Chapter 10: Parametric Equations and Polar Coordinates

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10.1 Curves Defined by Parametric Equations

Parametric Equations

$$x = f(t)$$
$$y = g(t)$$

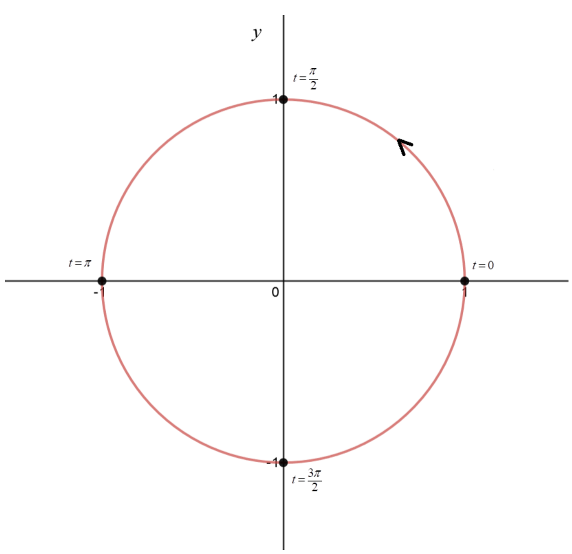
Set of parametric equations for a plane curve.

x and y are defined in terms of a common parameter, t.

Plane curve spans all points with coordinates $(f(t), g(t))$ for all values of t in the domain.

Example

$$x = \cos t$$
$$y = \sin t$$
$$0 \leq t \leq 2\pi$$



Curve has a direction.

Equivalent Cartesian equation:

$x^2 + y^2 = 1$

Sketch

$$x = t^2$$

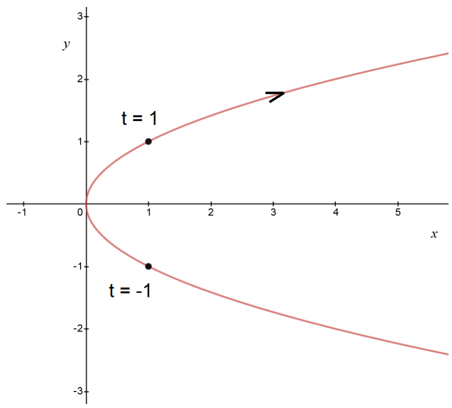
$$y = t$$

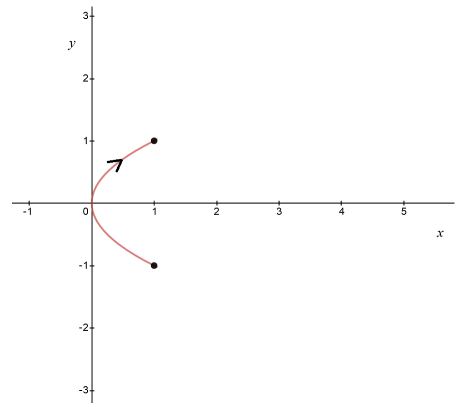
Note that $x = y^2$

$$x = t^2$$

$$y = t$$

$$-1 \leq t \leq 1$$



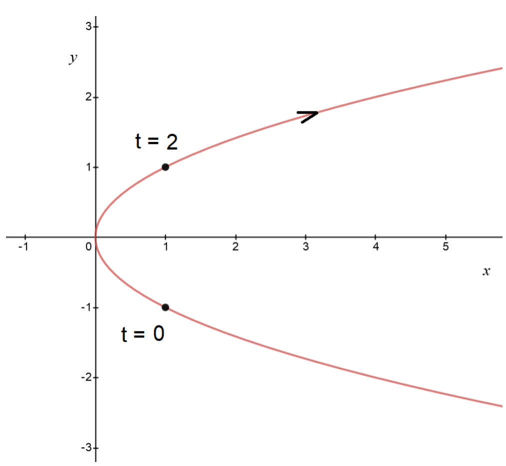


Alternative Parameterization

$$x = (t+1)^2$$

$$y = t + 1$$

Note that $x = y^2$ as before.



Same points are included as before, but the points are traced differently.

Try It 1

Sketch the curve defined by the parametric equations:

$x = \cos^2 t$

$y = \cos t$

$0 \leq t \leq 4\pi$

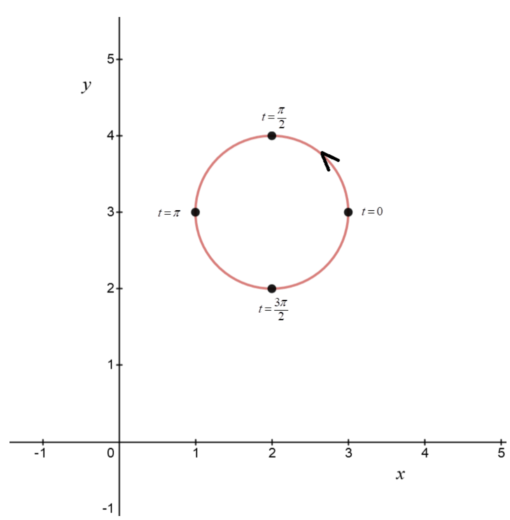
Example

Draw the parametric curve:

$x = 2 + \cos t$

$y = 3 + \sin t$

$0 \leq t \leq 2\pi$



Alternative

$x = 2 + \cos t$

$y = 3 + \sin t$

$0 \leq t \leq 2\pi$

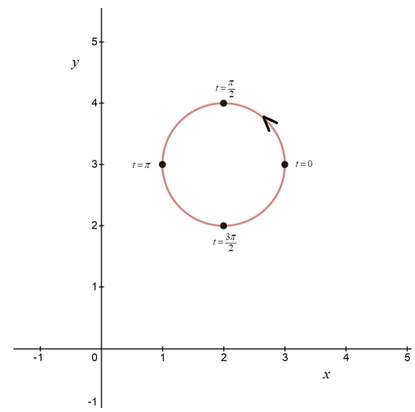
Eliminate parameter:

$x - 2 = \cos t$

$y - 3 = \sin t$

$(x-2)^2 + (y-3)^2 = \cos^2 t + \sin^2 t = 1$

$(x-2)^2 + (y-3)^2 = 1$



Example

Draw the parametric curve:
$x = 1 + t$

$y = 5 - 2t$

$-2 \leq t \leq 3$

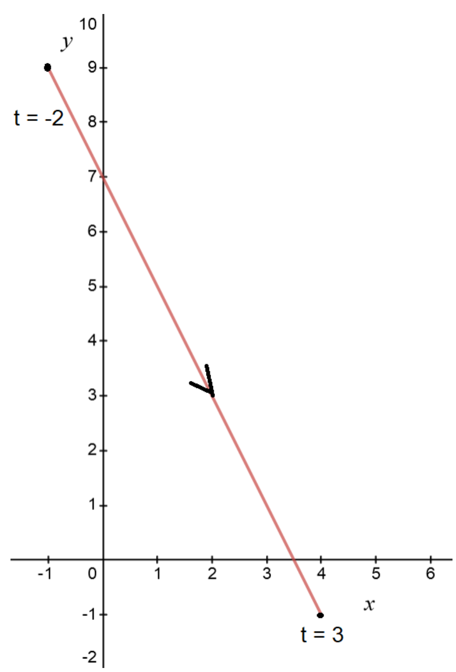
Eliminate parameter:

$t = x - 1$

$y = 5 - 2(x-1)$

$y = 7 - 2x$

$-1 \leq x \leq 4$



Alternative

$x = 1 + t$

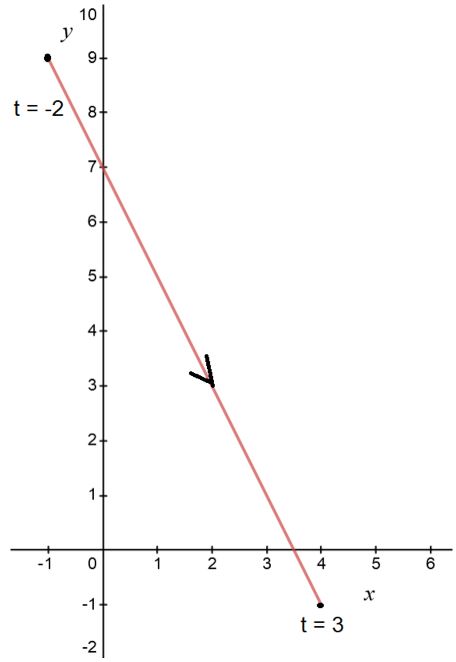
$y = 5 - 2t$

$-2 \leq t \leq 3$

Recognize that these are linear equations, so "curve" is a line segment. Plot endpoints and connect.

Endpoints are:

$(-1, 9)$ and $(4, -1)$



Try It 2

Draw the parametric curve:

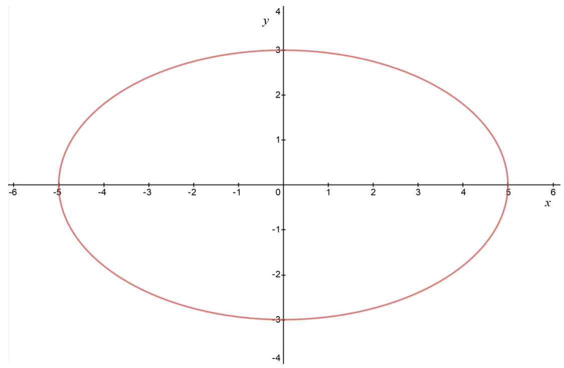
$x = e^t$

$y = e^{-t}$

Ellipse

Parameterize the following ellipse:

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$



Ellipse Example Solution

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

Use Pythagorean Theorem:

$$\frac{x^2}{25} = \cos^2 t$$

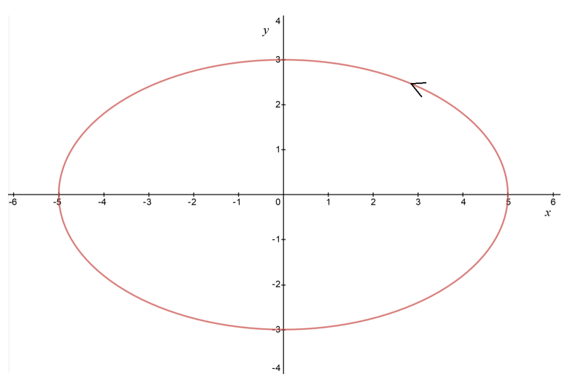
$$\frac{y^2}{9} = \sin^2 t$$

Then:

$x = 5\cos t$

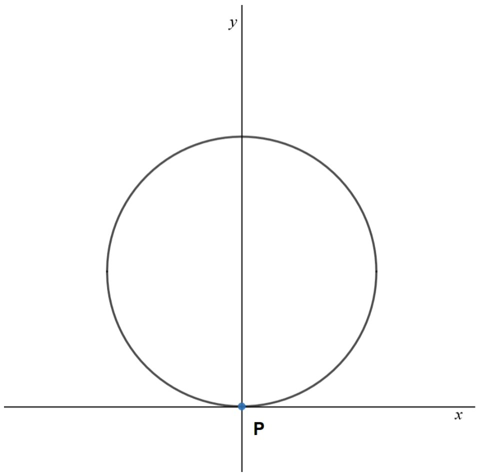
$y = 3\sin t$

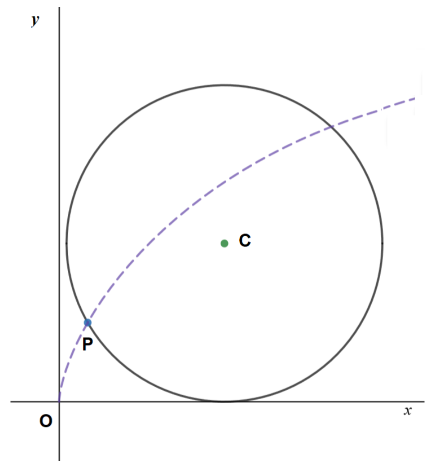
$0 < t < 2\pi$



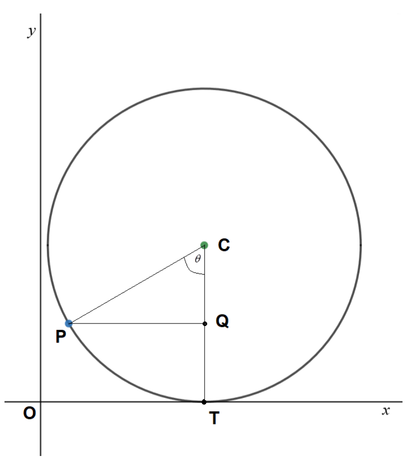
Cycloid

Cycloid – a curve traced by a point on a circle as the circle rolls on a straight horizontal line.





