2+5} \qquad \sum \frac{1}{n^2+5}$$

11.4 The Comparison Tests

Introduction

For $\sum_{n=2}^{\infty} \frac{1}{n-1}$, $\frac{1}{n-1}$ behaves like $\frac{1}{n}$ when n is large. Suspect divergence.

For $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$, $\frac{1}{2^n+1}$ behaves like $\frac{1}{2^n}$ when n is large. Suspect convergence.

Comparison Test

Suppose that there exists $M > 0$ such that $0 \leq a_n \leq b_n$ for $n \geq M$.

i) If $\sum b_n$ is convergent, then $\sum a_n$ is also convergent.

ii) If $\sum a_n$ is divergent, then $\sum b_n$ is also divergent.

Note:

1) Both series must have non-negative terms.

2) We only compare two series that converge or two series that diverge.

Example

For $\sum_{n=2}^{\infty} \frac{1}{n-1}$

$a_n = \frac{1}{n}$

$b_n = \frac{1}{n-1}$

$0 \leq a_n \leq b_n$ for all $n \geq 2$

$\sum a_n$ is a divergent p-series, $p = 1 \not> 1$

$\therefore \sum_{n=2}^{\infty} b_n$ is divergent by CT(Comparison Test)

Steps

1) Identify series to compare to.
2) Check criteria.
3) Execute test.
4) Show convergence results for comparison series.
5) State conclusion.

Repeat Example

$$\sum_{n=2}^{\infty} \frac{1}{n-1}$$

$a_n = \frac{1}{n}$; $b_n = \frac{1}{n-1}$ $\boxed{1}$

$0 \leq a_n \leq b_n$ for all $n \geq 2$ $\boxed{2}$ $\boxed{3}$

$\sum a_n$ is a divergent p-series, $p = 1 \not> 1$ $\boxed{4}$

$\therefore \sum_{n=2}^{\infty} b_n$ is divergent by CT(Comparison Test) $\boxed{5}$

Example 2

$$\sum_{n=2}^{\infty} \frac{1}{2^n+1}$$

$a_n = \frac{1}{2^n+1}$; $b_n = \frac{1}{2^n}$ $\boxed{1}$

$0 \le a_n \le b_n$ for all $n \ge 1$ $\boxed{2}$ $\boxed{3}$

$\sum b_n$ is a convergent geometric series, $|r| = \frac{1}{2} < 1$ $\boxed{4}$

$\therefore \sum_{n=2}^{\infty} a_n$ is convergent by CT(Comparison Test) $\boxed{5}$

Example 3

$$\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^3}$$

$0 \le \frac{\cos^2 n}{n^3} \le \frac{1}{n^3}$ for all n $\boxed{1}$ $\boxed{2}$ $\boxed{3}$

$\sum \frac{1}{n^3}$ is a convergent p-series, $p = 3 > 1$ $\boxed{4}$

$\therefore \sum_{n=1}^{\infty} \frac{\cos^2 n}{n^3}$ is convergent by CT(Comparison Test) $\boxed{5}$

Example 4

For $\sum_{n=3}^{\infty} \frac{1}{n^2-5}$, suspect convergence, but $\frac{1}{n^2-5} \not< \frac{1}{n^2}$.

What to do?

Limit Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Return to Example 4

$$\sum_{n=3}^{\infty} \frac{1}{n^2-5}$$

Compare $\frac{1}{n^2-5}$ to $\frac{1}{n^2}$ $\boxed{1}$

$\frac{1}{n^2-5} \geq 0$ and $\frac{1}{n^2} \geq 0$ $\boxed{2}$

$$\lim_{n\to\infty} \frac{\frac{1}{n^2-5}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2-5} = 1 \text{ (finite and not 0)} \quad \boxed{3}$$

$\sum \frac{1}{n^2}$ is a convergent p-series, $p = 2 > 1$ $\boxed{4}$

$\therefore \sum_{n=3}^{\infty} \frac{1}{n^2-5}$ is convergent by LCT (Limit Comparison Test) $\boxed{5}$

Example 5

$$\sum_{n=3}^{\infty} \frac{n^2+99n+3}{3n^3+n^2-n-1}$$

Compare $\frac{n^2+99n+3}{3n^3+n^2-n-1}$ to $\frac{n^2}{3n^3} = \frac{1}{3n}$ $\boxed{1}$

$\frac{n^2+99n+3}{3n^3+n^2-n-1} \geq 0$ and $\frac{1}{3n} \geq 0$ $\boxed{2}$

$$\lim_{n\to\infty} \frac{\frac{n^2+99n+3}{3n^3+n^2-n-1}}{\frac{1}{3n}} = \lim_{n\to\infty} \frac{3n^3+297n^2+9n}{3n^3+n^2-n-1} = 1 \text{ (finite and not 0)}$$

$\boxed{3}$

$\sum \frac{1}{3n}$ is a divergent p-series, $p = 1 \not> 1$ $\boxed{4}$

$\therefore \sum_{n=3}^{\infty} \frac{n^2+99n+3}{3n^3+n^2-n-1}$ is divergent by LCT (Limit Comparison Test)

$\boxed{5}$

Try It

$$\sum_{n=1}^{\infty} \frac{5+2n^3}{(1+n^2)^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{3^n-1}$$

$$\sum_{n=1}^{\infty} \frac{e^n}{n}$$

Try It

$$\sum_{n=1}^{\infty} \left(\frac{\sin n}{n}\right)^2$$

$$\sum_{n=1}^{\infty} \frac{n+8}{\sqrt{n^6+n^4+1}}$$

$$\sum_{n=1}^{\infty} -\left(\frac{n}{n^3+1}\right)$$

11.5 Alternating Series and Absolute Convergence

Recap

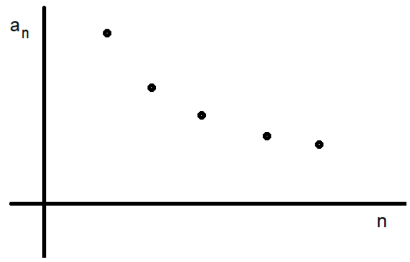
All of these tests require positive terms:

Integral Test

P-Series Test

Comparison Test

Limit Comparison Test



What If?

What if the terms are not positive?

Examples:

$\sum -(\frac{1}{n}) = -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots$

$\sum -(\frac{1}{n^2}) = -1 - \frac{1}{4} - \frac{1}{9} - \frac{1}{16} - \cdots$

What if the terms are alternating in sign?

