Specimen Analyseos Figuratae in Elementis Geometriae

A Sample of Figurate Analysis in the Elements of Geometry

G. W. Leibniz, 1683

Tr. David Jekel, Matthew McMillan

I call figurate analysis that which affords a method of representing figures by letters signifying points, and of discovering and demonstrating their effects and properties, so that not only magnitudes, as in the algebraic calculus, but also situations themselves are directly exhibited by this new type of calculus. Here we will give a sample of this technique[1] on the *Elements* of Euclid, and in the process we will take up the lemmas, axioms, definitions, and other propositions we need.

Now whatever is explained in the first 10 books of Euclid, this is to be understood *to be in one plane* — so we won't need to be constantly reminded of this.[2]

Regarding Book I of the *Elements*

Proposition 1. Over a given straight line with endpoints AB, to construct the equilateral triangle ABC.

(1) $AC = AB$ by hypothesis. (2) $BC = BA$ by hypothesis. And C is that which satisfies these. (3) Let $AM = AB$. (4) Then (by Postulate 1[3]) $\overline{M}$ becomes *the circumference of a circle with centre A and interval*[4] AB (by Definition 1). (5) Let $BR = BA$. (6) Then $\overline{R}$ becomes the circumference of a circle with centre B and radius BA (as in 4). (7) Now some R is M (by Lemma 1). (8) That is, $\overline{R}$ and $\overline{M}$ *meet each other*[5] (from 7 by

[1] Latin *artificii*. For Leibniz, 'art' or 'artifice' contrasts with labour and ingenuity. Cf. DAS par. 3 about Euclid: 'a work more of toil than of method and art, although now and then an art to the process seems to be concealed'.

[2] Leibniz believed that space is essentially 3-dimensional.

[3] Leibniz collected two lists of postulates, definitions, and axioms in the margins of the first two manuscript pages. We have provided the two lists in two boxes below.

[4] Latin *intervallo*. This frequently indicates a circle's radius.

[5] Leibniz crossed out *se secant* and replaced it with *occurrunt*. In 9 he replaced *intersectio* with *occursus*.

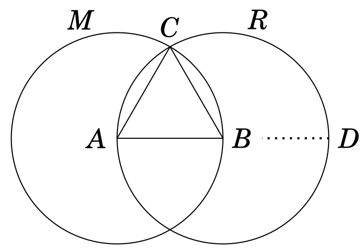

Figure 1

Definition 2). (9) Given curves $\overline{R}$ and $\overline{M}$ (by 4 and 6) meeting each other (by 7, 8), one has their meeting (by Postulate 2), namely an R which is an M. (10) But this is C (by 1 and 3 and by 2 and 5). (11) Therefore we have C. (12) AB is already given (by hypothesis). (13) Therefore we have ABC. Q.E.F.[6]

Definition 1: Circumference, centre, interval. Book 1, Proposition 1, item 4.
Postulate 1: To describe a circle with a given centre and interval.
Definition 2: The meeting. Item 8.
Postulate 2: Given things that meet, one has their meeting. Item 9.
Definition 3: Being inside a circle. Item 16.
Postulate 3: A line can be extended from one endpoint to any distance whatever. Items 17, 26.
Axiom 1: The whole is greater than the part. Item 20.
Definition 4: Being outside a circle. Item 21.
Axiom 2: A continuum that is inside and outside a figure meets its circumference. Item 24.
Definition 5: A straight line is the shortest path between extremes, i.e. the distance of two points. Item 32.
Axiom 3: All parts, having no common part, equal the whole. Item 34.
Axiom 4: What moves is not in multiple places simultaneously. Item 38.

[6] *Quod Erat Faciendum*, i.e. 'that which was to be made'. Here Leibniz had 'Qu. Er. Fac.'. We standardize Leibniz's various forms of this expression to 'Q.E.F.' throughout the essay. Similarly we standardize 'Quod erat demonstrandum' to 'Q.E.D.' throughout.

Lemma 1. If two circles, MB[7] from A and RA from B, each have their centre in the other's circumference, B in $\overline{M}$ and A in $\overline{R}$, their circumferences meet someplace, at C. [See Figure 1.]