Cycloid - Continued



θ is the parameter. "r" is a constant.

$x = |OT| - |PQ|$

$|OT| = r\theta$

$|PQ| = r\sin\theta$

$x = r\theta - r\sin\theta$

$y = |CT| - |CQ|$

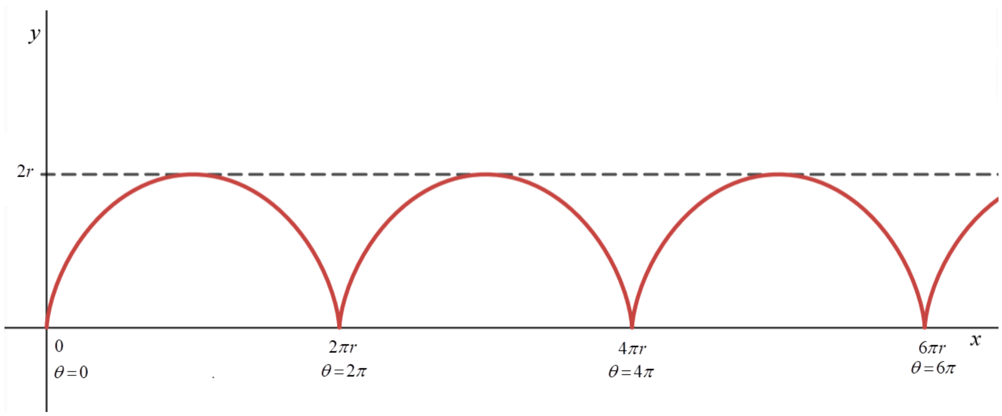
$|CT| = r$

$|CQ| = r\cos\theta$

$y = r - r\cos\theta$

Graph of Cycloid

$$x = r\theta - r\sin\theta$$
$$y = r - r\cos\theta$$



Quiz

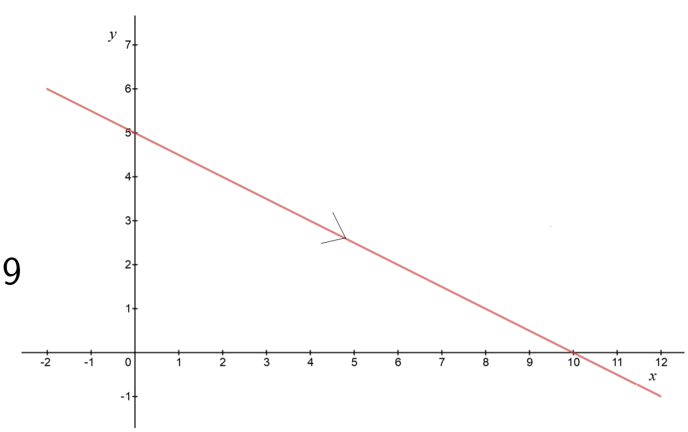
Which set of parametric equations corresponds to the figure?

A) $x = \sin t$
$y = -t$

B) $x = t^2 - 9$
$y = -t^3 + 8t$

C) $x = 1 - t$ $\quad y = t^2 - 9$

D) $x = 2t + 4$
$y = 3 - t$



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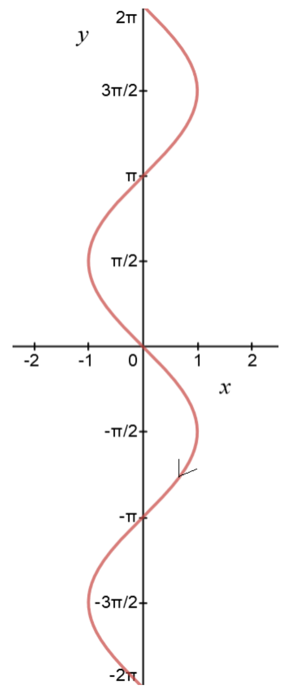

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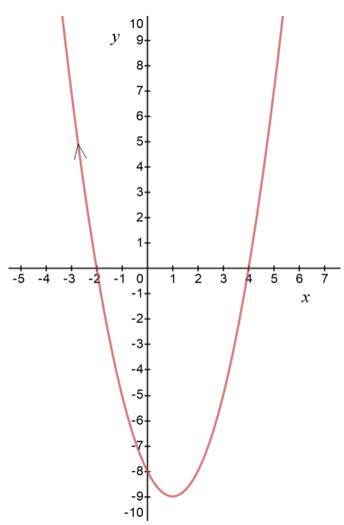

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D) $x = 2t + 4$

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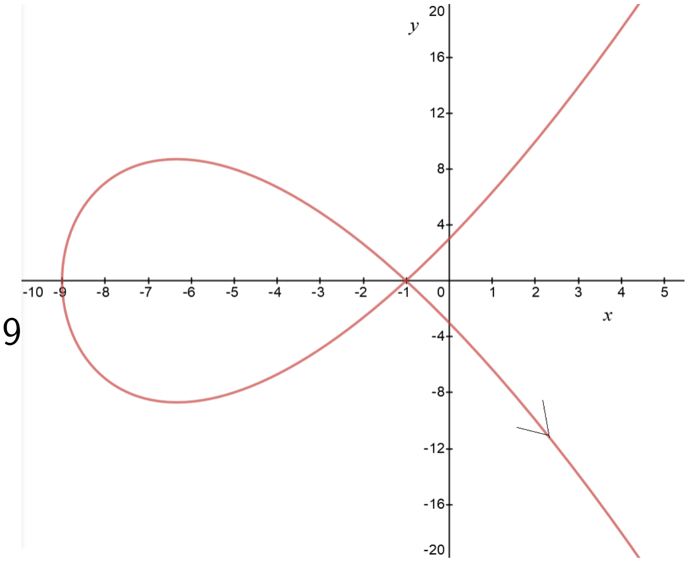
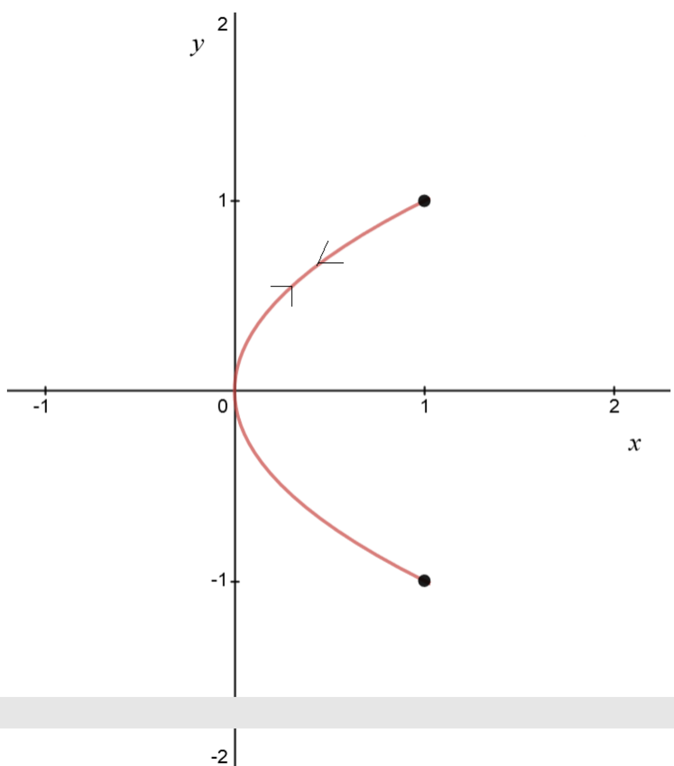


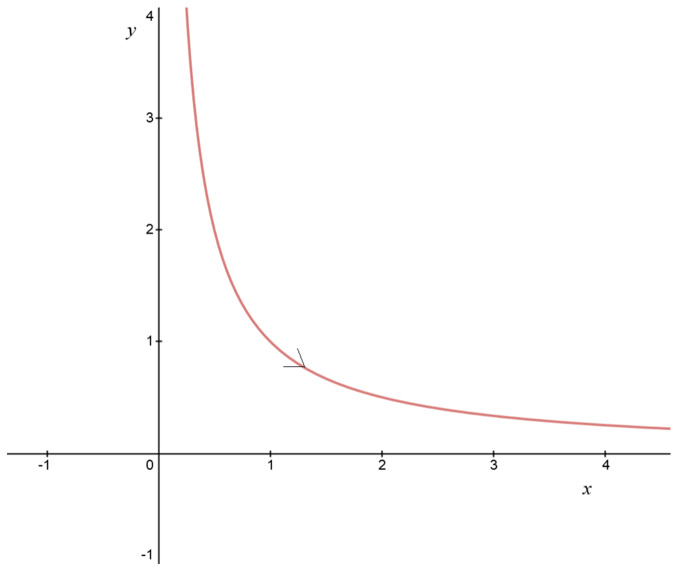
Try It 1 Solution

All points of the parametric curve lie on the curve, $x = y^2$, but $0 \leq x \leq 1$ and $-1 \leq y \leq 1$. The curve is traversed 4 times, twice in each direction, between $t = 0$ and $t = \pi$



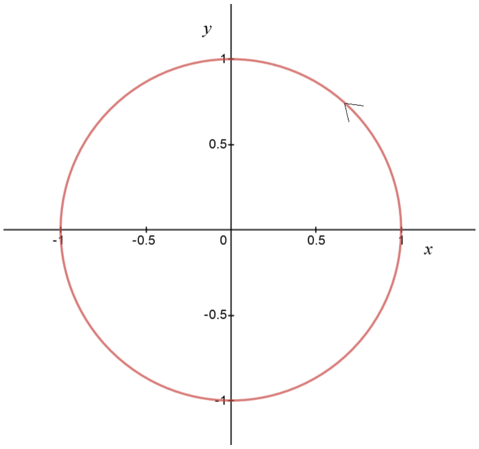
Try It 2 Solution

The parametric curve lies on $y = \frac{1}{x}$, but $x > 0$, and $y > 0$.



10.2 Calculus with Parametric Curves

How Parametric Equations Can Help



$\frac{dy}{dx} = ?$

For $y = \sqrt{1 - x^2}$,

$\frac{dy}{dx} = \frac{-x}{\sqrt{1-x^2}}$

For $y = -\sqrt{1 - x^2}$,

$\frac{dy}{dx} = \frac{x}{\sqrt{1-x^2}}$

First Derivative

If $x = f(t)$, and $y = g(t)$,

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, if $\frac{dx}{dt} \neq 0$

Horizontal tangents occur when $\frac{dy}{dt} = 0$, $\frac{dx}{dt} \neq 0$

Vertical tangents occur when $\frac{dx}{dt} = 0$, $\frac{dy}{dt} \neq 0$

Example

$x = \cos t$

$y = \sin t$

$\frac{dy}{dx} = \frac{\cos t}{-\sin t} = -\cot t$

Horizontal tangent:

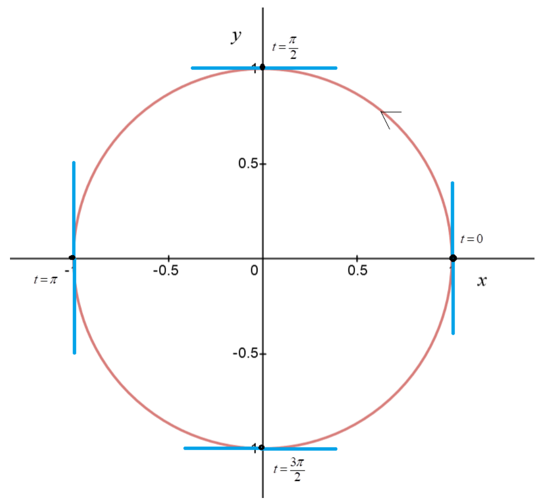
$\cos t = 0$ at $t = \frac{\pi}{2}, \frac{3\pi}{2}$

$-\sin t \neq 0$ at $t = \frac{\pi}{2}, \frac{3\pi}{2}$

Vertical tangent:

$-\sin t = 0$ at $t = 0, \pi$

$\cos t \neq 0$ at $t = 0, \pi$



Try It 1

For $x = \cos t, y = \sin t$, find an equation of the tangent line at $t = \frac{\pi}{6}$.