Examples:

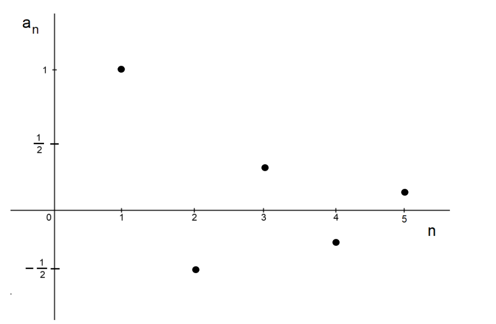
$\sum (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

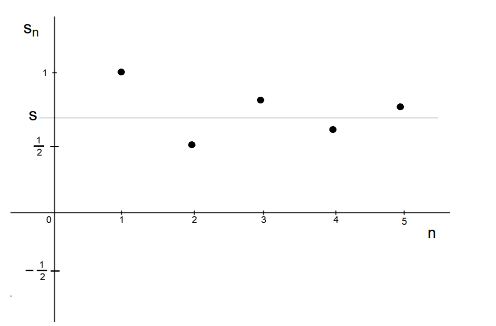
$\sum (-1)^{n-1} \frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$

Example

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

n	a_n	s_n
1	1	1
2	$-\frac{1}{2}$	$1 - \frac{1}{2} = \frac{1}{2}$
3	$\frac{1}{3}$	$\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$
4	$-\frac{1}{4}$	$\frac{5}{6} - \frac{1}{4} = \frac{7}{12}$
5	$\frac{1}{5}$	$\frac{7}{12} + \frac{1}{5} = \frac{47}{60}$
$\vdots$	$\vdots$	$\vdots$





Alternating Series Test

If $\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - \cdots,$

with $b_n > 0$,

satisfies

i) $b_{n+1} \leq b_n$
ii) $\lim_{n\to\infty} b_n = 0$ for all n

then the series is convergent.

(b_n is the absolute value of the series term.)

Note

The Alternating Series Test (AST) is a test for convergence only.

Example

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$$

$$b_n = |(-1)^{n-1}\frac{1}{n}| = \frac{1}{n}$$

$b_{n+1} \leq b_n$, $(\frac{1}{n+1} \leq \frac{1}{n})$, for all n

$\lim\limits_{n\to\infty} b_n = 0$, $(\lim\limits_{n\to\infty} \frac{1}{n} = 0)$

$\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ is convergent by the AST

Example 2

$$\sum_{n=1}^{\infty}\frac{\cos n\pi}{2n^2} = -\frac{1}{2} + \frac{1}{8} - \frac{1}{18} + \frac{1}{32} - \cdots$$

$$a_n = \frac{\cos n\pi}{2n^2} = \frac{(-1)^n}{2n^2}$$

$$b_n = |a_n| = \frac{1}{2n^2}$$

$b_{n+1} \leq b_n$ for all n, $(\frac{1}{2(n+1)^2} \leq \frac{1}{2n^2})$

$$\lim_{n\to\infty} b_n = 0$$

$\sum\limits_{n=1}^{\infty}\frac{\cos n\pi}{2n^2}$ is convergent by the AST

Example 3

For $\sum_{n=1}^{\infty} (-1)^n e^{(\frac{1}{n})}$

$a_n = (-1)^n e^{\frac{1}{n}} \qquad b_n = |a_n| = e^{\frac{1}{n}}$

Decreasing?

$f(x) = e^{(\frac{1}{x})}$

$f'(x) = -\frac{1}{x^2} e^{(\frac{1}{x})}$, (negative for all x in domain)

$b_{n+1} \leq b_n$ for all n in $[1, \infty)$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} e^{(\frac{1}{n})} = 1$

$\lim_{n\to\infty} (-1)^n e^{\frac{1}{n}} \neq 0 \qquad$ (does not exist)

$\sum_{n=1}^{\infty} (-1)^n e^{\frac{1}{n}}$ is divergent by Divergence Test. (Remember that AST is a convergence test only.)

Alternating Series Estimation Theorem

If $s = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$, where $b_n > 0$, is the sum of an alternating series that satisfies

i) $b_{n+1} \leq b_n$

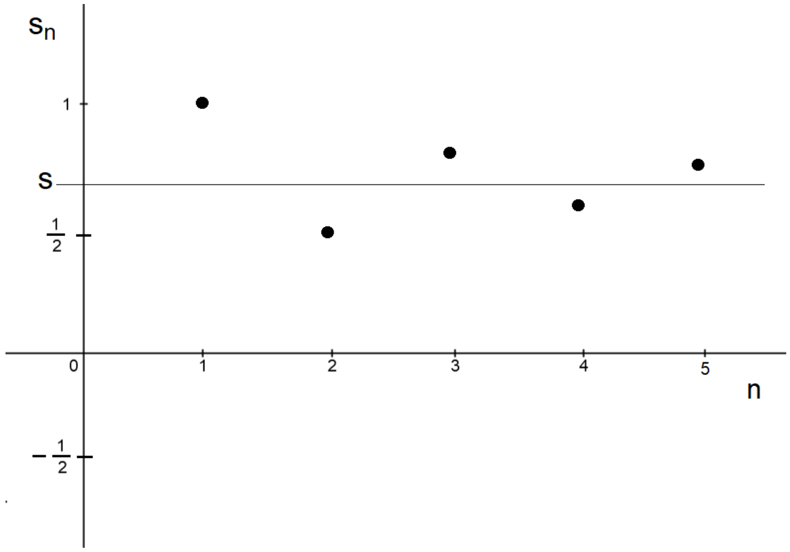
ii) $\lim_{n\to\infty} b_n = 0$ for all n

then $|E_n| = |s - s_n| \leq b_{n+1}$

where $|E_n|$ is the magnitude of the error in the estimate.

Interpretation

If we use s_n to estimate the sum, s, of an alternating series that converges by AST, then $|error|$ is less than the magnitude of the first term that is left out of the estimated sum.



Estimation Example

For $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$, what is the maximum error if 5 terms are used to approximation the sum?

$s = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$

$s \approx s_5 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}, \quad |error| \leq b_6$

$s \approx 0.78\bar{3}, \quad |error| \leq \frac{1}{6} = 0.16\bar{6}$

Try It

How large should n be (how many terms) to insure that $|error| \leq 0.01$, where s_n is used to approximate $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$?

Definition

A series $\sum a_n$ is absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent.

A series $\sum a_n$ is conditionally convergent if $\sum a_n$ is convergent, but $\sum |a_n|$ is divergent.

Examples

$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ is conditionally convergent

$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}$ is absolutely convergent

Theorem

Absolute convergence implies convergence.

(If $\sum |a_n|$ converges, then $\sum a_n$ converges.)

We can test for absolute convergence, using the tests that require positive terms.

Example

$\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$: Not alternating and not positive for all n; therefore CT, LCT, Integral Test are not options. (but suspect convergence)

$a_n = \frac{\cos n}{n^2}$

$|a_n| = \frac{|\cos n|}{n^2}$

$|a_n| \leq \frac{1}{n^2}$, positive terms

$\sum \frac{1}{n^2}$ is convergent p-series, $p = 2 > 1$

$\sum |a_n|$ is convergent by CT so $\sum a_n$ is absolutely convergent (and therefore convergent).

