Initia Mathematica: De Quantitate

First Mathematics: On Quantity

G. W. Leibniz, 1682

Tr. David Jekel, Matthew McMillan

Determiners are things that are simultaneously compatible with only a single one. As two extremes A, B are compatible with just one straight line.

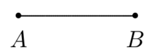

Figure 1

Coincidents are things that are completely the same and differ only by denomination, such as the path from A to B and the path from B to A.

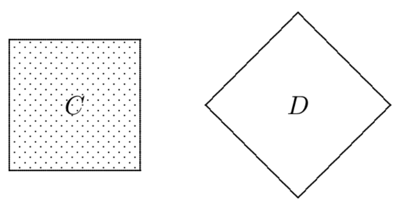

Figure 2

Congruents are things that, if they are distinct, cannot be distinguished except with respect to something external, such as the squares C and D, since of course they are in distinct places or situations at the same time, or because one C is in a gold material, the other D in silver. Likewise a gold pound and a silver pound are congruent, and the days today and yesterday. Any point is congruent to any other, as also any instant to any instant.

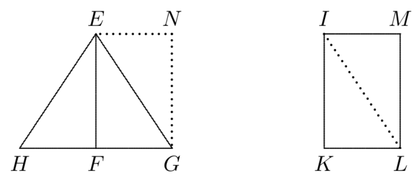

Figure 3

Equals are things that are either congruent (for example, triangles EFH, EFG, IKL, LMI, GNE, and likewise rectangles $EFGN$ et $IKLM$), or can be rendered congruent by a transformation (as triangle HEG to rectangle $IKLM$, since if part of HEG, namely EFH, is transposed to GNE, which can be done because they are congruent, then HEG is transformed to $FGEN$, congruent to $IKLM$; and so HEG and $IKLM$ are called equals). Therefore *equals* can be defined as things that can be resolved into separate parts, each one congruent individually to one part of the other.

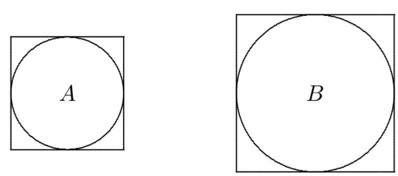

Figure 4

Similars are things in which nothing may be discovered by which they can be distinguished, if they are considered in themselves individually, such as two spheres or circles (or two cubes or two perfect squares) A and B. If the eye only, without any other member, is imagined to be now inside sphere A, now inside sphere B, it will not be able to distinguish them; it can, though, if both are seen simultaneously, or if another limb of the body or another measure is brought inside with it, which is applied now to one, now to the other. Thus, to distinguish similars we either need the co-presence of them with each other, or of a third thing with each in succession. [But in dissimilars, something noted in one by the proportion of its parts, which is not noted in the other, suffices for distinguishing them individually. More on this later.][1]

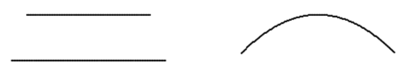

Figure 5

Homogeneous things are those which either are similar, or can be rendered similar by a transformation. Two lines are homogeneous since they are similar, but also a line and a circular arc are homogeneous since the circle can be stretched into a line.

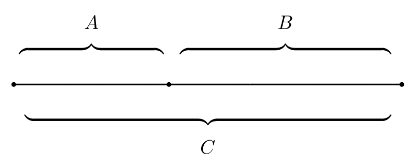

Figure 6

Suppose there is a plurality, say A and B, and a unity, say C, and they fit together in something, and nothing homogeneous is in A and B in common, but all homogeneous things in them are common to C; then the plurality are called *integrating parts* and the unity is called the *whole*.[2]

We could also define *homogeneous* things as those which fit together in something, and in which others also fit together that can be assumed in either one of them indefinitely.

[1] Brackets original.

[2] The manuscript has a variety of deletions and insertions in this sentence.

The *lesser* is that which is equal to a part of the other the (*greater*). *Quantity* is that which belongs to a thing insofar as it has all its parts, or on account of which it may be said to be equal to, greater than, or less than another thing (any homogeneous thing), i.e. it may be compared.