Second Derivative

$$\frac{d^2y}{dx^2} = \frac{d(\frac{dy}{dx})}{dx} = \frac{d(y')}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d(\frac{dy}{dx})}{dt}}{\frac{dx}{dt}} = \frac{\frac{d(y')}{dt}}{\frac{dx}{dt}}$$

Example

$x = \cos t$

$y = \sin t$

$x = \cos t$

$\frac{dy}{dx} = y' = -\cot t$

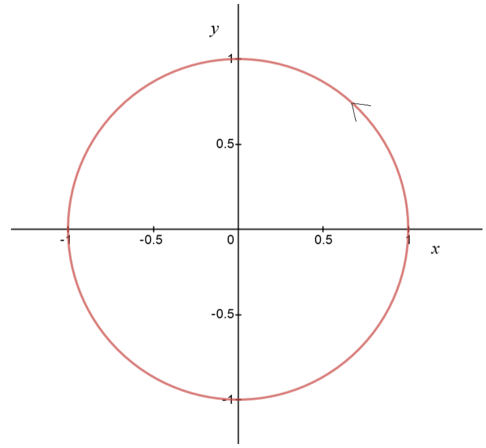
$$\frac{d^2y}{dx^2} = \frac{d(\frac{dy}{dx})}{dx} = \frac{\frac{d(\frac{dy}{dx})}{dt}}{\frac{dx}{dt}} = \frac{\frac{d(y')}{dt}}{\frac{dx}{dt}} = \frac{\csc^2 t}{-\sin t} = \frac{-1}{\sin^3 t}$$

Example - Continued

$\frac{d^2y}{dx^2} = \frac{-1}{\sin^3 t}$

Concave down when $\sin t > 0$, $0 < t < \pi$

Concave up when $\sin t < 0$, $\pi < t < 2\pi$



Quiz

Compute $\frac{d^2y}{dx^2}$ for the parametric curve, $x = t^3$, $y = t^2 - 1$.

A) $\frac{-2}{9t^4}$

B) $\frac{4}{9t^2}$

C) $\frac{1}{3t}$

Arc Length

$ds = \sqrt{1 + (\frac{dy}{dx})^2}dx$

If $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$,

then

$ds = \sqrt{1 + \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2}\frac{dx}{dt}dt$

$ds = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt$

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Requirements: f' and g' are continuous on $[\alpha, \beta]$ and the curve is traversed only once as t increases from α to β.

Example

$x = \cos t$

$y = \sin t$

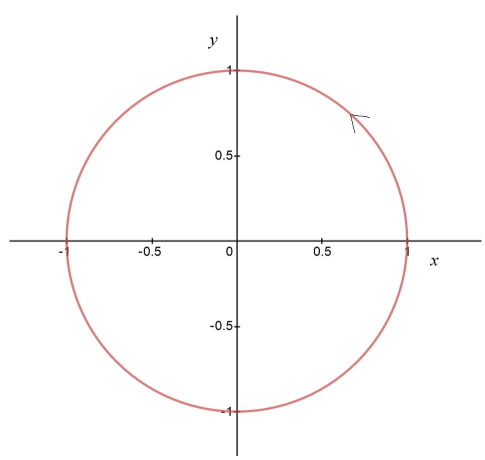
$0 \leq t \leq 2\pi$

$L = \int_{\alpha}^{\beta} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt$

$L = \int_{0}^{2\pi} \sqrt{\sin^2 t + \cos^2 t}dt$

$= \int_{0}^{2\pi} dt$

$= 2\pi$



Surface Area

Still only one formula:

$S = \int 2\pi r ds$

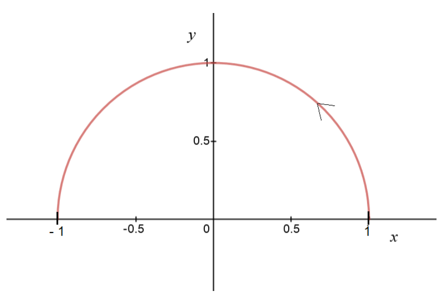
$ds = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt$

$r = y = g(t)$ for revolution about x-axis

$r = x = f(t)$ for revolution about y-axis

Example

Find the area of the surface generated by revolving the following curve about the x-axis.



$x = \cos t$

$y = \sin t$

$0 \leq t \leq \pi$

$S = \int 2\pi r ds$

$ds = \sqrt{\sin^2 t + \cos^2 t} dt$

$r = y = \sin t$

$$S = \int_0^\pi 2\pi \sin t \sqrt{\sin^2 t + \cos^2 t} dt$$
$$= \int_0^\pi 2\pi \sin t dt$$
$$= -2\pi \cos t \Big|_0^\pi$$
$$S = \boxed{4\pi}$$

Quiz

Which of the following integrals will correctly compute the area of the surface generated by rotating the curve, $x = t^3$, $y = t^2 - 1$, $0 \le t \le 1$, around the x-axis?

A) $S = \int_0^1 2\pi t^4 \sqrt{9t^2 + 4} dt$

B) $S = \int_0^1 2\pi (t^3 - t)\sqrt{9t^2 + 4} dt$

C) $S = \int_0^1 2\pi (t^2 - 1)\sqrt{9t^2 + 4} dt$

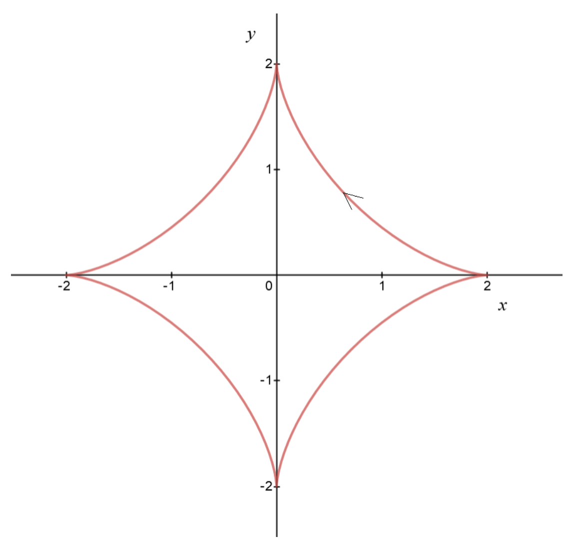
Try It 2

Find the length of the astroid defined by

$x = 2\cos^3\theta$

$y = 2\sin^3\theta$

$0 \le \theta \le 2\pi$



Distance

If $x = f(t)$ and $y = g(t)$ give the position of a particle as a function of t, time, then the distance travelled by the particle from the starting time, t_0, until time, t, is given by

$s(t) = \int_{t_0}^{t} \sqrt{(f'(u))^2 + (g'(u))^2} du$

Speed

If $x = f(t)$ and $y = g(t)$ give the position of a particle as a function of t, time, then the speed of the particle is

$\frac{ds}{dt} = \sqrt{(f'(t))^2 + (g'(t))^2}$

Try It 3

If a particle travels on the curve,

$x = t$

$y = \frac{2}{3}t^{\frac{3}{2}}$

for 8 seconds, starting at $t = 0$, what is the distance travelled? What is the displacement? What is the speed?

(All distances are in meters.)

Try It 1 Solution

To find the equation for any line, find a point and find the slope.

At $t = \frac{\pi}{6}$, $x = \cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$, and
$y = \sin(\frac{\pi}{6}) = \frac{1}{2}$

The slope is $\frac{dy}{dx}|_{t=\frac{\pi}{6}}$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t}{-\sin t}$

$\frac{dy}{dx}|_{t=\frac{\pi}{6}} = -\sqrt{3}$

Using point-slope form:

$y - \frac{1}{2} = -\sqrt{3}(x - \frac{\sqrt{3}}{2})$ OR $y = -\sqrt{3}x + 2$

Try It 2 Solution

Using symmetry, find arc length for $0 \le \theta \le \frac{\pi}{2}$ and multiply by 4.

$\frac{dx}{d\theta} = 6\cos^2\theta(-\sin\theta)$ $\frac{dy}{d\theta} = 6\sin^2\theta(\cos\theta)$

$$l = 4\int_0^{\frac{\pi}{2}} \sqrt{36\cos^4\theta\sin^2\theta + \sin^4\theta\cos^2\theta d\theta}$$

$$l = 4\int_0^{\frac{\pi}{2}} \sqrt{36\cos^2\theta\sin^2\theta(cos^2\theta + \sin^2\theta)}d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} |6\cos\theta\sin\theta| d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} 6\cos\theta\sin\theta d\theta$$

$$= 24\left(\frac{\sin^2\theta}{2}\right)\Big|_0^{\frac{\pi}{2}}$$

$$l = \boxed{12}$$

Try It 3 Solution 1

This curve is traversed only once between $t = 0$ and $t = 8$, so distance travelled is the same as arc length.

$$d = \int_0^8 \sqrt{(\tfrac{dx}{dt})^2 + (\tfrac{dy}{dt})^2}dt$$

$$= \int_0^8 \sqrt{1 + (t^{\frac{1}{2}})^2}dt$$

$$= \int_0^8 \sqrt{1+t}dt$$

$$= \tfrac{2}{3}(1+t)^{\frac{3}{2}}|_0^8$$

$$= \tfrac{2}{3}(27-1)$$

$$= \boxed{\tfrac{52}{3}\text{ m}} \text{ (distance)}$$

Try It 3 Solution 2

The displacement is the length of the line segment from the starting point to the ending point:

Initial point: $(0, 0)$

Terminal point: $(8, \frac{32\sqrt{2}}{3})$

$$
\begin{aligned}
l &= \sqrt{8^2 + (\tfrac{32\sqrt{2}}{3})^2} \\
&= \sqrt{64 + \tfrac{2048}{9}} \\
&= \sqrt{\tfrac{576+2048}{9}} \\
&= \boxed{\tfrac{\sqrt{2624}}{3}\text{m}} \text{ (displacement)}
\end{aligned}
$$

Try It 3 Solution 3

The speed is $\frac{ds}{dt}$, computed at $t = 8$.

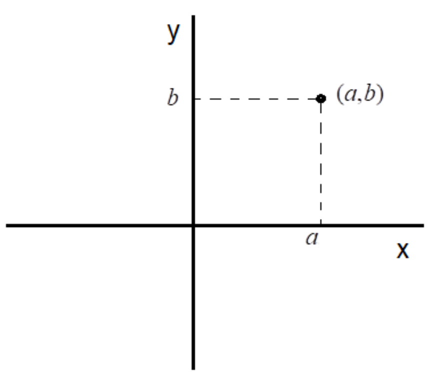
$$
\begin{aligned}
\frac{ds}{dt} &= \sqrt{(\tfrac{dx}{dt})^2 + (\tfrac{dy}{dt})^2} \\
&= \sqrt{1 + (t^{\frac{1}{2}})^2} \\
&= \sqrt{1 + t}
\end{aligned}
$$

$$
\begin{aligned}
\frac{ds}{dt}\Big|_{t=8} &= \sqrt{1 + 8} \\
&= 3 \text{ m/s (speed at } t = 8)
\end{aligned}
$$

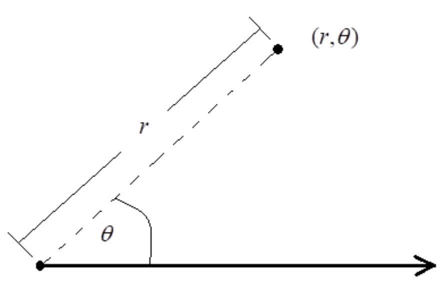
10.3a Polar Coordinates

Definition

Polar coordinates can be used to locate a point in a plane:



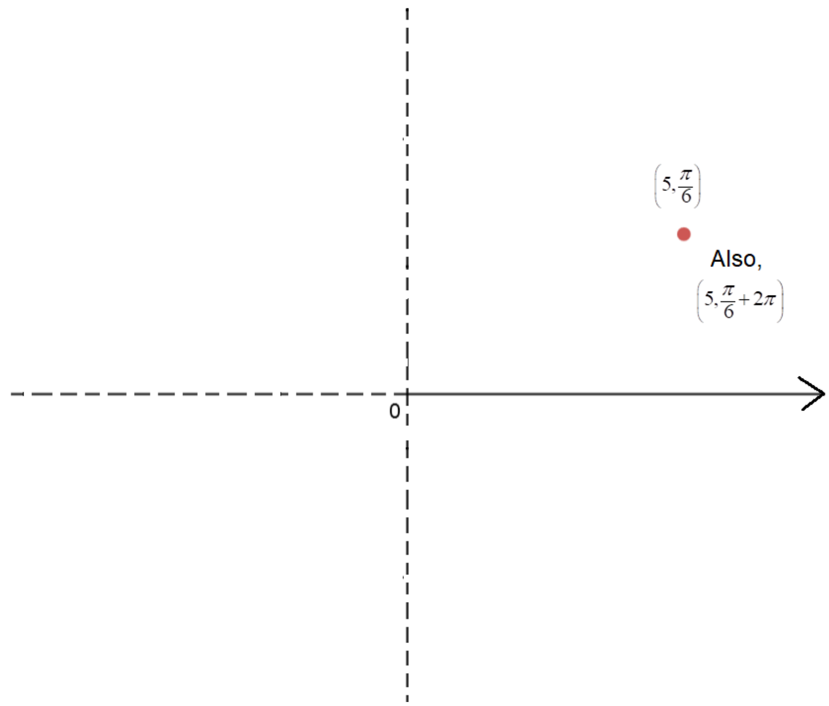
Rectangular Coordinates



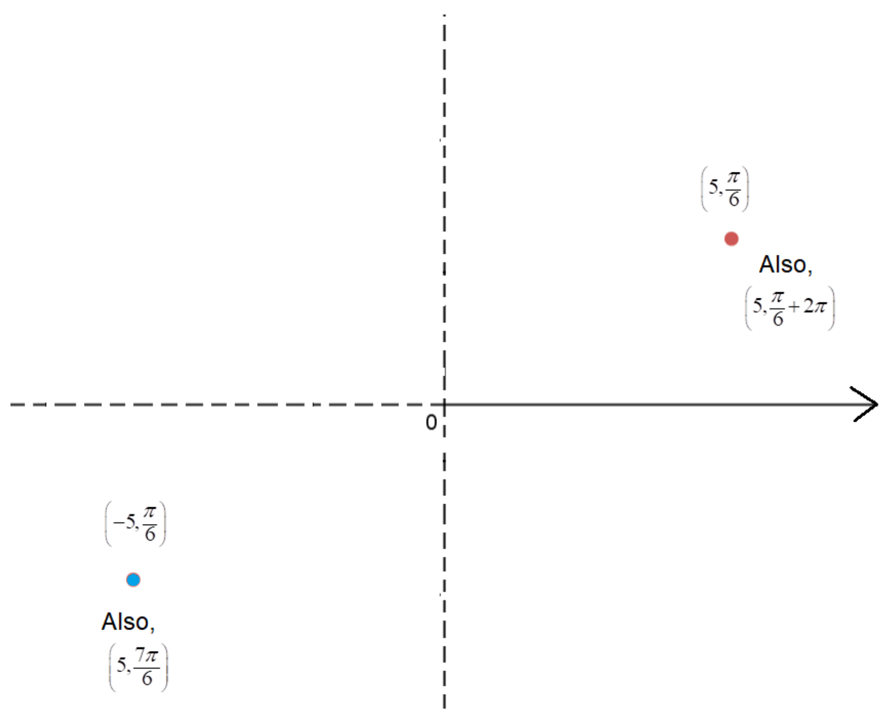
Polar Coordinates

Example

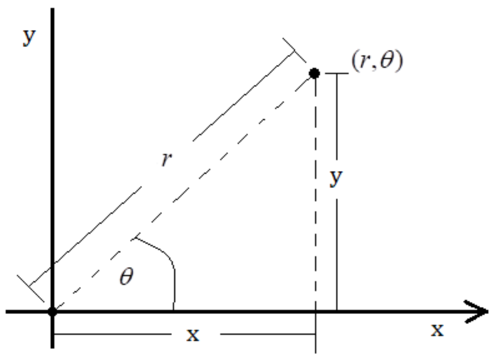
Graph $(5, \frac{\pi}{6})$



Negative r



Relationship to (x, y)



Conversion Equations:

$r^2 = x^2 + y^2$

$\tan\theta = \frac{y}{x}$

$x = r\cos\theta$

$y = r\sin\theta$

Quiz

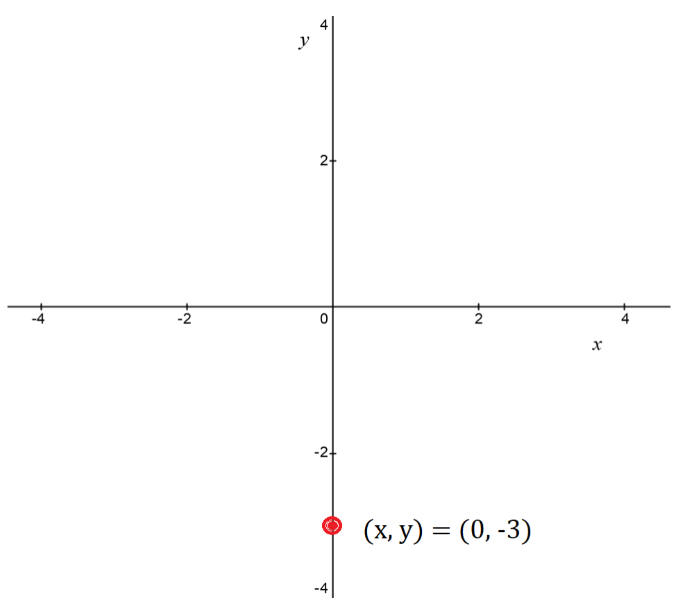
What are the polar coordinates, (r, θ),of the point indicated by the Cartesian coordinates $(0, -3)$?

A) $(0, -3)$

B) $(3, \pi/2)$

C) $(3, 3\pi/2)$

D) $(-3, -\pi/2)$



Examples

Polar Equation	Rectangular Equivalent
$r = 5$	$x^2 + y^2 = 25$
$r\cos\theta = 2$	$x = 2$
$r\sin\theta = 3$	$y = 3$

Try It 1

Find polar equation for $x^2 + (y - 3)^2 = 9$

10.3b Limaçons

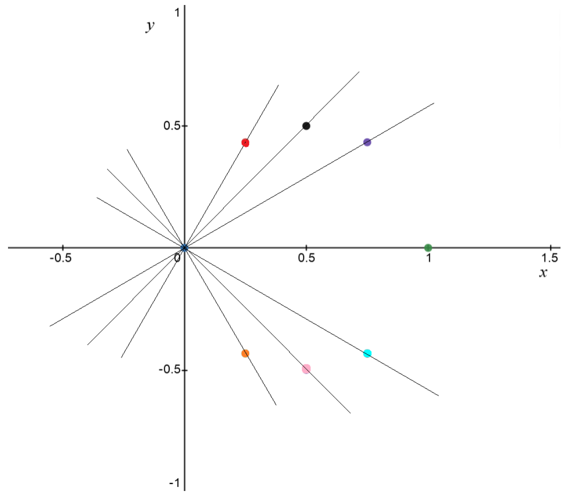
Drawing Polar Curves

Graph $r = \cos\theta$

First plot a few points...

Drawing Polar Curves - Cont

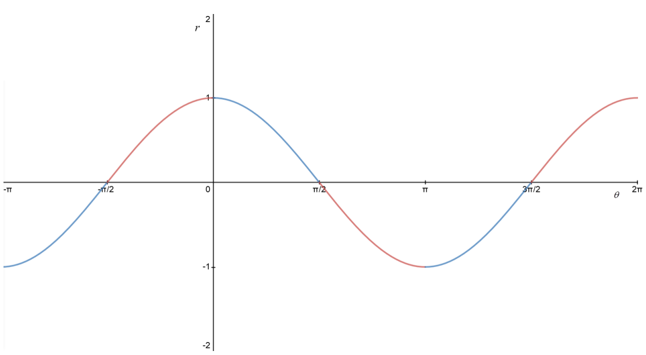
θ	r		θ	r
0	1		π	-1
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$		$\frac{7\pi}{6}$	$-\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$		$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$		$\frac{4\pi}{3}$	$-\frac{1}{2}$
$\frac{\pi}{2}$	0		$\frac{3\pi}{2}$	0
$\frac{2\pi}{3}$	$-\frac{1}{2}$		$\frac{5\pi}{3}$	$\frac{1}{2}$
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$		$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2}$		$\frac{11\pi}{6}$	$\frac{\sqrt{3}}{2}$
π	- 1		2π	1



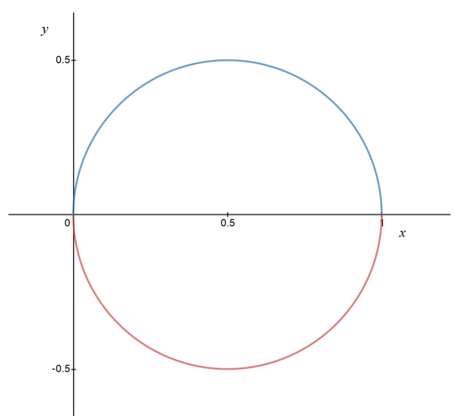
Technique

First graph $r = \cos\theta$ as a Cartesian equation.

Then transfer information to polar graph.