Supposing the same things as before, (14) $BA = BA$ (by itself). (15) Therefore some R is A (by 5). (16) And so some R is inside the circle $A.\overline{M}$ (by Definition 3, since we say *a point is inside a circle* if its distance to the centre is less than the radius). (17) Let AB be extended from B to D (by Postulate 3) (18) so that $BD = BA$ (by Lemma 2). (19) Therefore $AD = AB + BD$ (by Lemma 3). (20) Therefore $AD \sqcap AB$ (the whole [is greater] than the part, by Axiom 1). (21) Therefore D is outside $A.\overline{M}$ (by Definition 4, for we say *a point is outside a circle* if its distance from the centre is greater than the radius). (22) But D is R (by 18 and 5). (23) Therefore, some R is outside $A.\overline{M}$. (24) And so (by 16 and 23) some R is on $\overline{M}$ (by Axiom 2, every continuum $\overline{R}$ which is both inside and outside the figure $A.\overline{M}$, is also on its circumference $\overline{M}$. *Scholium:* From here, although nothing would follow formally from pure particulars, materially, in continua, 24 follows from 16 and 23.[8])

(25) Therefore, (from 24 by Definition 2 at 8) $\overline{M}$ and $\overline{R}$ meet each other. Q.E.D.

Lemma 2. Line AB from centre B (indeed from any point inside the circle) can be extended so that it also meets the circumference of the circle $\overline{R}$ someplace, at D.

(26) For it can be extended to an arbitrarily large distance (by Postulate 3 at 17). (27) Therefore [it can be extended] to E such that $BE \sqcap BA$. (28) Therefore (by Definition 4 at 21), the line has been extended out-

[7] MA in the manuscript, seemingly by mistake, though it is also possible that he intended 'MA from A and RB from B'.

[8] A *scholium* is a sort of interpretive marginal note or explanatory comment added to a primary text. They were commonplace in the scholastic period. This one is a reference to the form-matter distinction applied to syllogisms; the form is the argument structure or type; the matter is the actual contents or subject. Also in connection with syllogisms, 'particular' premises ('some A is B') were said not to have formal consequences. So Leibniz is noting that 24 is not inferred from the logical structure of 16 and 23 alone, but results after his Axiom 2 about the continuum is additionally assumed.

side the circle $B.\overline{R}$. (29) The same line is in the circle at its centre B (by Definition 3 at 16). (30) Therefore (by Axiom 2 at 24[9]), it meets its circumference someplace, at D.

Corollary of Lemma 2.[10] A line AD passing through a point B inside a circle meets the circle twice, at A and D. For the line AB extended from B (receding from A) meets the circumference someplace, at D, by Lemma 2. And DB extended from B (receding from D) will meet the circle someplace, at A.

Lemma 3. If three points A, B, D[11] are on a line, the distance of some two of them, say AD, coincides with $AB+BD$, the sum of the distances of a third one B from A, D.[12]

(31) The three points are on a line (by hypothesis). (32) The *line* AD is the shortest path, or the distance, between the extremes A and D (by Definition 5). (33) Therefore the point B on the line is less distant from the extreme A than the extremes A and D are from each other. (34) And $AB+BD=AD$ (all the parts having no common part [are equal] to the whole by Axiom 3). (35) But AB is the distance between A and B, and BD the distance between B and D (by Definition 5 at 32). (36) It remains for us to show that when three points exist on a line, one can get a line which ends in two of them while including the third. (37) Indeed let us suppose that some point runs along the line which they

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[9] The manuscript has '23'.

[10] In the manuscript, this is written in the margin beside Lemma 3.

[11] The manuscript has '$A.B.D$', which appears to be an intrusion of a notation with dots that Leibniz uses elsewhere but has otherwise avoided in this essay.