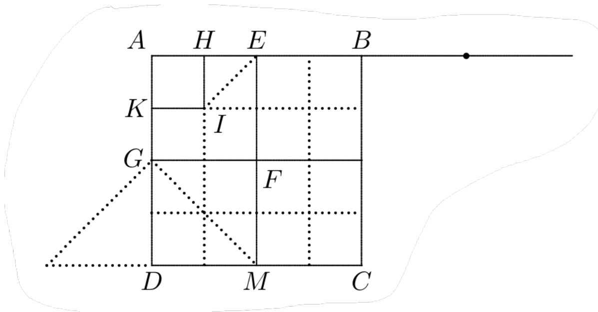

Figure 7

The *quantity* of a thing, for example of the area $ABCD$ in Figure 7, is expressed by a number, for example four, with another thing such as the square foot $AEFG$ being taken for a primary measure or real unit. Indeed, $ABCD$ is four square feet. But if another unit $AHIK$ is assumed, which is the square of the half foot AH, the quantity of the area $ABCD$ will be 16. And so various numbers will result for the same quantity according to the unit assumed. And hence a quantity is not a definite number, but the material of a number, or an indefinite number, to be defined when a fixed measure is assumed. Quantities, therefore, are expressed by definite numbers, like 1, 2, or indefinite, by letters or other characters a, b, ⊙, ☽.

A *number* is something homogeneous with unity and thus can be compared with unity and added to and subtracted from it. It is either an aggregate of unities, which is called an *integer*, such as 2 (or $1+1$), and likewise 3, 4 (or $2+1$ or $1+1+1$), or else an aggregate of some parts of unity, which is called a *fraction*; if a unity, for example a foot AB, is divided in four quarter parts, then something such as the line BH, which has three quarters of a foot, or three $\frac{1}{4}$, will thus be expressed

$\frac{3}{4}$; and sometimes a fraction may be reduced to an integer, as AB or 2 is four of AH, or 4 halves, or $\frac{4}{2}$; or, finally, a number is determined in some other way by relation to unity, and in fact these relations could be infinite, but the most customary are by radicals. Indeed, for the number 4 (for the square $ABCD$, Figure 7), suppose its square root is sought (or the side AB); it is the number which, multiplied by itself, makes 4; the number is 2, and so since $2 \cdot 2$ or aa is 4, $\sqrt{4}$ ($\sqrt{aa}$) is 2 (a). And in this case the radical can be reduced to a common number or a *rational*.

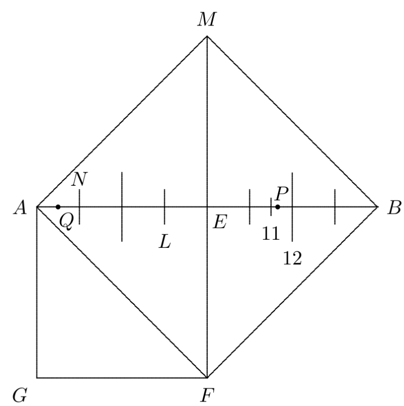

Figure 8

But sometimes this reduction does not succeed. For example, if a number is sought which multiplied by itself would make 2, it certainly is not an integer (for otherwise, since it is necessarily less than 2, it would be unity, but unity multiplied by itself makes 1), and neither is it a fraction, since every fraction multiplied by itself produces another fraction, as $\frac{3}{2}$ produces $\frac{9}{4}$ or $2+\frac{1}{4}$; and so the number is nothing but *irrational*, as they say, or rather ineffable, αλογος, *surd*, which is written another way as $\sqrt{2}$, or $\sqrt{q}\ 2$, or $\sqrt[2]{2}$, that is, the quadratic root of 2, for with this number set as y, then its square yy or y^2 will be 2. And so that it is clear that this number is in the nature of things, in

Figure 8 let the diagonal AF be drawn in the square $AEFG$. Let AG be 1, namely one foot, whose square is 1 (namely the space $AEFG$, or one square foot), then AF, which we call y, will be $\sqrt{2}$; for its square yy, $AFBM$, is 2 (namely double a square foot); the square $AFBM$ is in fact double the square $AEFG$, for its half, triangle AFB, is equal to the whole $AEFG$. Since, therefore, we define number as something homogeneous to unity, certainly there should be some number having the ratio to unity that the line AF has to the line AG, or with AG set as 1, the number expressed by the quantity of AF should be what is called $\sqrt{2}$; AB, moreover, will be 2.