Try It

Show whether these series are absolutely convergent, conditionally convergent, or divergent.

1) $\sum (-1)^n \frac{1}{\sqrt{n}}$

2) $\sum (-1)^{n+2} \frac{1}{n^3+2}$

3) $\sum (-1)^n \frac{5^n}{n^5}$

Quiz

Absolutely Convergent (A), conditionally convergent (C), or Divergent (D)?

1) $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n+1}$

2) $\sum_{n=1}^{\infty} \frac{1}{n+1}$

3) $\sum_{n=1}^{\infty}(-1)\frac{1}{n+1}$

4) $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n+1}$

5) $\sum_{n=1}^{\infty}(-1)^{n-1} \tan^{-1} n$

6) $\sum_{n=1}^{\infty}(-1)^{n-1} \sin(\frac{1}{n})$

Quiz

For what values of p does the series, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^p}$, converge?

A) $p \geq 0$

B) $p > 0$

C) $p \geq 1$

D) $p > 1$

Quiz

For what values of p is the series, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^p}$, absolutely convergent?

A) $p \geq 0$

B) $p > 0$

C) $p \geq 1$

D) $p > 1$

Quiz

For what values of p is the series, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^p}$, conditionally convergent?

A) $p > 0$

B) $0 \leq p < 1$

C) $0 < p \leq 1$

D) $p > 1$

11.6 The Ratio Test & Root Test

Ratio Test

i) If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L < 1$, then $\sum a_n$ is absolutely convergent.

ii) If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L > 1$ or $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \infty$, then $\sum a_n$ is divergent.

iii) If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$, then the Ratio Test is inconclusive.(The series may be convergent or divergent.)

When to Use

Use the Ratio Test when there is a factorial in the series terms or when there is a combination of geometric and power factors.

Example

For $\sum_{n=1}^{\infty} \frac{3^n}{n!}$, $a_n = \frac{3^n}{n!}$, $|a_n| = \frac{3^n}{n!}$, $|a_{n+1}| = \frac{3^{n+1}}{(n+1)!}$

$$|\frac{a_{n+1}}{a_n}| = \frac{3^{n+1}}{(n+1)!} \times \frac{n!}{3^n}$$

$$= \frac{3^{n+1}}{3^n} \times \frac{n!}{(n+1)!}$$

$$= \frac{3}{n+1}$$

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 0 < 1$$

$\sum a_n$ is absolutely convergent by the Ratio Test(and therefore convergent).

Example

For $\sum_{n=1}^{\infty}(-1)^n\frac{e^n}{n^{100}}$, $a_n = (-1)^n\frac{e^n}{n^{100}}$, $|a_n| = \frac{e^n}{n^{100}}$, $|a_{n+1}| = \frac{e^{n+1}}{(n+1)^{100}}$

$$|\frac{a_{n+1}}{a_n}| = \frac{e^{n+1}}{(n+1)^{100}} \times \frac{n^{100}}{e^n}$$

$$= \frac{e^{n+1}}{e^n} \times \frac{n^{100}}{(n+1)^{100}}$$

$$= e(\frac{n}{n+1})^{100}$$

$$\lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| = e > 1$$

$\sum a_n$ is divergent by the Ratio Test.

Example

For $\sum_{n=1}^{\infty} 4(\frac{2}{3})^n$, $a_n = 4(\frac{2}{3})^n$

$$|\frac{a_{n+1}}{a_n}| = \frac{4(\frac{2}{3})^{n+1}}{4(\frac{2}{3})^n}$$

$$= \frac{2}{3} < 1$$

$\sum a_n$ is convergent by Ratio Test, but Geometric Series Test is faster.

Example

For $\sum_{n=1}^{\infty} \frac{n^2}{n^3+3n}$, $a_n = \frac{n^2}{n^3+3n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^2}{(n+1)^3+3(n+1)} \times \frac{n^3+3n}{n^2}$$

$$= \left(\frac{n+1}{n}\right)^2 \times \frac{n^3+3n}{(n+1)^3+3(n+1)}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$$

The Ratio Test is inconclusive.

The Ratio Test is inconclusive when the series is like a p-series.

Use LCT to prove that the series is divergent.

Root Test

i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then $\sum a_n$ is absolutely convergent.

ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then $\sum a_n$ is divergent.

iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then the Root Test is inconclusive.(The series may be convergent or divergent.)

When to Use

Use the Root Test for a series with terms of the form, $|a_n| = [f(n)]^n$, so that $\sqrt[n]{|a_n|} = f(n)$.

Example

For $\sum_{n=1}^{\infty} (\frac{1}{n})^n$, $a_n = (\frac{1}{n})^n$

$$\sqrt[n]{|a_n|} = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{1}{n} = 0 < 1$$

$\sum a_n$ is absolutely convergent by the Root Test (and therefore convergent).

Example

For $\sum_{n=1}^{\infty}(-1)^n(\frac{n}{2n+1})^n$, $a_n = (-1)^n(\frac{n}{2n+1})^n$

$$|a_n| = (\frac{n}{2n+1})^n$$

$$\sqrt[n]{|a_n|} = \frac{n}{2n+1}$$

$$\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} < 1$$

$\sum a_n$ is absolutely convergent by the Root Test (and therefore convergent).

Example

For $\sum_{n=1}^{\infty}(\frac{3}{2})^n$, $a_n = (\frac{3}{2})^n$

$$\sqrt[n]{|a_n|} = \frac{3}{2} > 1$$

$\sum a_n$ is divergent by Root Test, but Geometric Series Test is faster.

Try It

Test whether the following series are absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty} \left(\frac{3}{n}\right)^n \qquad \sum_{n=1}^{\infty} \frac{(-1)^n (n+2)!}{10^n} \qquad \sum_{n=1}^{\infty} (-1)^n \frac{n}{n^3+2}$$

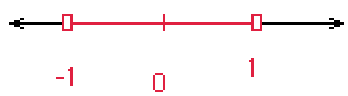
$$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{(2n+1)!} \qquad \sum_{n=1}^{\infty} \frac{n^2 4^n}{5^{n+2}} \qquad \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$$

11.8 Power Series

Example

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

Note: This is both a power series and a geometric series.
Convergent for $|x| < 1$



Interval of convergence: $(-1, 1)$

Radius of convergence, $R = 1$

In General

Power Series in $(x - a)$

Power Series centered at a

Power Series about a

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

For $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$

$$a = 0;\ c_n = 1 \text{ (for all } n)$$

Example

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Find the radius and interval of convergence.