[12] Leibniz wrote two versions of this lemma and proof. In the first version, A and B are the endpoints, while in the second version, A and D are the endpoints. He cancelled the first version's proof and the second version's statement. In addition, he mixed up B and D in two places in the first version's statement. We have opted to make A and D the endpoints for both statement and proof. Finally, at 'of a third one B from A, D', Leibniz actually wrote 'tertiae D ab ipsis A, B', apparently assigning 'tertia' to an elided feminine noun (perhaps 'distantia') instead of the neuter 'punctum' D as is really necessary to make sense of the clause. It is worth noting that he crossed out two different drafts of a marginal figure he drew showing three line segments with various permutations of A, B, D at the endpoints.

are in [*insunt*[13]]. (38) It will reach them successively (by Axiom 4). (39) Therefore let A be the first, B the second, D the third. (40) Therefore the portion of the line it traverses between A and D will terminate in A and D but include B.

Addition 1. If two circles, $A\overline{M}$, $B\overline{R}$, of equal radii have radius greater than half the distance of their centres $A.B$,[14] they will meet each other in C outside the line through their centres.

(41) The line AB intersects $\overline{R}$ twice, at E and D (by the corollary of Lemma 2). [See Figure 2.] (42) And similarly $\overline{M}$ at F. (43) Let E be [on the line] from B toward A (44) and let D be [on the line] from B moving away from A (45) and let F be [on the line] from A toward B, (46) then AE will be less than AF (see 58 shortly). (47) Since (in view of 43) E falls between B and A, or B between E and A, (48) AE will be (by Lemma 3) the difference between AB and BE, (49) i.e. between AB and AF, ((50) since $AF = BE$ (by the hypothesis)). (51) Now if $AF \sqcap AB$, (52) we will have $AF = AB + AE$ (by 48, 49). (53) Therefore, $AF \sqcap AE$. (54) But if $AB \sqcap AF$, then $AB - AF = AE$ (by 48, 49). (55) Now $AF \sqcap \frac{1}{2}AB$ (by the hypothesis). (56) Therefore $AE \sqcap \frac{1}{2}AB$. (57) Therefore $AF \sqcap AE$. (58) Either way, therefore, $AE \sqcap AF$ as claimed in item 46. (59) In turn, AD is greater than AF. (60) For $AD = AB + BD$ (by Lemma 3), (61) $= AB + AF$ (because $BD = AF$[15] by hypothesis). (62) Therefore, some R is inside $\overline{M}$, namely E (since $AE \sqcap AF$, i.e. AM, by 58). (63) Some R is outside $\overline{M}$, namely D (since by 59, $AD \sqcap AM$, i.e. AF[16]). (64) Therefore (by Axiom 2), some R is on $\overline{M}$, say C.

Addition 2. Over a given base AB, to construct an isosceles triangle whose legs AC or BC are of a given magnitude G, which should be greater than half the base AB. [See Figure 2.]

(65) With centres A and B, (66) interval $= G$ (by Proposition 2 demon-

[13] This may or may not be used here in the technical sense that Leibniz developed for *inesse*, as part of his mereological calculus, around the period of this essay or not long after. Cf. SGL 'Some part of the doctrine about'.

[14] The '$A.B$' was inserted later, which may explain the dot.

[15] The manuscript has '$BD = BF$' but the argument calls for $BD = AF$.

[16] 'AM' in the manuscript.

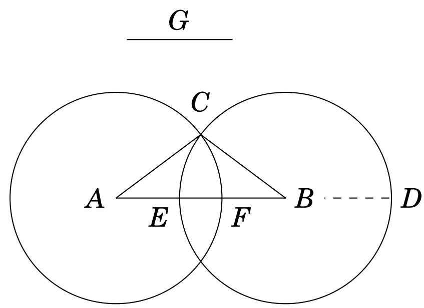

Figure 2

strated independently of this), (67) let circles be described (Postulate 1) which intersect someplace, at C. By Addition 1 we will have $AC = G$ and $BC = G$. Q.E.F.

Proposition 2. At a given point A, to put a line AG equal to a given line BC. [See Figure 3.]