Therefore if a *scale* AB (Figure 8) is divided into two equal parts, and four, eight, sixteen, etc., and further subdivided as much as one pleases, then certainly any line smaller than the scale can be placed on the scale and thus expressed explicitly by numbers [*explicari numeris*], either exactly or approximately. *Exactly* when, beginning at the start of the scale A, it falls on some point of division L, as AL, whose number is thus obtained in the parts of the scale. For example, with AE being 1, then AL is $\frac{3}{4}$, and then the line AL is commensurable with the scale, that is, there is a common measure for them, i.e. the line AN ($\frac{1}{4}$), which produces [*efficit*] by repetition, or *measures*, the scale AB as well as the line AL. On the other hand, a line applied to a scale can be *approximately* expressed explicitly by numbers when it does not fall on a point of division however much the scale is subdivided and in whatever way the division is established. And a line such as AF is incommensurable with the scale to the extent that it cannot be expressed explicitly by a rational or effable number except approximately. Nevertheless it is necessary that AF placed on the scale, i.e. translated to AP, at least falls between two division points, as of course here it falls between 11 and 12, supposing the scale AB to be divided into sixteen equal parts any of which is the eighth part of the unit or the foot AE.[3] Hence if AG or AE, the side of a square, is a foot, then the diagonal AF or AP falls between $\frac{12}{8}$ (or $\frac{3}{2}$) and $\frac{11}{8}$ of a foot, and thus if you allot $\frac{3}{2}$ to AF, i.e. $\sqrt{2}$ or

[3] Leibniz edited this sentence heavily. We have ignored a leading *'quoniam'* that it appears he forgot to cancel.

y, it is too much, and if $\frac{11}{8}$, too little is allotted (for the square of $\frac{3}{2}$ is $\frac{9}{4}$, that is, more than $\frac{8}{4}$ or 2 or yy, and the square of $\frac{11}{8}$ is $\frac{121}{64}$, which is less than $\frac{128}{64}$ or 2); nonetheless the error is always smaller than a smallest part into which unity in the scale is divided here, that is, less than $\frac{1}{8}$. And $\frac{1}{8}$ or AQ will be approximately a common measure for the unity (or indeed the scale) and AF. The more a scale is subdivided, the smaller the error will be, and so it will be as small as anyone wants, i.e. it can be rendered smaller than any assignable error. Therefore even if AF and AG are incommensurable, they are nevertheless *homogeneous* or comparable, and one can discover a common measure sufficiently precisely that the error or remainder is smaller than a given quantity. And in fact this is the foundation of *approximations*, and *tabular computations*, and likewise the binary or sexagesimal or even *decimal* logistic, when indeed a scale is divided into ten parts, and any of them subdivided into another ten, continuing as far as desired. In fact, even if one cannot always divide the scale sufficiently with instruments, nevertheless, with the mind or by calculation, one can at least proceed to the highest precision that could be desired *in practice*. How this is done will appear later.

[The next portion was cancelled.]