$a = 0$; $c_n = \frac{1}{n!}$

Use the Ratio Test: $|\frac{a_{n+1}}{a_n}| = \frac{|x|^{n+1}}{(n+1)!} \times \frac{n!}{|x|^n}$

$$= \frac{|x|^{n+1}}{|x|^n} \times \frac{n!}{(n+1)!}$$

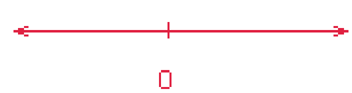
$$= |x|(\frac{1}{n+1})$$

Example - Continued

$$|\frac{a_{n+1}}{a_n}| = |x|(\frac{1}{n+1})$$

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 0 \text{ for all } x$$

Interval of convergence: $(-\infty, \infty)$



Radius of convergence, $R = \infty$

Example

$$\sum_{n=0}^{\infty} n!(x-3)^n = 1 + (x-3) + 2!(x-3)^2 + 3!(x-3)^3 \cdots$$

Find the radius and interval of convergence:

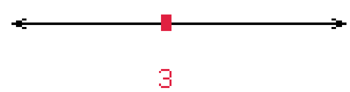
$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!|x-3|^{n+1}}{n!|x-3|^n}$$

$$= (n+1)|x-3|$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty \text{ unless } |x-3| = 0$$

Interval of convergence: $x = 3$

Radius of convergence, $R = 0$



Example

$$\sum_{n=0}^{\infty} \frac{3(x-2)^n}{n+1}$$

Find the interval and radius of convergence:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{3|x-2|^{n+1}}{n+2} \times \frac{n+1}{3|x-2|^n}$$

$$= \frac{3|x-2|^{n+1}}{3|x-2|^n} \times \frac{n+1}{n+2}$$

$$= |x-2| \times \frac{n+1}{n+2}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x-2|$$

Example - Continued

$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = |x-2|$

Convergent for $|x-2| < 1$

Divergent for $|x-2| > 1$

What about $|x-2| = 1$?

Ratio Test is inconclusive. Use another test.

Example - Continued

$\sum_{n=0}^{\infty} \frac{3(x-2)^n}{n+1}$ Checking $|x-2| = 1$

$x-2=-1$

$\sum_{n=0}^{\infty} \frac{3(-1)^n}{n+1}$

Convergent

$x-2=1$

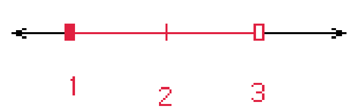
$\sum_{n=0}^{\infty} \frac{3}{n+1}$

Divergent

Series is convergent for $-1 \le x-2 < 1$; $1 \le x > 3$

Interval of convergence: $[1,3)$

Radius of convergence, $R=1$



Theorem

For a given power series, $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities:

i) The series converges only when $x = a$.

ii) The series converges for all x.

iii) There is a positive number, R, such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

Summarize

Series	Radius of Convergence	Interval of Convergence
$\sum_{n=0}^{\infty} x^n$	$R = 1$	$(-1, 1)$
$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$R = \infty$	$(-\infty, \infty)$
$\sum_{n=0}^{\infty} n!(x-3)^n$	$R = 0$	$x = 3$
$\sum_{n=0}^{\infty} \frac{3(x-2)^n}{n+1}$	$R = 1$	$[1, 3)$

Interpret

The possible intervals of convergence for a power series centered at a, $\sum_{n=0}^{\infty} c_n(x-a)^n$, are:

$x = a$

$(a - R, a + R]$

$[a - R, a + R)$

$[a - R, a + R]$

$(a - R, a + R)$

$(-\infty, \infty)$

Try It 1

$\sum_{n=1}^{\infty} \frac{10^n (x-1)^n}{n^2}$

Find the radius and interval of convergence.

Try It 2

$$\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n}$$

Find the radius and interval of convergence.

Try It 3

If $\sum_{n=0}^{\infty} b_n 8^n$ is convergent and $\sum_{n=0}^{\infty} b_n 10^n$ is divergent, then what about

$\sum_{n=0}^{\infty} b_n 2^n$

$\sum_{n=0}^{\infty} b_n (-10)^n$

$\sum_{n=0}^{\infty} b_n (-2)^n$

$\sum_{n=0}^{\infty} b_n 11^n$

$\sum_{n=0}^{\infty} b_n (-8)^n$

Try It 1 Solution 1

$$|\frac{a_{n+1}}{a_n}| = \frac{10^{n+1}|x-1|^{n+1}}{(n+1)^2} \times \frac{n^2}{10^n|x-1|^n}$$

$$= \frac{10^{n+1}}{10^n} \times \frac{n^2}{(n+1)^2} \times \frac{|x-1|^{n+1}}{|x-1|^n}$$

$$= 10(\frac{n}{n+1})^2|x-1|$$

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 10|x-1|$$

Series is convergent for $10|x-1| < 1$ OR $|x-1| < \frac{1}{10}$

Series is divergent for $10|x-1| > 1$ OR $|x-1| > \frac{1}{10}$

$R = \frac{1}{10}$

Try It 1 Solution 2

Check $10|x-1| = 1$ OR $|x-1| = \frac{1}{10}$ (Ratio test inconclusive.)

This corresponds to $x - 1 = -\frac{1}{10}$ AND $x - 1 = \frac{1}{10}$

(That is, $x = \frac{9}{10}$ AND $x = \frac{11}{10}$)

For $x = \frac{9}{10}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

Convergent

For $x = \frac{11}{10}$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Divergent

$I = [\frac{9}{10}, \frac{11}{10}]$, $R = \frac{1}{10}$

Try It 2 Solution 1

Rewrite as $\sum_{n=1}^{\infty} \frac{4^n(x+\frac{1}{4})^n}{n}$, $|\frac{a_{n+1}}{a_n}| = \frac{4^{n+1}|x+\frac{1}{4}|^{n+1}}{n+1} \times \frac{n}{4^n|x+\frac{1}{4}|^n}$

$$= \frac{4^{n+1}}{4^n} \times \frac{n}{n+1} \times \frac{|x+\frac{1}{4}|^{n+1}}{|x+\frac{1}{4}|^n}$$

$$= 4(\frac{n}{n+1})|x+\frac{1}{4}|$$

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 4|x+\frac{1}{4}|$$

Series is convergent for $4|x+\frac{1}{4}| < 1$ OR $|x+\frac{1}{4}| < \frac{1}{4}$

Series is divergent for $4|x+\frac{1}{4}| > 1$ OR $|x+\frac{1}{4}| > \frac{1}{4}$

$R = \frac{1}{4}$

Try It 2 Solution 2

Check $4|x+\frac{1}{4}| = 1$ OR $|x+\frac{1}{4}| = \frac{1}{4}$ (Ratio test inconclusive.)