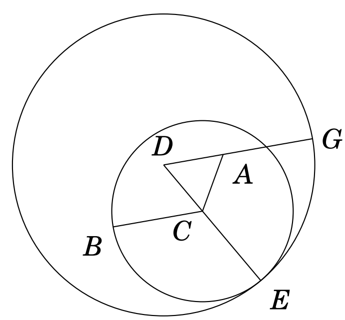

Figure 3

Solution. (1) Let AC be connected. (2) Let equilateral triangle ACD be constructed upon it (by 1, I[17]). (3) Let a circular circumference BE

[17]'*1. prim*' in the manuscript, literally '1 of the first', an elliptical reference to Proposition 1, Book I of Euclid's *Elements*. More such references follow.

be described with centre C, radius BC (by Postulate 1), (4) which the line DC extended from C (by Postulate 3) (5) meets someplace, at E (by Proposition 1, Lemma 2). (6) Let a circle be described with centre D, radius DE (Postulate 1), (7) which the line DA extended from A meets someplace, at G (by the assertion in Lemma 2). (8) Then $AG = BC$. (9) For $DC + CE = DE$ (by 5 and Lemma 3 of Proposition 1). (10) $= DG$ (by 6 and 7), (11) $= DA + AG$ (by 7 and Lemma 3 of Proposition 1). (12) $= DC + AG$ (by 2). (13) Therefore (by 9 and 12) $AG = CE$, (14) $= BC$ (by 3, 4).

Scholium to Proposition 2. The analysis by which this construction can be discovered is like this: A line [segment] is to be put at point A equal to the line put at point C. By a line put at point C one can understand not only BC but also any other, such as CE, equal to CB, i.e. drawn from C to the circumference of the circle CBE. Since, therefore, points A and C should be treated in the same way, the line AC should also be treated in such a way that with respect to C it is treated just as it has been treated with respect to A. Therefore, let some point D be sought relating in the same way to A and C, which is made by by constructing an equilateral or isosceles triangle ADC over AC. The line drawn from D, extended through C, will meet the circle in E; in the line from D extended through A let DG be taken equal to DE, and we will have $AG = CE$, because G is found in the same way with respect to A as E with respect to C.

Proposition 3. Given two lines, A and a larger one BC, from [BC] cut off BE equal to A. [See Figure 4.]

(1) Form $BD = A$ (by 2, I[18]) (2) and form $\overline{M}$ such that $BM = BD$ (by Postulate 1). (3) Now BC is inside $\overline{M}$ at B (as in item 16 of Proposition 1) (4) and outside $\overline{M}$ at C ((5) because $BC \sqcap A$ by hypothesis, (6) therefore $\sqcap BM$ by 1, 2). (6) Therefore (by Axiom 2) BC meets $\overline{M}$ someplace, at E. (7) Therefore BE is a part of BC. (8) And $BE = BD = A$.

Also thus:

(3) Let $\overline{Y} \infty BC$. (4) Then $BY + YC = BC$ (by Lemma 3 of Proposition 1). (5) $BC \sqcap A$ (by hypothesis). (6) Therefore some $BY = A$ (indeed by

[18] Leibniz wrote '3. prim.', but it seems he meant Proposition 2.

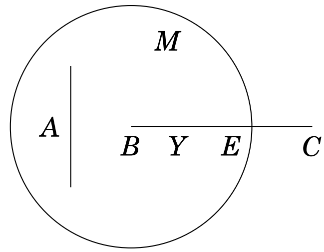

Figure 4

> *Definition 6:* If A is equal to a part of B, then A is said to be *less*, B *greater*. Book 1, Proposition 3. Item 6.
> *Definition 7: Rectilinear angles* are said to be *equal* if they are congruent. Proposition 4. Item 3.
> *Axiom 5:* Given the points at which the angles of a figure stand, the figure is given. [Proposition 4.] Item 9. This can be considered a postulate.
> *Axiom 6:* Those which are given (determinately) in the same way from congruent givens are congruent. [Proposition 4,] item 11.

Definition 6. Greater and less. A is *less* when some part BY of the other one, BC, which is called *greater*, is equal to it). (7) Therefore some $BY = BM$ (by 6, 1, 2). (8) Therefore some Y is M. (9) Let it be E. Therefore E is given (by Postulate 2). (10) Therefore also $BE = A$ (by 6), (11) a part of BC (by 4). Q.E.F.