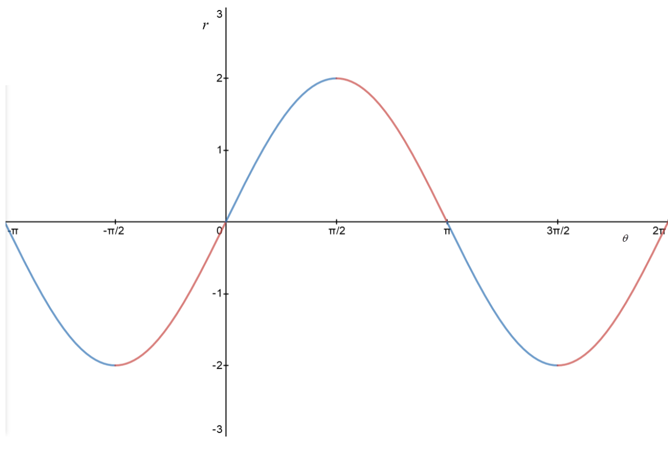
Graphing Tool



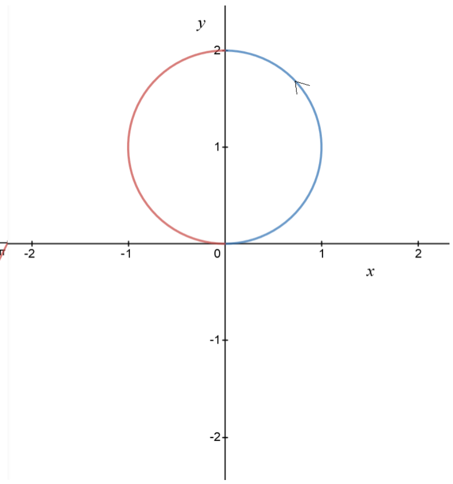
Polar Graph

Example

Graph $r = 2\sin\theta$



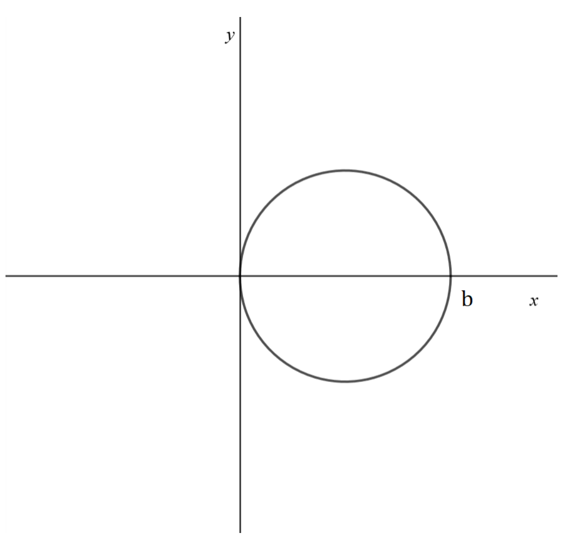
Graphing Tool



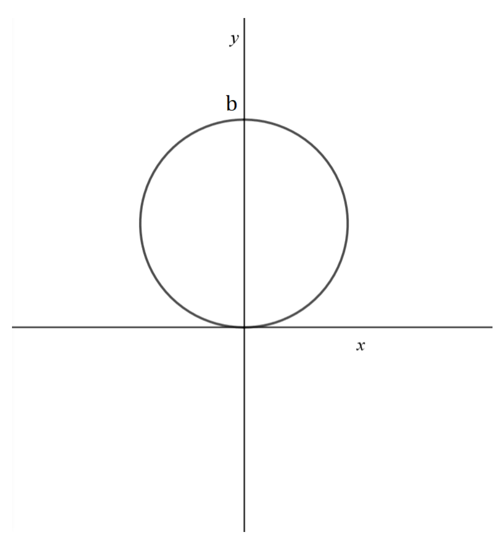
Polar Graph

Generalize

Graph $r = b\cos\theta$



Graph $r = b\sin\theta$



Limaçon

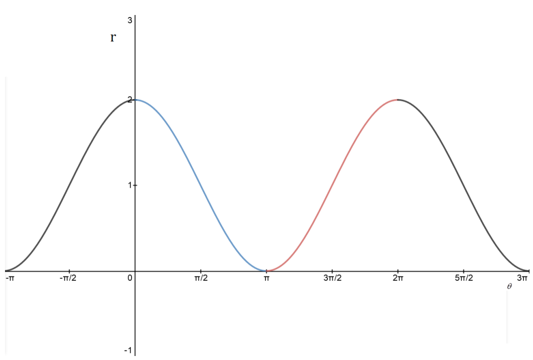
$r = a + b\cos\theta$

OR

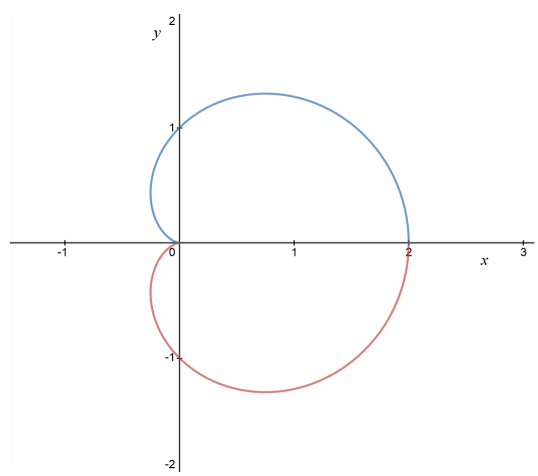
$r = a + b\sin\theta$

Example

Graph $r = 1 + \cos\theta$



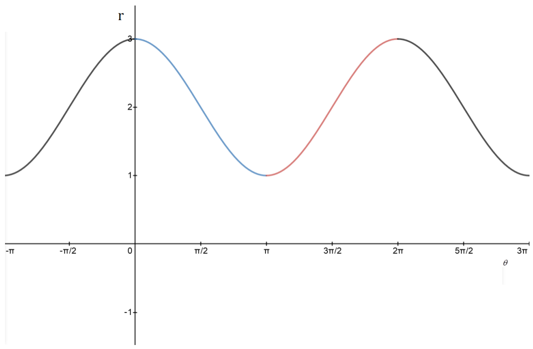
Graphing Tool



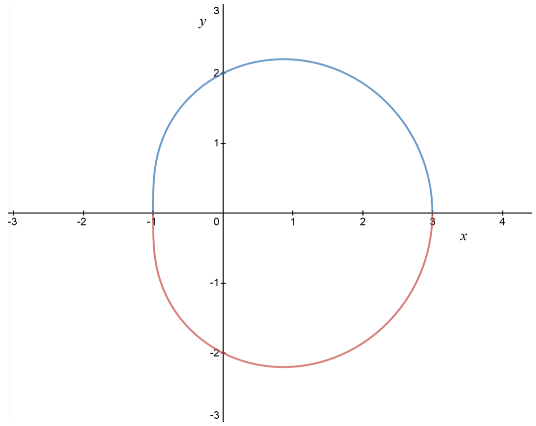
Polar Graph

Example

Graph $r = 2 + \cos\theta$



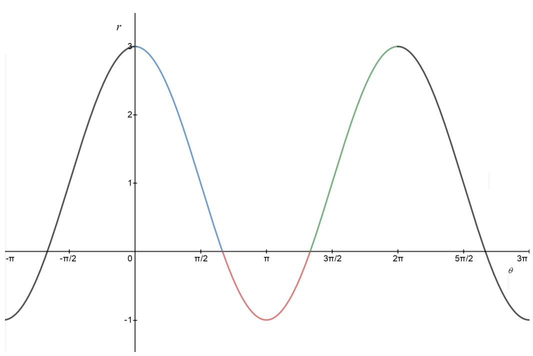
Graphing Tool



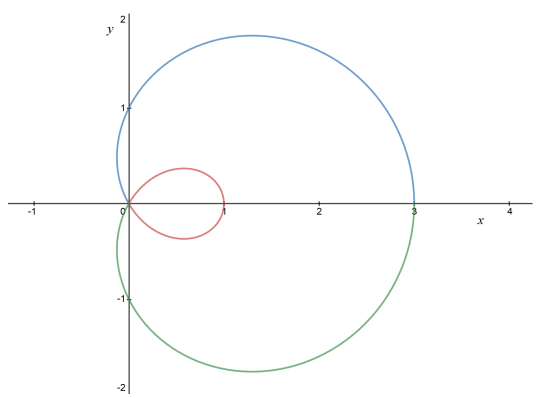
Polar Graph

Example

Graph $r = 1 + 2\cos\theta$



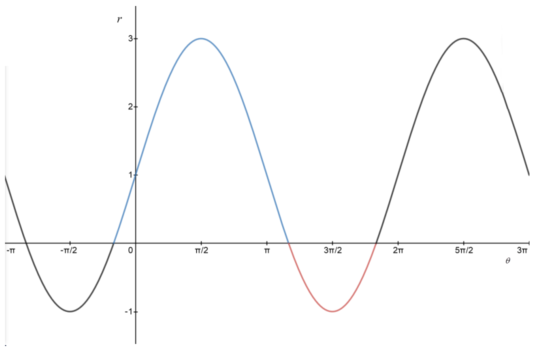
Graphing Tool



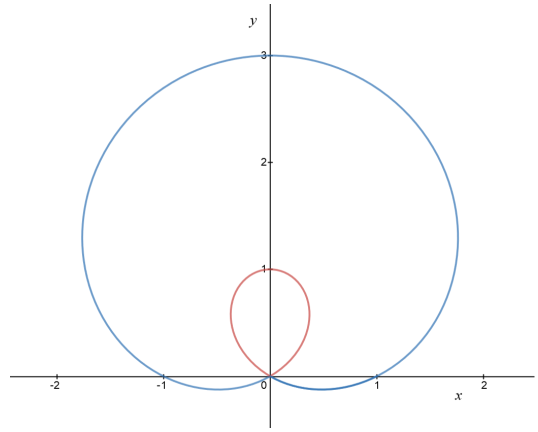
Polar Graph

Example

Graph $r = 1 + 2\sin\theta$

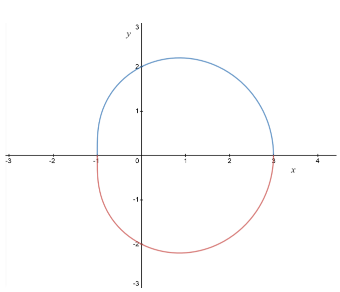


Graphing Tool

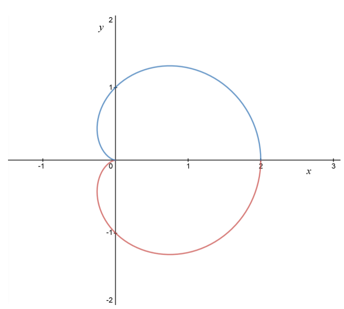


Polar Graph

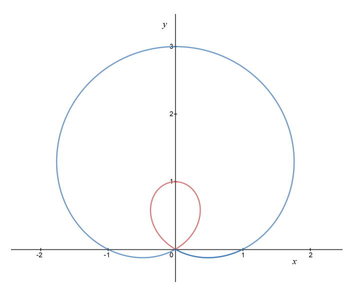
Limaçon Summary



$r = 2 + \cos\theta$



$r = 1 + \cos\theta$



$r = 1 + 2\sin\theta$

Limaçon Summary Cont

$r = a + b\cos\theta$ OR $r = a + b\sin\theta$

If $a < b$, there is a loop.

If $a = b$, the graph is a cardiod (with a cusp).

If $b < a < 2b$, there is a dimple in a simple, closed curve.

If $a = 2b$, there is a point of 0 curvature (flat spot) in a simple, closed curve.

If $a > 2b$, the simple, closed curve is convex.

Try It 2

Graph the polar curve, $r = 3 + 2\sin\theta$

Try It 3

Graph the polar curve, $r = 2 + 4\sin\theta$

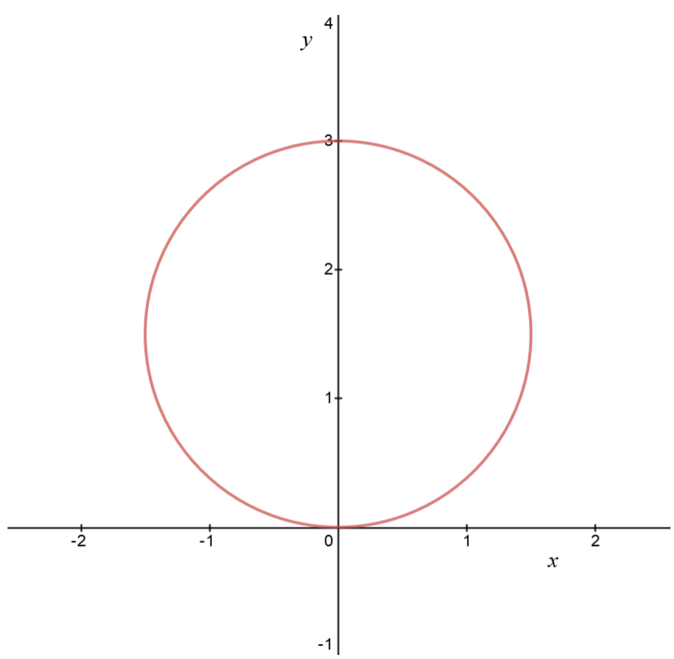
Quiz

Which of the following polar equations matches the figure?

A) $r = 3$

B) $r = 3\cos\theta$

C) $r = 3\sin\theta$



Quiz

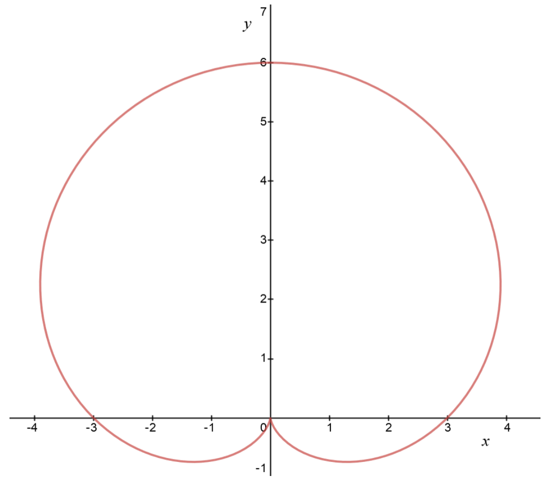
Which of the following polar equations matches the figure?