This corresponds to $x+\frac{1}{4} = -\frac{1}{4}$ AND $x+\frac{1}{4} = \frac{1}{4}$

(That is, $x = -\frac{1}{2}$ AND $x = 0$)

For $x = -\frac{1}{2}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

Convergent

For $x = 0$

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Divergent

$I = [-\frac{1}{2}, 0)$, $R = \frac{1}{4}$

Try It 3 Solutions

If $\sum_{n=0}^{\infty} b_n 8^n$ is convergent and $\sum_{n=0}^{\infty} b_n 10^n$ is divergent, then what about

$\sum_{n=0}^{\infty} b_n 2^n$	C	$\sum_{n=0}^{\infty} b_n(-10)^n$	?
$\sum_{n=0}^{\infty} b_n(-2)^n$	C	$\sum_{n=0}^{\infty} b_n 11^n$	D
$\sum_{n=0}^{\infty} b_n(-8)^n$	?		

11.9 Representations of Functions

Geometric Power Series

$1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \frac{1}{1-x}$ if $|x| < 1$

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n$ if $|x| < 1$

What does it mean that a function is equal to a series?

Substitution

We find more functions with power series representations by substitution into a known function and associated power series:

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n} = 1 + x^2 + x^4 + \cdots$$

Then use substitution to find interval of convergence:

$|x^2| < 1$

$|x| < 1$

More Substitution

$\frac{1}{1+2x} = \frac{1}{1-(-2x)} = \sum_{n=0}^{\infty}(-2x)^n$

$\frac{1}{1+2x} = \sum_{n=0}^{\infty}(-1)^n 2^n x^n$ for $|-2x| < 1$ or $|x| < \frac{1}{2}$

Multiplication

We find more functions with power series representations by multiplying a known function and associated power series by a constant or by a power of x. This does not affect the interval of convergence.

$\frac{5}{1-x} = 5(\frac{1}{1-x}) = 5\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} 5x^n$ for $|x| < 1$

$\frac{x}{1-x} = x(\frac{1}{1-x}) = x\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+1}$ for $|x| < 1$

Multiplication and Substitution

$\frac{1}{3-x} = \frac{1}{3(1-\frac{x}{3})} = \frac{1}{3}(\frac{1}{1-\frac{x}{3}}) = \frac{1}{3}\sum_{n=0}^{\infty}(\frac{x}{3})^n$

$\frac{1}{3-x} = \sum_{n=0}^{\infty}\frac{1}{3^{n+1}}x^n$ for $|\frac{x}{3}| < 1$ or $|x| < 3$

Try It 1

Find power series representations for the following functions, and determine the interval of convergence:

$\frac{1}{1-8x^3}$ $\frac{7}{2+3x}$

$\frac{3}{2-5x}$ $\frac{2x^3}{1-x^2}$

Theorem

If the power series, $\sum_{n=0}^{\infty} c_n(x-a)^n$, has radius of convergence, $R > 0$, then the function, $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$,is differentiable, and therefore continuous, on the interval $(a-R, a+R)$ and

i) $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

ii) $\int f(x)dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$

The radii of convergence of the power series in equations i) and ii) are also R, but the interval of convergence might be different from that of the original power series.

Interpretation

Power series representations of functions can be integrated or differentiated term by term to yield power series for other functions

Differentiation Example

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad R = 1,\ I = (-1, 1)$

$\frac{d(\frac{1}{1-x})}{dx} = 1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots \qquad R = 1$

$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n, \qquad R = 1$

Integration Example

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad R = 1,\ I = (-1, 1)$

$\int \frac{1}{1-x} dx = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n} + \cdots \qquad R = 1$

$-\ln|1-x| = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}, \qquad R = 1$

Now find C...

Integration Example Cont.

$$-\ln|1-x| = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}, \qquad R = 1$$

Substitute $x = 0$:

$$-\ln|1-0| = C + \sum_{n=0}^{\infty} \frac{0^{n+1}}{n+1}$$

$$0 = C + 0$$

$$C = 0$$

So, $\ln|1-x| = \sum_{n=0}^{\infty} -\frac{x^{n+1}}{n+1}, \qquad R = 1$

Integration Example Cont.

What about endpoints?

At $x = -1$:

$$\sum_{n=0}^{\infty} -\frac{(-1)^{n+1}}{n+1}$$

Series is convergent.

At $x = 1$:

$$\sum_{n=0}^{\infty} -\frac{1}{n+1}$$

Series is divergent.

Interval of convergence for series is $[-1, 1)$ but series is only guaranteed to be equal to the function within $R, (-1, 1)$.

It can be shown that the series converges to the function for x in $[-1, 1)$.

Note

$\ln(1-x) = \sum_{n=0}^{\infty} -\frac{x^{n+1}}{n+1}$ on $[-1, 1)$

Also, $\ln(1-x) \approx \sum_{i=0}^{n} -\frac{x^{i+1}}{i+1}$ for x close to 0.

Example

Find a series representation for $\ln(2/3)$.

$\ln(1-x) = \sum_{n=0}^{\infty} -\frac{x^{n+1}}{n+1}, \qquad R = 1$

$\ln(1-\frac{1}{3}) = \sum_{n=0}^{\infty} -\frac{(\frac{1}{3})^{n+1}}{n+1}$

$\ln\frac{2}{3} = \sum_{n=1}^{\infty} \frac{-1}{3^n n}$

Example

What is the sum of the convergent series: $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$

Solution:

$-\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1} = \ln(1-x)$ for x in $[-1,1)$ (Previously derived.)

$\sum_{n=1}^{\infty}-\frac{x^n}{n} = \ln(1-x)$ (Adjust the index.)

$\sum_{n=1}^{\infty}-\frac{(-1)^n}{n} = \ln(1-(-1))$ (Plug in $x=-1$.)

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = \ln 2$$

Quiz 1

Find the sum of the series $\sum_{n=0}^{\infty}-\frac{(\frac{1}{4})^{n+1}}{n+1}$

A) $-\ln(1/4)$

B) $\ln(1/4)$

C) $\ln(3/4)$

Quiz 2

Find the sum of the series $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{5^{n+1}(n+1)}$

A) $\ln(4/5)$

B) $\ln(6/5)$

C) $\ln(5/6)$

Example

Find a power series representation for $\tan^{-1} x$

First, find a power series representation for $\frac{1}{1+x^2}$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Then, integrate to find power series representation for $\tan^{-1} x$

Example Cont.

$\tan^{-1} x = C + \int \frac{1}{1+x^2} dx$

$= C + \int (\sum_{n=0}^{\infty} (-1)^n x^{2n}) dx = C + \int (1 - x^2 + x^4 - x^6 + \cdots) dx$

$\tan^{-1} x = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

Solve for C using $x = 0$:

$\tan^{-1} 0 = C + 0 - \frac{0^3}{3} + \frac{0^5}{5} - \frac{0^7}{7} + \cdots$

$C = 0$

Example Cont.