Proposition 4. If two triangles BAC, EDF have two sides of the one, BA, AC equal to two corresponding sides of the other, ED, DF respectively, and angle A of the one equal to angle D of the other, subtended by the equal straight lines, then the one triangle is congruent to the other. [See Figure 5.]

(1) Let us suppose that the angles are given in position, LAM and NDP, (2) equal by the hypothesis, (3) hence congruent. (For *equal rectilinear angles* are defined in Definition 7 as those which are congruent.) (4) And [suppose] G [has] the magnitude of AC and DF (5) and H the

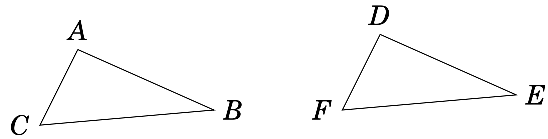

Figure 5

magnitude of AB and DE. (6) Finally, it is given on which sides of the angles these lines are to be taken, namely AB in AM, AC in AL, DE in DP, DF in DN. (7) Therefore points B, C, likewise E, F are given (by 3, I). (8) Therefore A, B, C [are given]; likewise D, E, F (by 1 and 7). (9) Therefore, triangles ABC and DEF [are given] (for, given the points at which the angles of the figure stand, the figure is given by Axiom 5). (10) And indeed both are exhibited determinately in the same way from congruent givens (by the whole procedure). (11) Therefore they are congruent (by Axiom 6). Q.E.D.

Scholium. If one applies superposition, it comes to the same thing; for if the congruent givens become actually congruent [*actu congruentia*], i.e. coincident, through superposition, then those which are given determinately[19] from them will also coincide; otherwise not one but several things satisfying these givens could be found, contrary to hypothesis. From this one understands the reason for the sixth axiom.

Porism. A triangle is given in magnitude and shape[20] if an angle and the sides enclosing it are given in magnitude (by 1 through 9). Indeed, for it to be given in position, one only needs an angle to be given in position, as well as which legs of the angle are assigned to which magnitudes of sides; this changes nothing in the magnitude and shape, since

[19] Leibniz says that something given by a geometric procedure or construction is given *determinately* when it is unique among things satisfying the conditions, *semideterminately* when it is one of a finite (or discrete) set of things, and *indeterminately* when it is one of an infinite (or continuous) set of things satisfying the conditions. Cf. SGL 'There is also the semidetermined'.

[20] Leibniz's word *species* can mean both 'appearance' in the sense of form or shape, and 'species' as a refinement of a 'genus'.

either leg relates in the same way at the angle.

Scholium. I call *porism*[21] something inferred from a demonstration, and *corollary* something inferred from a proposition.

Proposition 5. A triangle ABC which has two sides AB and AC equal also has their two angles B and C on the remaining side BC equal. [See Figure 6.]

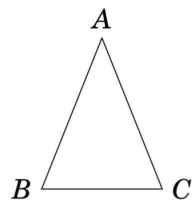

Figure 6

(1) Suppose BC is given in position. (2) Let G also be given, equal to BA, CA, (3) and the *side* on which [*partes ad quas*] A should fall (this is by Definition 8: the plane being cut into two parts by the line BC extended indefinitely, let it be given in which part A should be[22]). (4) A is given if it is possible, (5) since, with centres B and C and interval equal to G (by 2, I), (6) circles are described (Postulate 1), (7) and now A is in the circumference of both (by 2), if it is possible, of course. (8) Therefore the circles meet each other at A if A is possible. (9) Therefore (by Postulate 2) A is given. (10) Therefore (by Axiom 5) ABC is given. (10) Therefore also the angles ABC, ACB (since by Axiom 7, when something is given, its requisites are given) (11) and indeed, in the same manner (by the procedure explained). (12) Therefore (by Axiom 6) the angles are congruent. (13) And hence equal. (For congruents are equals by Axiom 8.)