A) $r = 3 + 3\sin\theta$

B) $r = 6 + 3\sin\theta$

C) $r = 3 + \sin\theta$

D) $r = 3 + 3\cos\theta$



Try It 1 Solution

$x^2 + (y-3)^2 = 9$

$x^2 + y^2 - 6y + 9 = 9$

$(x^2 + y^2) - 6y = 0$

$r^2 - 6r\sin\theta = 0$

$r(r - 6\sin\theta) = 0$

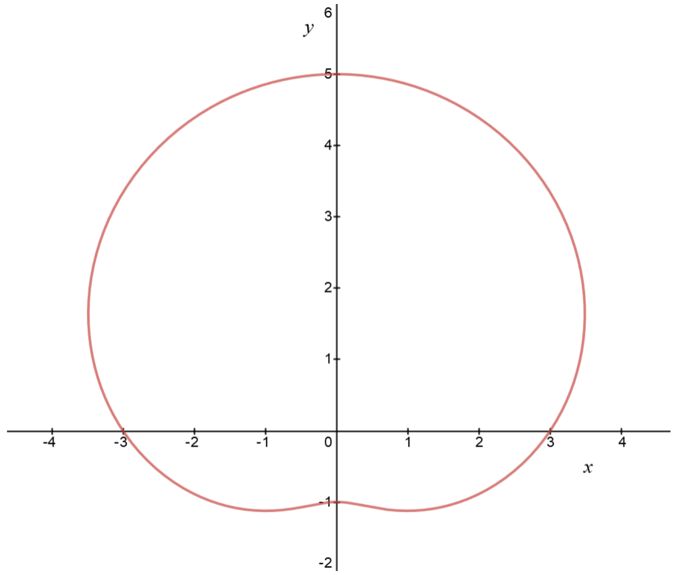
$r = 0$ OR $r = 6\sin\theta$

$r = 0$ is redundant because the pole is included in $r = 6\sin\theta$

$r = 6\sin\theta$

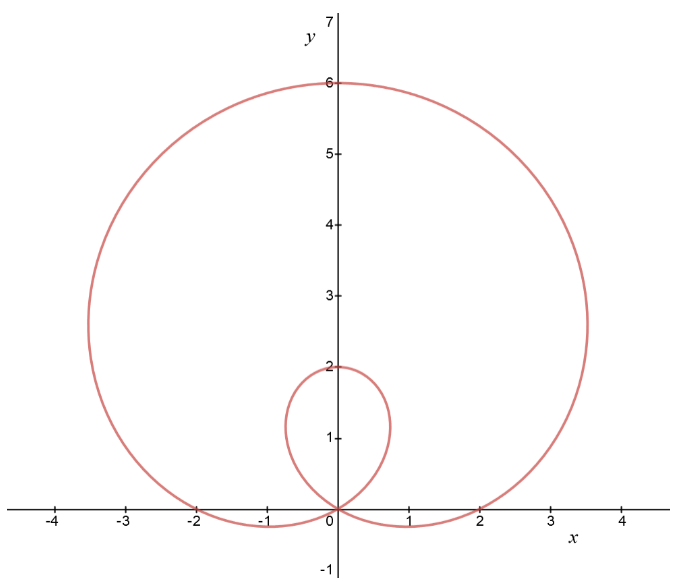
Try It 2 Solution

$r = 3 + 2\sin\theta$



Try It 3 Solution

$r = 2 + 4\sin\theta$



10.3c Roses

Rose

Roses have the form:

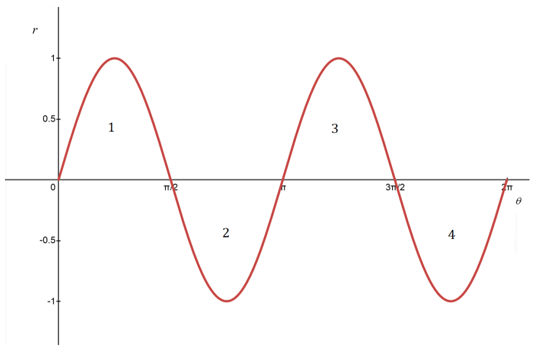
$r = a \cos n\theta$

OR

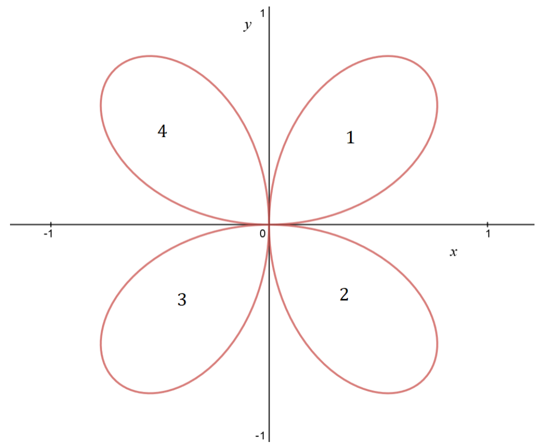
$r = a \sin n\theta$

Example 1

Graph the polar curve $r = \sin 2\theta$.



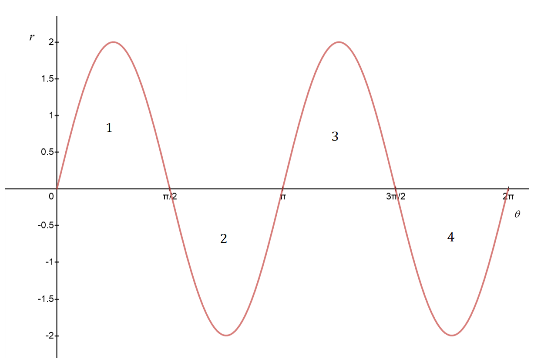
Graphing Tool



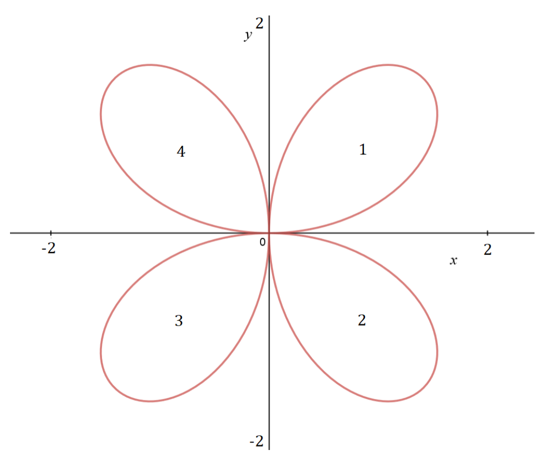
Polar Graph

Example 2

Graph the polar curve $r = 2 \sin 2\theta$.



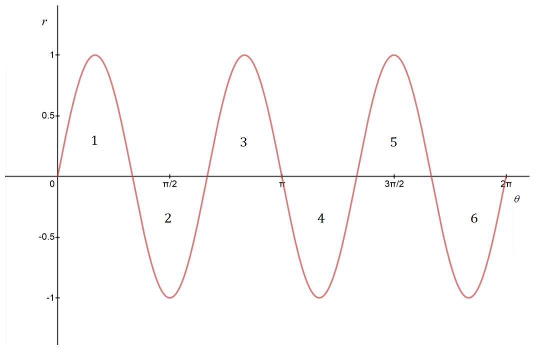
Graphing Tool



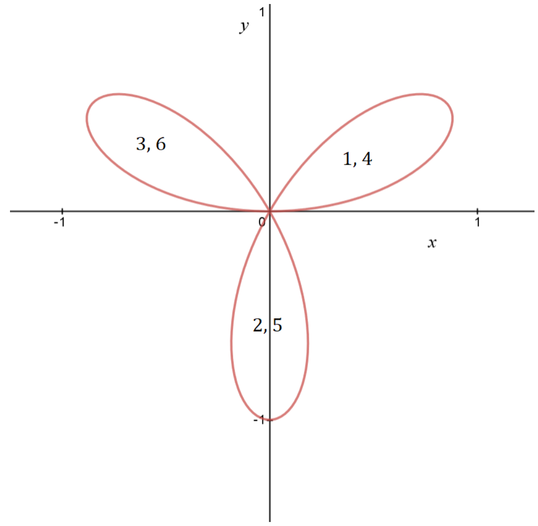
Polar Graph

Example 3

Graph the polar curve $r = \sin 3\theta$.



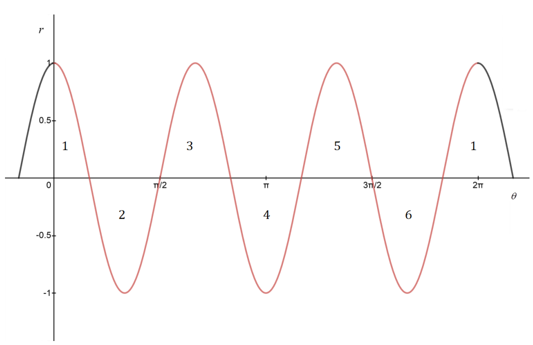
Graphing Tool



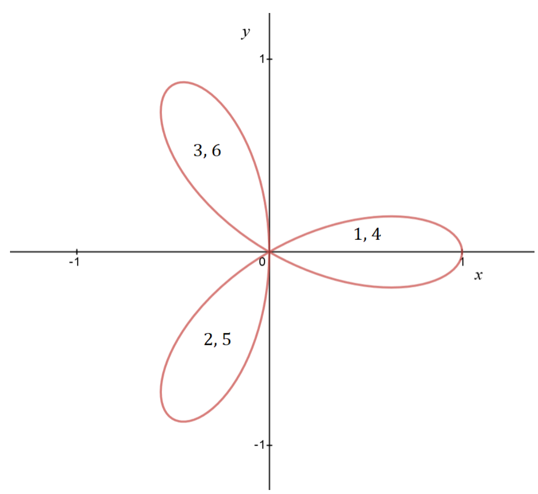
Polar Graph

Example 4

Graph the polar curve $r = \cos 3\theta$.

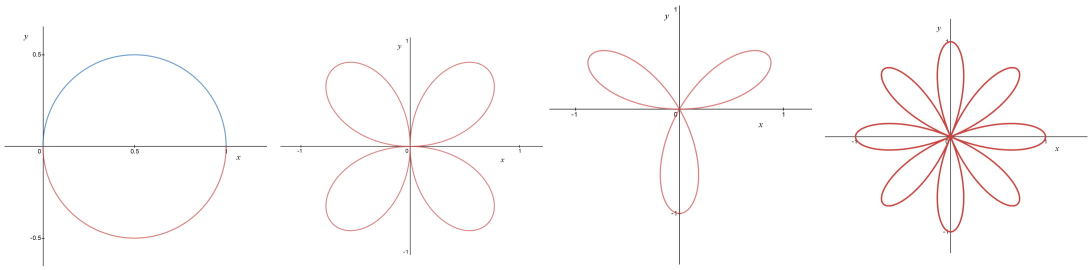


Graphing Tool



Polar Graph

Summary



$n = 1$	$n = 2$	$n = 3$	$n = 4$
1 petal	4 petals	3 petals	8 petals

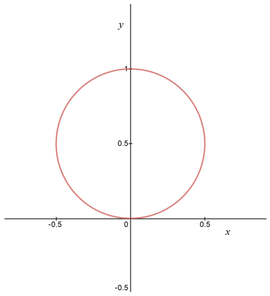
What is the rule???

Try It 1

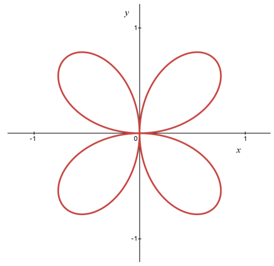
Draw the polar curve $r = 3\cos 2\theta$.

Quiz

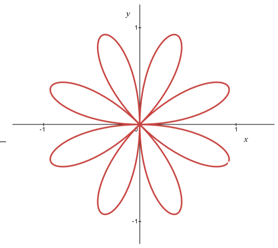
Which figure is the graph of the polar curve $r = \sin 4\theta$?