$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \qquad R = 1$

Note: It can be proven that this series converges to the function within the interval $[-1, 1]$. Thus

$\tan^{-1} 1 = \sum_{n=0}^{\infty} (-1)^n \frac{1^{2n+1}}{2n+1}$

$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$

$\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots) = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots$

Try It 2

$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots$

How many terms need to be added to approximate π to within 0.001?

Quiz

Find the sum of the series $\sum_{n=0}^{\infty} (-1)^n \frac{\sqrt{3}^{2n+1}}{3^{2n+1}(2n+1)}$

A) No sum (divergent)

B) $\frac{\pi}{6}$

C) $\frac{\pi}{3}$

Try It 1 Solutions

$\frac{1}{1-8x^3} = \sum_{n=0}^{\infty} 8^n x^{3n}, I = (-\frac{1}{2}, \frac{1}{2})$

$\frac{3}{2-5x} = \sum_{n=0}^{\infty} \frac{3(5^n)}{2^{n+1}} x^n, I = (-\frac{2}{5}, \frac{2}{5})$

$\frac{7}{2+3x} = \sum \frac{7(-1)^n 3^n}{2^{n+1}} x^n, I = (-\frac{2}{3}, \frac{2}{3})$

$\frac{2x^3}{1-x^2} = \sum_{n=0}^{\infty} 2x^{2n+3}, I = (-1, 1)$

Try It 2 Solution

$\pi = \sum_{n=0}^{\infty} \frac{(-1)^n 4}{2n+1}$

Find the term that has a magnitude of less than or equal to 0.001. That will be the error term.

For what values of n is $\frac{4}{2n+1} \leq \frac{1}{1000}$?

$2n + 1 \geq 4000$

$2n \geq 3999$

$n \geq 1999.5$

$n \geq 2000$

So, the n = 2000 term will be the error term. Include terms 0 through 1999. That is 2000 terms.

$\pi \approx \sum_{n=0}^{1999} \frac{(-1)^n 4}{2n+1}$

11.10 Taylor and Maclaurin Series

Other Functions

How can we find power series representations for functions that are not related to $\frac{1}{1-x}$?

Answer: Use Taylor or MacLaurin Series.

Theorem

If a function, $f(x)$, has a power series about a, then the series will have the form,

$T(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ where $c_n = \frac{f^{(n)}(a)}{n!}$

This is a Taylor Series.

If $a = 0$, then the series is also known as a Maclaurin Series.

Why?

If $f(x) = T(x)$, then all derivatives should be equal at a.

$$
\begin{aligned}
f(x) = T(x) &= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots \\
f(a) &= T(a) \rightarrow f(a) = c_0 \\
T'(x) &= c_1 + 2c_2(x-a) + 3c_2(x-a)^2 + 4c_4(x-a)^3 + \cdots \\
f'(a) &= T'(a) \rightarrow f'(a) = c_1 \\
T''(x) &= 2c_2 + 3 \cdot 2c_3(x-a) + 4 \cdot 3c_4(x-a)^2 + \cdots \\
f''(a) &= T''(a) \rightarrow f''(a) = 2c_2 \rightarrow c_2 = \frac{f''(a)}{2} \\
T^{(3)}(x) &= 3 \cdot 2c_3 + 4 \cdot 3 \cdot 2c_4(x-a) + \cdots \\
f^{(3)}(a) &= T^{(3)}(a) \rightarrow f^{(3)}(a) = 3 \cdot 2c_3 \rightarrow c_3 = \frac{f^{(3)}(a)}{3 \cdot 2}
\end{aligned}
$$

Coefficients

$c_0 = f(a)$

$c_1 = f'(a)$

$c_2 = \frac{f''(a)}{2}$

$c_3 = \frac{f^{(3)}(a)}{3 \cdot 2}$

In general, $c_n = \frac{f^{(n)}(a)}{n!}$

These are the coefficients that insure that the derivatives of $T(x)$ and of $f(x)$ at a are equal. Finding a Taylor/Maclaurin Series for a function is a matter of finding these coefficients.

Example

Find the Maclaurin series for $f(x) = e^x$

Solution:

$$T(x) = \sum_{n=0}^{\infty} c_n x^n$$

Now find coefficients...

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$c_n = \frac{f^{(n)}(0)}{n!}$
0	e^x	1	1
1	e^x	1	1
2	e^x	1	$\frac{1}{2}$

$c_n = \frac{1}{n!}$

$$T(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

Theorem

Let $I = (a - R, a + R)$, where $R > 0$. Suppose there exists $K > 0$ such that all derivatives of f are bounded by K on I:

$|f^{(i)}(x)| \leq K$ for all $i \geq 0$ and $x \in I$

Then $f(x)$ is represented by its Taylor series in I:

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n$ for all $x \in I$

Radius of Convergence

$f(x) = e^x$

$f^{(n)}(x) = e^x$

For all $R > 0$, $|f^{(i)}(x)| \leq e^{a+R}$ for $x \in (a - R, a + R)$

Use $K = e^{a+R}$

$T(x)$ converges to $f(x)$ for all $x \in (a - R, a + R)$

Since R is arbitrary, $T(x)$ converges to $f(x)$ for all x.

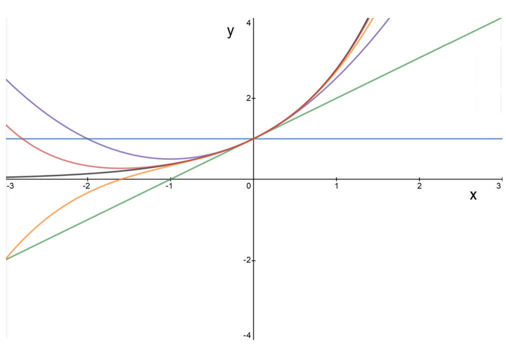
Summary

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad R = \infty$$

Interpretation

$$e^x = T(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \text{ for all } x$$

$$e^x \approx T_n(x) = \sum_{i=0}^{n} \frac{x^i}{i!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \text{ for } x \text{ close to } 0$$



$f(x) = e^x$

$T_0(x) = 1$

$T_1(x) = 1 + x$

$T_2(x) = 1 + x + \frac{x^2}{2}$

$T_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$

$T_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$

Interpretation Cont.

$$e = e^1 = T(1) = \sum_{n=0}^{\infty} \frac{1^n}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \cdots$$

$e \approx 1$

$e \approx 1 + 1 = 2$

$e \approx 1 + 1 + \frac{1}{2} = 2.5$

$e \approx 1 + 1 + \frac{1}{2} + \frac{1}{6} = 2.\overline{6}$

The more terms that are included, the better the approximation.