Scholium. The same thing could have been shown by superposition: if some triangle DEF had been taken congruent to this one, and now

[21]The word 'porisma' in Greek literally means *bonus* or *windfall*.

[22]This definition did not make it into the marginal lists.

ABC would be placed onto DEF, now ACB onto DEF; thus now angle ABC, now angle ACB would be congruent to the same DEF; therefore they would be congruent to each other.

Proposition 6. A triangle ABC that has two angles B and C equal will also have the two sides AB and AC belonging to the remaining angle A equal.

This is demonstrated in the same way, (1) since, given side BC and angles B, C and the side on which A should be, A is given. (2) For line BC and the angle to it of another line BA being given in position, the line BA itself is given indefinitely (for the angle being given in position, by Axiom 7 the sides are given), in the same way CA [is given] indefinitely; (3) these will intersect each other in A if A is possible. (4) Therefore A is given, and thus both [sides are given] in the same way, therefore they are congruent. Q.E.D.

Scholium. The same thing could have been demonstrated from the preceding. As also by superposition in the preceding fashion.

Proposition 7. If triangles ABC, DEF have sides equal to the corresponding sides of the other, the triangles will be congruent. [See Figure 7.]

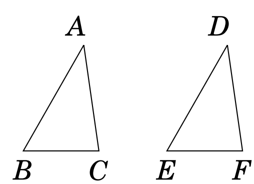

Figure 7

This is demonstrated by the same method, since, with one side of one triangle being given in position, and [one side] of the other being equal to it, and the remaining ones being given in magnitude, then by describing circles from the extremes of the sides given in position and with the magnitudes as intervals, each of the triangles will be given, and in the same way; therefore by Axiom 6 they are congruent.

Proposition 8. To bisect a given angle BAC. [See Figure 8.[23]]

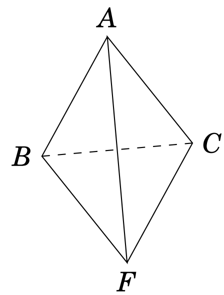

Figure 8

Let FA be the bisecting line, and $FAB = FAC$. It relates in the same way to BA and to CA. We have one point of the line FA, namely A; let us further seek another, F, to which this line relates as to A. This will happen if (supposing BA, CA are equal so that each side is treated in the same way with respect to FA) we translate BAC to BFC by describing circles with centres B and C and radii equal to BA or CA[24], which intersect each other someplace, at F (since triangle BFC is congruent to BAC by 7, I on account of the same base BC and equal sides, certainly [F] is possible). Therefore line FA, relating in the same way to AB and AC, will certainly bisect the angle.

The same will obtain if we construct any other isosceles triangle BFC at all over the base BC of the isosceles triangle ABC, by Addition 2 to Proposition 1. For the locus of all points relating in the same way to the two sides of the same angle or to the two extremes of the same line is a line passing through the two apices of the two isosceles triangles relating in the same way to the proposed angle or line.

Proposition 9.[25] To bisect a given line BC. [See Figure 9.]

Two points should be sought relating in the same way to B and C. This will happen if two isosceles triangles BAC, BDC of any sort

[23] Leibniz originally had dashed lines AF, BF, and CF, but then he drew solid lines

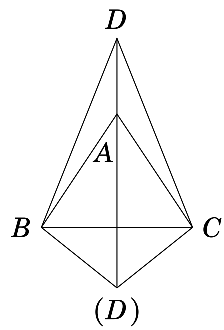

Figure 9

are constructed over the base BC; the line drawn through the angles opposite the base, i.e. through their apices, will bisect the base.

Scholium. Euclid uses an equilateral triangle for Propositions 8 and 9, but it is better to apply a more general construction.

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over them.

[24]'CB' in the manuscript.

[25]The manuscript has 'Prop. 8' which repeats the number 8. It is not clear why, although Proposition 7 in Book 1 of Euclid is missing from Leibniz's presentation, so Leibniz's numbering does not align with Euclid at Propositions 7, 8, 9 above. It is possible that Leibniz considered his Proposition 8 to incorporate both of Euclid's Propositions 7 and 8.