A *ratio* is something homogeneous with equality, and so if equality is viewed as the unit, since then the consequent will be contained in the antecedent only once, the ratio will be the number which arises from the division of the antecedent by the consequent, or the number by which the antecedent would be expressed if the consequent were the primary measure by which others should be expressed, or the unit. Even if now perhaps another primary measure or unit is adopted. Thus the ratio of three feet to a foot is triple the ratio a foot has to itself, or triple the unit, that is, three. For if a foot is 1, three feet will be 3. Unity, moreover, is the ratio of a thing to itself, or to an equal, when the one quantity is contained in the other only once. But the ratio of one foot to three, that is, to one triple-foot [*tripode*] (three feet being taken for the unit) is sub-triple, i.e. the third part of the ratio which the triple-foot has to itself, or the third part of the unit. Indeed if the triple-foot is 1, the foot would of course be $\frac{1}{3}$ of the unit, namely of the triple-foot. The ratio of the foot to the double-foot [*dipode*] is sub-double, or half of the ratio which the double-foot has to itself, that is, [half] of the unit. It is therefore $\frac{1}{2}$. For instance, if the double-foot is 1, the foot would of course be $\frac{1}{2}$ of the unit, namely the double-foot. And the ratio of three feet to the double-foot, or 3 to 2, is three times the ratio of one foot to the double-foot, or triple the sub-double, or three of the half, or $\frac{3}{2}$. And thus if the double-foot is 1, then three feet would be three of the half, or $\frac{3}{2}$, of one double-foot.

Things are *proportional* whose ratio is the same. As [the ratio] of the number 3 to 2 is the same as that of the number 6 to 4. For the ratio of the line AC to the line AB is always the same, and therefore the ratio of the numbers by which AC and AB are expressed will also be the same even if a different unit is assumed.[4]

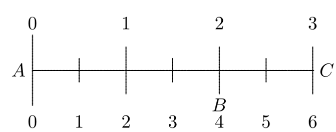

Figure 9

[Cancelled portion ends here.]

Dimensions are distinct (sometimes heterogeneous) quantities that are understood to be compounded by each other,[5] so that the whole of one is applied [*applicetur*] to each part of the other. For example, by drawing a latitude AB of two inches to a longitude BC of three linear inches (so that the angles A, B, C, e, f are always congruent, i.e. are the same, what are called right, which is the simplest way of drawing

[4] Leibniz wrote BC instead of AB in this sentence.

[5] Latin *in se invicem duci*, literally 'drawn to themselves mutually'.

a straight line to a straight line) the rectangle ABC is formed, which has two dimensions, and is six inches—but square inches.

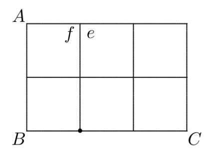

Figure 10

By drawing a longitude CB, 2, latitude BA, 3, and altitude AD, 4, to each other, a solid rectangle $CBAD$ is formed (Figure 10), which has three dimensions, or twenty four cubic inches (2 $\widehat{\text{by}}$ 3 $\widehat{\text{by}}$ 4); indeed on any of the six little squares or square inches of the base ABC (like the little square AEF) there stand four cubes, or cubic units, or cubic inches (namely a column or prism $FEAD$ consisting of four cubic inches placed on each other). Nor should one think, as has been believed up to now, that dimension is observed in figures alone, and thus that something of higher degree, or of more dimensions than three dimensions, is imaginary. Even though space *per se* has only three dimensions, a body can nevertheless have many more, for example two bodies, one of gold, another of silver, have, beyond the consideration of the bulk or space that they occupy, yet another consideration of specific gravity, which is observed in any part of the bulk. Thus, setting the specific gravity of a cubic inch of silver at 55, and of gold at 99 ounces (that is nearly the proportion), the weight of the solid $CBAD$, if it is gold, will be $2^\frown 3^\frown 4^\frown 99$ (or $24^\frown 99$) or 2376 ounces; but if the solid is silver, it will be $2^\frown 3^\frown 4^\frown 55$ ounces or ($24^\frown 55$) 1320 ounces. Therefore those weights have four dimensions, namely by compounding the three-dimensional bulk or space with the body itself, or the weight. Besides the dead weight, one can also add an impetus from the descent of a heavy body continued over some time; from this arises an impact that has five dimensions: from the three-dimensional bulk, the weight of the body, and the elapsed time, compounded with each other. In this way if one square arm of cloth is worth three imperial coins, two arms will be worth twice three