Quiz

Use substitution and multiplication to find the Maclaurin series for $f(x) = x^2e^{-5x}$

A) $\sum_{n=0}^{\infty} \frac{(-5)^n x^{n+2}}{n!}$

B) $\sum_{n=0}^{\infty} \frac{(-5)^n x^{2n}}{n!}$

C) $\sum_{n=0}^{\infty} \frac{-5x^{n+2}}{n!}$

Quiz

$f(x) = x^2 e^{-5x}$

Find $f^{(25)}(0)$

A) 0

B) $\frac{1}{25!}$

C) $\frac{-(25!)5^{23}}{23!}$

Maclaurin Series for cos(x)

i	$f^{(i)}(x)$	$f^{(i)}(0)$	$c_i = \frac{f^{(i)}(0)}{i!}$
0	$\cos x$	1	1
1	$-\sin x$	0	0
2	$-\cos x$	-1	$\frac{-1}{2}$
3	$\sin x$	0	0
4	$\cos x$	1	$\frac{1}{4!}$

Coefficients for cos(x)

$\cos x = 1 + 0 - \frac{1}{2}x^2 + 0 + \frac{1}{4!}x^4 + 0 + \cdots$

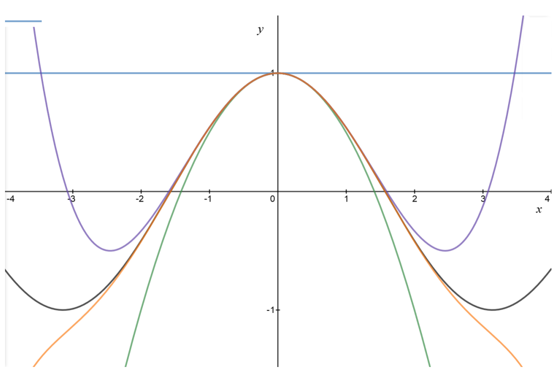
$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \cdots$

Note that the Taylor series includes only even terms, so it is an even function, just as $\cos x$ is an even function. It can be shown that $R = \infty$.

$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, R = \infty$

Graphical Interpretation

The partial sums are Taylor Polynomials.



$f(x) = \cos x$

$T_0(x) = 1$

$T_2(x) = 1 - \frac{x^2}{2!}$

$T_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

$T_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$

Example

$$\cos(0.01) = \sum_{n=0}^{\infty} (-1)^n \frac{(0.01)^{2n}}{(2n)!} = 1 - \frac{(0.01)^2}{2} + \frac{(0.01)^4}{4!} - \frac{(0.01)^6}{6!} + \cdots$$

$\cos(0.01) \approx T_0(0.01) = 1$

$\cos(0.01) \approx T_2(0.01) = 1 - \frac{(0.01)^2}{2} = .99995$

$\cos(0.01) \approx T_4(0.01) = 1 - \frac{(0.01)^2}{2} + \frac{(0.01)^4}{24} = .9999500004$

Using $T_4(0.01), |error| \le \frac{(0.01)^6}{6!} = 1.4 \times 10^{-15}$

Quiz

Find the Maclaurin Series for $f(x) = x^2 \cos x^3$

A) $T(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{12n}}{(12n)!}$

B) $T(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{12n}}{(2n)!}$

C) $T(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+2}}{(2n)!}$

Quiz

If $f(x) = x^2 \cos x^3$, find $f^{(43)}(0)$

A) $\frac{1}{43!}$

B) $-\frac{1}{43!}$

C) 0

Example

Find the Maclaurin Series for $\sin(x)$:

Solution: Instead of starting from scratch, we will use differentiation.

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, R = \infty$$

Now, differentiate...

Example Cont.

$$\begin{aligned}
\tfrac{d}{dx}(\cos x) &= \tfrac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}\right] \\
&= \sum_{n=1}^{\infty}(-1)^n \frac{1}{(2n)!}(2n)x^{2n-1} \\
&= \sum_{n=1}^{\infty}(-1)^n \frac{x^{2n-1}}{(2n-1)!} \\
&= \sum_{n=0}^{\infty}(-1)^{n+1} \frac{x^{2(n+1)-1}}{(2(n+1)-1)!} \\
-\sin x &= \sum_{n=0}^{\infty}(-1)^{n+1} \frac{x^{2n+1}}{(2n+1)!} \\
\sin x &= \sum_{n=0}^{\infty}(-1)^{n} \frac{x^{2n+1}}{(2n+1)!} \quad R=\infty
\end{aligned}$$

Summarize

$$\begin{aligned}
\tfrac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \quad R=1 \quad (-1,1) \\
\ln(1-x) &= \sum_{n=0}^{\infty} -\frac{x^{n+1}}{n+1} \quad R=1 \quad [-1,1) \\
\tan^{-1} x &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} \quad R=1 \quad [-1,1] \\
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad R=\infty \\
\cos x &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} \quad R=\infty \\
\sin x &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad R=\infty
\end{aligned}$$

Try It 1

Find the sum of the following series:

$$\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{4^{2n+1}(2n+1)!}$$

$$\sum_{n=0}^{\infty} \frac{2^{2n}}{n!}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n(\pi)^{2n+2}}{3^{2n+1}(2n)!}$$

Try It 1 Solution

Find the sum of the following series:

$$\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{4^{2n+1}(2n+1)!} = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$\sum_{n=0}^{\infty} \frac{2^{2n}}{n!} = e^{(2^2)} = e^4$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n(\pi)^{2n+2}}{3^{2n+1}(2n)!} = \frac{\pi^2}{3} \sum_{n=0}^{\infty} \frac{(-1)^n(\pi)^{2n}}{3^{2n}(2n)!} = \frac{\pi^2}{3} \cos\frac{\pi}{3} = \frac{\pi^2}{6}$$

Applications of Taylor Polynomials

Linear Approximation

We know that we can approximate a function, $y = f(x)$, near $x = a$, by finding an equation for the line tangent to the curve at $x = a$.

$$L(x) = f(a) + f'(a)(x - a)$$

This linearization has the same value and the same first derivative at $x = a$ as the function, $y = f(x)$.

Linearization

If $L(x) = f(a) + f'(a)(x - a)$,

then $L(a) = f(a)$ and $L'(a) = f'(a)$.

Same value, same slope.

Also, $f(x) \approx L(x)$ near $x = a$.

Example

If $f(x) = \ln x$, then $f(1) = 0$, $f'(x) = \frac{1}{x}$, and $f'(1) = 1$.