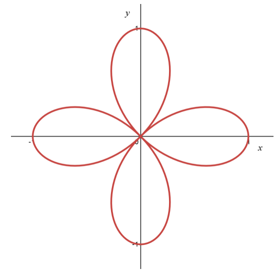
A



B



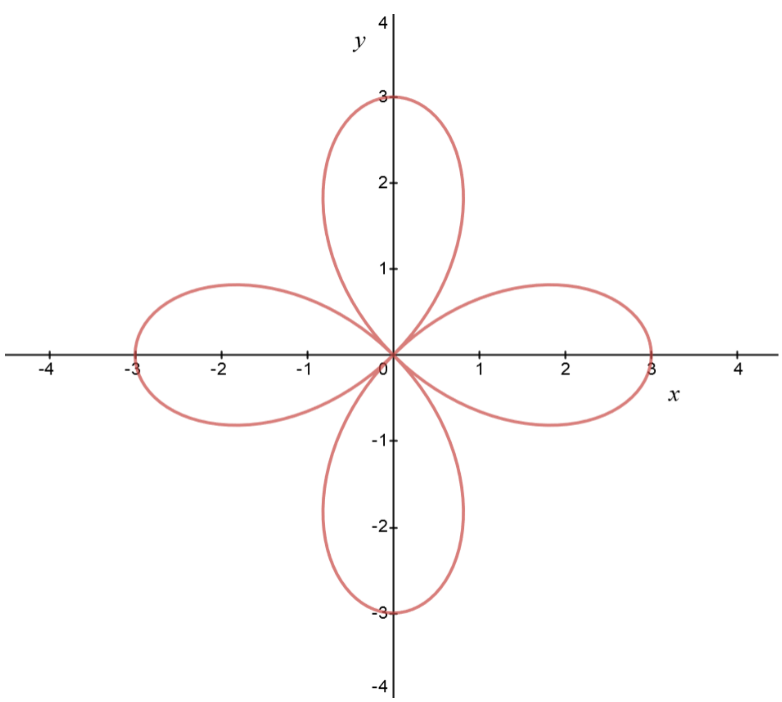
C



D

Try It 1 Solution

$r = 3\cos 2\theta$.



10.4 Areas and Lengths in Polar Coordinates

Polar to Parametric

A polar function, $r = f(\theta)$, can be written as a set of parametric equations in which θ is the parameter.

$x = f(\theta)\cos\theta$

$y = f(\theta)\sin\theta$

So, $r = 1 + \sin\theta$ can be written as

$x = (1 + \sin\theta)\cos\theta$

$y = (1 + \sin\theta)\sin\theta$

Why?

Why would we want to write a polar function as a set of parametric equations?

Because we can then use the calculus formulas for parametric equations to compute slope, arc length, etc.

Example

Find the horizontal and vertical tangents of the polar curve, $r = 1 + \sin\theta$

Solution:

Start by writing the polar function as a set of parametric equations.

$x = (1 + \sin\theta)\cos\theta$

$y = (1 + \sin\theta)\sin\theta$

Solution

$y = (1 + \sin\theta)\sin\theta$

$\frac{dy}{d\theta} = \cos\theta\sin\theta + (1 + \sin\theta)\cos\theta$

$\frac{dy}{d\theta} = \cos\theta(1 + 2\sin\theta)$

$\frac{dy}{d\theta} = 0$ when $\cos\theta = 0$ or when $1 + 2\sin\theta = 0$

$\frac{dy}{d\theta} = 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

$x = (1 + \sin\theta)\cos\theta$

$\frac{dx}{d\theta} = \cos\theta\cos\theta + (1 + \sin\theta)(-\sin\theta)$

$\frac{dx}{d\theta} = \cos^2\theta - \sin^2\theta - \sin\theta$

$\frac{dx}{d\theta} = \cos 2\theta - \sin\theta$

$\frac{dx}{d\theta} = 0$ when $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

Horizontal tangents at $\theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

Vertical tangents at $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

What about $\theta = \frac{3\pi}{2}$?

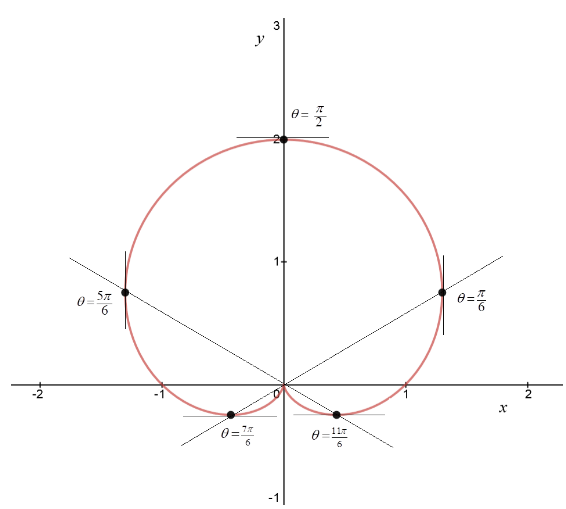
Solution - Continued

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos\theta(1+2\sin\theta)}{\cos 2\theta - \sin\theta}$$

$$\lim_{\theta \to \frac{3\pi}{2}^+} \frac{dy}{dx} = -\infty$$

$$\lim_{\theta \to \frac{3\pi}{2}^-} \frac{dy}{dx} = \infty$$

Vertical cusp at $\theta = \frac{3\pi}{2}$



Quiz

The tip of one petal of the rose, $r = \sin 2\theta$, occurs at $\theta = \pi/4$. What is an equation of the line that is tangent to the polar curve at that point?

A) $y = -x + \sqrt{2}$

B) $y = \frac{\sqrt{2}}{2}$

C) $y = x - \frac{\sqrt{2}}{2}$

Arc Length of Polar Curve

Start with parametric equations and parametric formula for arc length:

$x = r\cos\theta, y = r\sin\theta$, where $r = f(\theta)$,

$ds = \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}d\theta$

$\frac{dx}{d\theta} = r'\cos\theta - r\sin\theta$, $\frac{dy}{d\theta} = r'\sin\theta + r\cos\theta$

$ds = \sqrt{(r'\cos\theta - r\sin\theta)^2 + (r'\sin\theta + r\cos\theta)^2}d\theta$

$ds = \sqrt{(r')^2(\cos^2\theta + \sin^2\theta) + r^2(\cos^2\theta + \sin^2\theta)}d\theta$

$ds = \sqrt{(r')^2 + r^2}d\theta$

Arc Length of Polar Curve

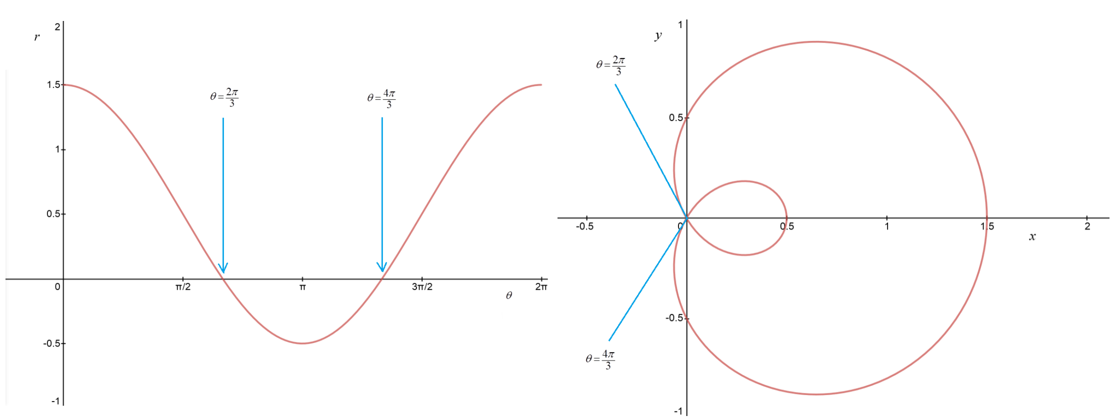
$$s = \int_{\alpha}^{\beta} \sqrt{r^2 + (\frac{dr}{d\theta})^2} d\theta$$

where $\alpha \leq \theta \leq \beta$

Example

Set up an integral to find the length of the inner loop of the polar curve, $r = \frac{1}{2} + \cos\theta$.

Solution:



Example - continued

$$s = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \sqrt{(\frac{1}{2} + \cos\theta)^2 + (-\sin\theta)^2 d\theta}$$

$$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \sqrt{\frac{1}{4} + \cos\theta + \cos^2\theta + \sin^2\theta d\theta}$$

$$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \sqrt{\frac{5}{4} + \cos\theta d\theta}$$

Quiz

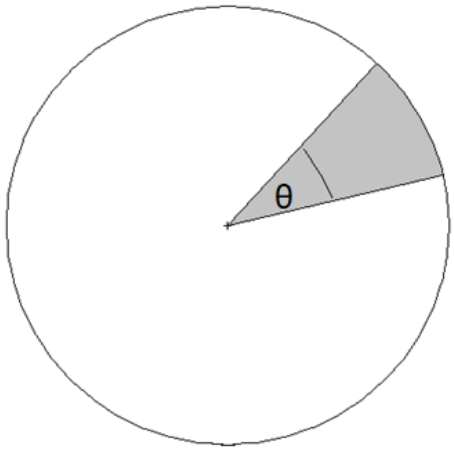
Which integral computes the length of one loop of the polar curve, $r = 3\sin 4\theta$?

A) $s = \int_0^{\frac{\pi}{4}} \sqrt{9\sin^2 4\theta + 144\cos^2 4\theta d\theta}$

B) $s = \int_0^{\frac{\pi}{4}} \sqrt{1 + 144\cos^2 4\theta} d\theta$

C) $s = \int_0^{\frac{\pi}{2}} \sqrt{9\sin^2 4\theta + 144\cos^2 4\theta d\theta}$

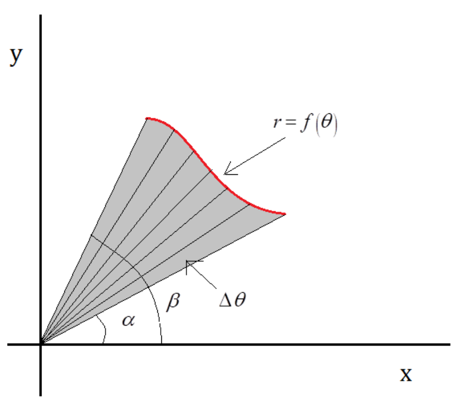
Area of a Sector of a Circle



$A = \frac{\theta}{2\pi}(\pi r^2)$

$A = \frac{1}{2}\theta r^2$

Area of a General Polar Region



$$A = \sum_{i=1}^{n} A_i$$

$$A_i \approx \frac{1}{2}(f(\theta_i^*))^2 \Delta\theta$$

$$A \approx \sum_{i=1}^{n} \frac{1}{2}(f(\theta_i^*))^2 \Delta\theta$$

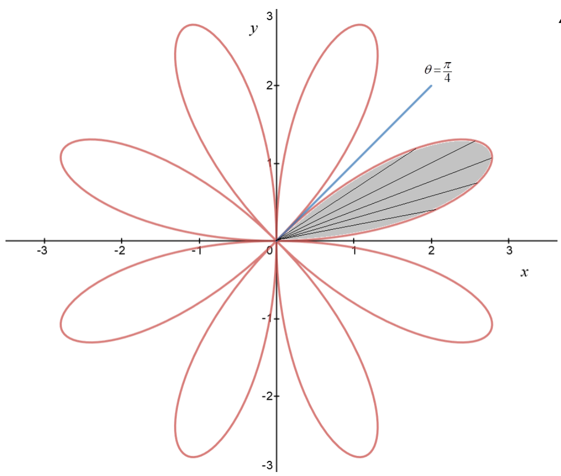
$$A = \int_{\alpha}^{\beta} \frac{1}{2}(f(\theta))^2 d\theta$$

Note: The boundary of the region is defined by angles.

Example

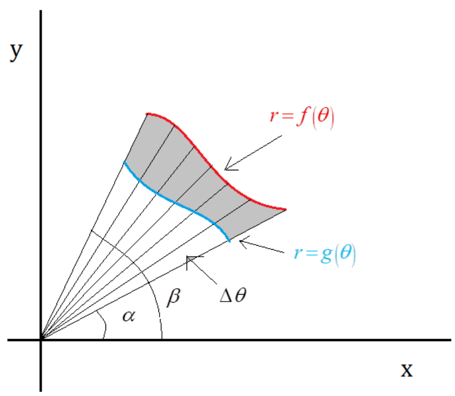
Find the area inside one loop of the polar curve $r = 3\sin 4\theta$

Solution:



$$A = \int_0^{\frac{\pi}{4}} \frac{1}{2}(3\sin 4\theta)^2 d\theta$$
$$= \frac{9}{2}\int_0^{\frac{\pi}{4}} \sin^2 4\theta d\theta$$
$$= \frac{9}{4}\int_0^{\frac{\pi}{4}} (1-\cos 8\theta) d\theta$$
$$= \frac{9}{4}(\theta - \frac{1}{8}\sin 8\theta)\Big|_0^{\frac{\pi}{4}}$$
$$= \boxed{\frac{9\pi}{16}}$$

Area of a Region Between Two Polar Curves



$A_i = A_{fi} - A_{gi}$

$A_i = \frac{1}{2}(f(\theta_i^*))^2\Delta\theta - \frac{1}{2}(g(\theta_i^*))^2\Delta\theta$

$$A = \int_\alpha^\beta \frac{1}{2}[(f(\theta))^2 - (g(\theta))^2]d\theta$$

Try It 1

Find the area of the region inside the polar curve $r = 1 + \cos\theta$ and outside $r = 1$.