$L(x) = 0 + 1(x - 1)$

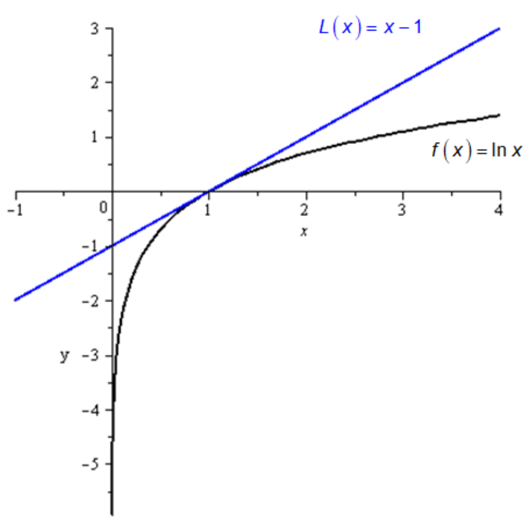
$L(x) = x - 1$

Same value, same slope at $x = 1$.

$L(x) \approx f(x)$ near $x = 1$.

$L(0.9) = -0.100000$

$f(0.9) = -0.105361$



Improve?

Can we make a better approximation by making a quadratic function, $Q(x)$, such that

$Q(a) = f(a)$

$Q'(a) = f'(a)$

and $Q''(a) = f''(a)$???

Yes.

Quadratic Approximation

If $f(x) = \ln x$, then $Q(x) = 0 + 1(x-1) - \frac{1}{2}(x-1)^2$

or $Q(x) = x - 1 - \frac{1}{2}(x-1)^2$

Compare derivatives:

$f(x) = \ln x$ $\qquad Q(x) = x - 1 - \frac{1}{2}(x-1)^2$

$f(1) = 0$ $\qquad Q(1) = 0$

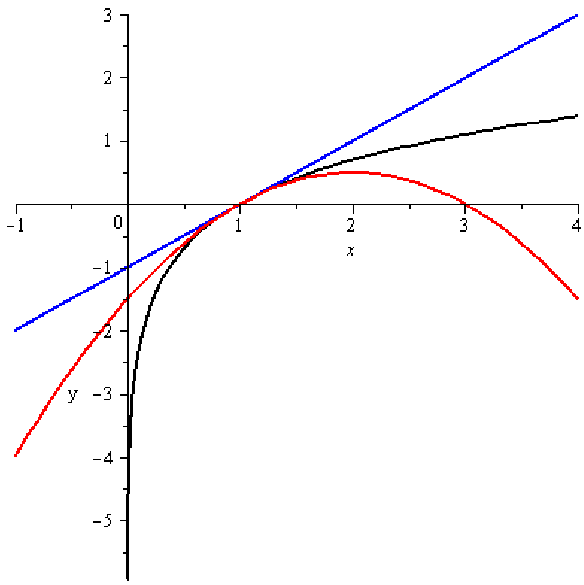
$f'(x) = \frac{1}{x}$ $\qquad Q'(x) = 1 - (x-1)$

$f'(1) = 1$ $\qquad Q'(1) = 1$

$f''(x) = -\frac{1}{x^2}$ $\qquad Q''(x) = -1$

$f''(1) = -1$ $\qquad Q''(1) = -1$

Continued



$Q(x) \approx f(x)$ near $x = 1$

Compare:

$L(0.9) = -0.100000$

$Q(0.9) = -0.105000$

$f(0.9) = -0.105361$

$|error| \approx 0.000361$

Improvement?

Can we improve this approximation by making a polynomial of degree n, such that the derivatives of the polynomial and the function agree at $x = a$ through the n^{th} derivative?

Yes. This is the Taylor polynomial of degree n.

Taylor Polynomial

The Taylor Polynomial approximation of degree n for the function $f(x)$, near $x = a$, is

$$T_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n$$

where $c_n = \frac{f^n(a)}{n!}$, and these are the coefficients that make the derivatives agree.

The Taylor polynomial is a partial sum of the Taylor series.

Requirements

$f(x)$ is defined on an open interval, I.

All derivatives, $f^{(k)}(x)$, exist on I.

$a \in I$

Error Bound Theorem

Assume that $f^{(n+1)}(x)$ exists and is continuous. Let M be a number such that $|f^{(n+1)}(u)| \leq M$ for all u between a and x, then

$$|R_n(x)| = |f(x) - T_n(x)| \leq M \frac{|x-a|^{n+1}}{(n+1)!}$$

where $T_n(x)$ is the nth Taylor polynomial centered at $x = a$.

Estimation Example 1

If $T_2(0.9)$ is used to approximate $\ln(0.9)$, what does the error bound formula guarantee?

Solution:

$T_2(x) = x - 1 - \frac{1}{2}(x-1)^2$ (centered at $a = 1$)

Find M:
$f^{(3)}(x) = \frac{2}{x^3}$

$|f^{(3)}(u)| = \frac{2}{u^3} \leq \frac{2}{0.9^3}$ for u between 1 and 0.9

so $M = \frac{2000}{9^3}$

Solution - Continued

$|f(0.9) - T_2(0.9)| \leq \frac{2000}{9^3} \frac{|0.9-1|^3}{(2+1)!}$

$|error| \leq \frac{1}{9^3 * 3} = 0.000457$

We found that $|error| \approx 0.000361$.

$0.000361 < 0.000457$

Estimation Example 2

Now, if a Maclaurin polynomial is to be used to approximate $\sin(0.02)$ to within 10^{-6}, how many terms must be included?

Solution:

$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

This is an alternating series so use the alternating series estimation theorem.

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$\sin(0.02) = 0.02 - \frac{0.02^3}{3!} + \frac{0.02^5}{5!} - \cdots$

$= 0.02 - 1.\bar{3} \times 10^{-6} + 2.\bar{6} \times 10^{-11} - \cdots$

Two terms needed.

$\sin(0.02) \approx 0.019998\bar{6}, \qquad |error| \leq 2.\bar{6} \times 10^{-11}$

Estimation Example 3

Approximate $\int_0^{0.3} e^{-x^2}\,dx$ with $|error| \le 0.00001$ using a Taylor polynomial.

Solution:

$$\int_0^{0.3} e^{-x^2}\,dx = \int_0^{0.3} \left[\sum_{n=0}^{\infty} \frac{(-x^2)}{n!}\right] dx$$

$$= \int_0^{0.3} \left[1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots\right] dx$$

$$= \left[x - \tfrac{x^3}{3} + \tfrac{x^5}{5\cdot 2} - \tfrac{x^7}{7\cdot 6} + \cdots\right]\Bigg|_0^{0.3}$$

$$= \left[0.3 - \tfrac{0.3^3}{3} + \tfrac{0.3^5}{5\cdot 2} - \tfrac{0.3^7}{7\cdot 6} + \cdots\right] - 0$$

$$\approx 0.3 - \tfrac{0.3^3}{3} + \tfrac{0.3^5}{5\cdot 2} \qquad |error| \le \tfrac{0.3^7}{7\cdot 6} < 1 \times 10^{-5}$$

$$\int_0^{0.3} e^{-x^2}\,dx \approx 0.291243$$