Try It 2

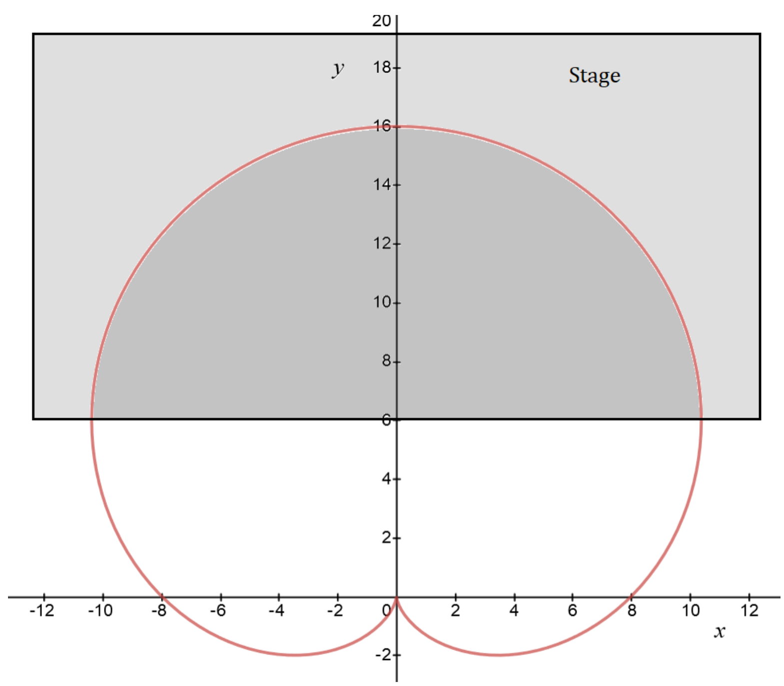
Find area inside the polar curve $r = 1 + \sin\theta$ and outside the polar curve $r = 2\sin\theta$.

Try It 3

Find the area inside both of the polar curves, $r = \cos\theta$ and $r = \sin\theta$.

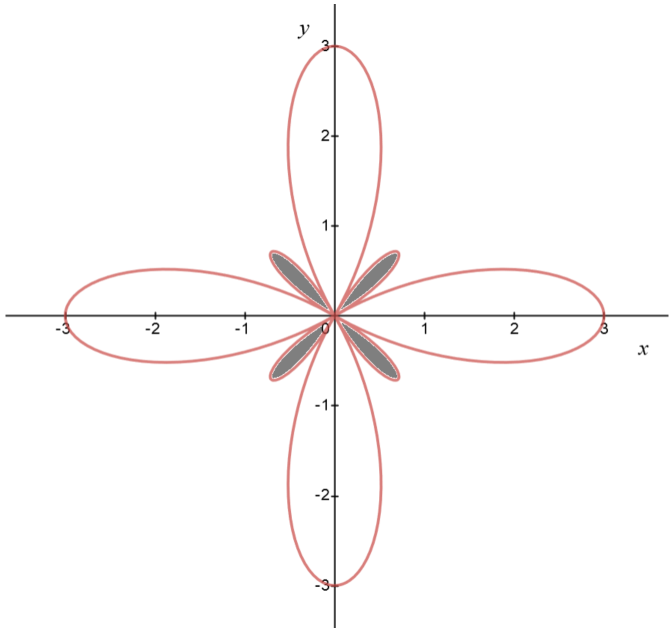
Try It 4

Find the area of the rectangular stage shown that lies within the pickup region of the microphone, which is the polar region given by the equation, $r = 8 + 8\sin\theta$



Try It 5

The graph of the polar curve, $r = 1 + 2\cos 4\theta$, is shown. Find the shaded (black) area.



Try It 1 Solution

The appropriate area is shaded below.

The intersection points are $\theta = -\frac{\pi}{2}$ and $\theta = \frac{\pi}{2}$

$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(r_0^2 - r_i^2)d\theta$$

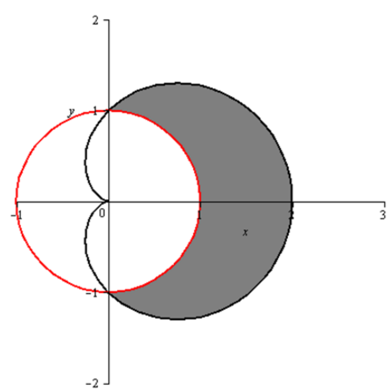
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}((1+\cos\theta)^2 - 1^2)d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(1 + 2\cos\theta + \cos^2\theta - 1)d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(2\cos\theta + \frac{1}{2}(1+\cos 2\theta))d\theta$$

$$= \frac{1}{2}(2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 2 + \frac{\pi}{4}$$



Try It 2 Solution 1

The appropriate region is shaded below.

Intersection points:

$1 + \sin\theta = 2\sin\theta$

$\sin\theta = 1$

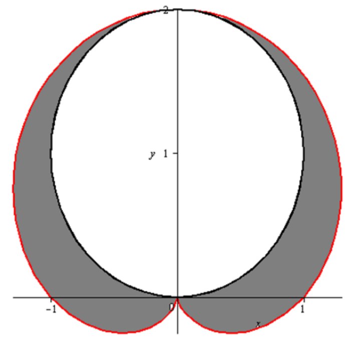
$\theta = \frac{\pi}{2}$

Also, the pole is an intersection.

Subtract area of circle from area inside red curve.

$A = \int_0^{2\pi} \frac{1}{2}(1 + \sin\theta)^2 d\theta - \int_0^{\pi} (2\sin\theta)^2 d\theta$

$= \int_0^{2\pi} \frac{1}{2}(1 + 2\sin\theta + \sin^2\theta) d\theta - \pi$ (area of circle)



Try It 2 Solution 2

$A = \int_0^{2\pi} \frac{1}{2}(1 + 2\sin\theta + \sin^2\theta) d\theta - \pi$

$= \int_0^{2\pi} \frac{1}{2}(1 + 2\sin\theta + \frac{1}{2}(1 - \cos 2\theta)) d\theta - \pi$

$= \frac{1}{2}(\frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\sin 2\theta)|_0^{2\pi} - \pi$

$= \frac{\pi}{2}$

Try It 3 Solution 1

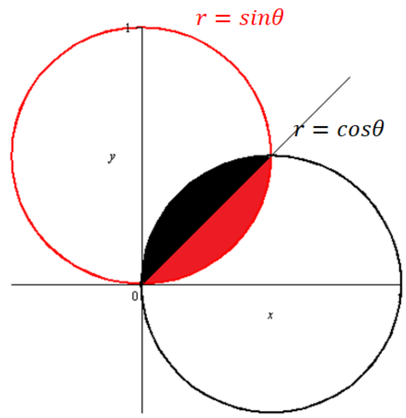
The curves are shown, with the appropriate area shaded.

The red region is the area within $r = \sin\theta$ from $\theta = 0$ to $\theta = \frac{\pi}{4}$.

The black region is the area within $r = \cos\theta$ from $\theta = \frac{\pi}{4}$ to $\theta = \frac{\pi}{2}$.

Total Area:
$A = \int_0^{\frac{\pi}{4}} \frac{1}{2}\sin^2\theta d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2}\cos^2\theta d\theta$ But

the red area and the black area are equal because of symmetry, so
$A = 2\int_0^{\frac{\pi}{4}} \frac{1}{2}\sin^2\theta d\theta$



Try It 3 Solution 2

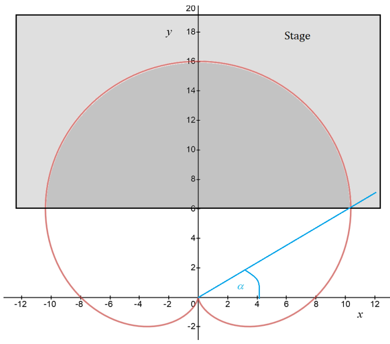
$A = 2\int_0^{\frac{\pi}{4}} \frac{1}{2}\sin^2\theta d\theta$

$= \int_0^{\frac{\pi}{4}} \frac{1}{2}(1 - \cos 2\theta) d\theta$

$= [\frac{1}{2}(\theta - \frac{1}{2}\sin 2\theta)]_0^{\frac{\pi}{4}}$

$= \frac{1}{2}(\frac{\pi}{4} - \frac{1}{2})$

Try It 4 Solution 1



Horizontal line at $y = 6$ has the polar equation, $r = 6\csc\theta$

Area is 2 times the right-hand area:

$A = 2\int_{\alpha}^{\frac{\pi}{2}} \frac{1}{2}[(8+8\sin\theta)^2 - (6\csc\theta)^2]d\theta$

Find α:

On outer curve, $y = (8+8\sin\theta)\sin\theta$ and $y = 6$ at $\theta = \alpha$, so

$(8+8\sin\alpha)\sin\alpha = 6$

$8\sin\alpha + 8\sin^2\alpha = 6$

$4\sin^2\alpha + 4\sin\alpha - 3 = 0$

$\sin\alpha = \frac{-4\pm\sqrt{16-(4)(4)(-3)}}{8}$

$\sin\alpha = \frac{1}{2}$ OR $-\frac{3}{2}$

Only $\alpha = \frac{\pi}{6}$ is a solution.

Try It 4 Solution 2

$$A = 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2}[(8+8\sin\theta)^2 - (6\csc\theta)^2]d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} [64 + 128\sin\theta + 64\sin^2\theta - 36\csc^2\theta]d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} [64 + 128\sin\theta + 32(1-\cos 2\theta) - 36\csc^2\theta]d\theta$$

$$= [96\theta - 128\cos\theta - 16\sin 2\theta + 36\cot\theta]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= 48\pi - 0 - 0 - 0 - [16\pi - 64\sqrt{3} - 8\sqrt{3} + 36\sqrt{3}]$$

$$= 32\pi + 36\sqrt{3}$$

Try It 5 Solution 1

Find the area of one small petal and multiply by 4.

$A = 4\int_{\alpha}^{\beta} \frac{1}{2}(1 + 2\cos 4\theta)^2 d\theta$

Find α and β:

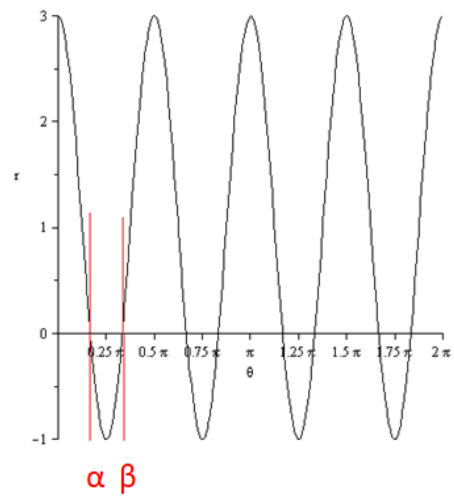
$1 + 2\cos 4\theta = 0$

$\cos 4\theta = -\frac{1}{2}$

$4\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \cdots$

$\theta = \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \cdots$

$\alpha = \frac{\pi}{6}$, $\beta = \frac{\pi}{3}$



Try It 5 Solution 2

$A = 4\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2}(1 + 2\cos 4\theta)^2 d\theta$

$= 2(\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 1 + 4\cos 4\theta + 4\cos^2 4\theta)d\theta$

$= 2\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (1 + 4\cos 4\theta + 4(\frac{1}{2})(1 + \cos 8\theta))d\theta$

$= 2[\theta + \sin 4\theta + 2(\theta + \frac{1}{8}\sin 8\theta)]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= 2[3\theta + \sin 4\theta + \frac{1}{4}\sin 8\theta]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= 2[\pi - \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{8} - (\frac{\pi}{2} + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{8})]$

$= \pi - \frac{3\sqrt{3}}{2}$