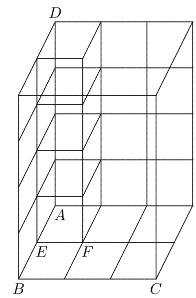

Figure 11

imperials, or six. And this price has two dimensions, for if the same cloth is four arms on the side, its price will be $2 \cdot 3 \cdot 4$ or 24 imperials, and so for three dimensions by compounding the longitude, latitude, and price with each other, that is, the price, or intrinsic value, with the quantity, or extrinsic value. In this way the price of a heap has four dimensions, for one sees in it a longitude of 100 feet, a latitude of 12, altitude of 20, and firmness or intrinsic value such that a cubic foot is worth ten coins. Then its value will be $100\frown 12\frown 20\frown 10$ coins or 240000, whence compounding a dimension by a dimension is a real exhibit of a mental multiplication.

From these definitions the following axioms can be derived.

Things which are determined (that is, in the same manner) by the same things (or by coincidents) are coincident. As two lines coincide whose two extremes coincide.

Things which coincide are even more so congruent. I.e. the same thing is congruent to itself.

Things which are determined by congruents (that is, in the same manner) are congruent. So, since a triangle is given when its three sides are given, then if the three sides of a triangle are respectively congruent to corresponding ones, the triangles will be congruent.

Things which are congruent are even more so equal.

Equals are expressed by the same number with the same measure being taken, i.e. they are of the same quantity; indeed since they can be rendered congruent to each other, they can be made congruent in the same way to the same primary measure, or unity, repeated in the same way. The same number results from this.

Equals treated in the same way according to quantity yield [*exhibent*] equals.

Similars treated similarly yield similars, and for that reason things determined similarly by similars are similar. By 'determined' I mean things that are designated by conditions which can hold simultaneously in only one thing. And so what is thus determined is clearly yielded.

Things simultaneously similar and equal are congruent. Indeed nothing remains by which they could be distinguished, whether individually or observed simultaneously, unless they are related to something external, such as place and time and other accidents.

There is no ratio except among homogeneous things; that is clear from the definition.

Things of which one is greater than, less than, or equal to the other, are homogeneous. This is obvious for equals, since they can be rendered congruent and thus also similar. The lesser is also homogeneous with the greater, since it is equal to, and thus homogeneous with, its part, and the part is moreover homogeneous with the whole. And for that reason we will not say that a curve, or its part, is less than a surface, nor that a contact angle is part of a rectilinear one, or less than it. Nonetheless, if one takes 'part' more broadly as anything that has quantity and is in [*inest*] something having quantity, then one can say that a line is less than a surface.

A part is less than the whole. For it is equal to a part of it, namely to itself.

The whole is equal to all of the co-integrant parts. For they coincide; or at least if they are conjoined, since they compose the whole, they will be rendered coincident. And so also congruent.

A part of a part is a part of the whole. So indeed something less than the lesser is less than the greater. For it is equal to a part of the lesser, therefore also to a part of the greater, that is, to a part of a part of the greater.

[The next portion was cancelled.]

The homologous [components] to one of [two] assumed similars have the same ratio to each other.

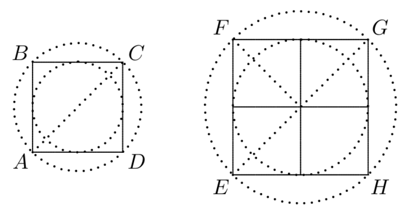

Figure 12

For example, the squares $ABCD$ and $EFGH$ are similar. In the first square, we have side AB, boundary $ABCDA$, diagonal AC. Inscribed circle. Circumference of a circumscribed circle. Area of the square. Area of the circle. In the other square we have correspondingly: Side EF. Diagonal EG. Boundary $EFGHE$. The circumscribed circle. Its area. The area of the square itself. We could also inscribe circles in either case, and carry out many other things. Hence I say that a side in the one is in the ratio to its diagonal in which also the side of the other is to its diagonal. Otherwise someone who is in the one square, and afterwards separately in the other, could distinguish them, noting that the ratio of side and diagonal in the one is different from what it is in the other. We have defined similars, though, as things which cannot be distinguished when viewed separately. For the same reason the periphery of one circle is to its diameter as the periphery of the other circle is to its. Likewise the area of one circle is to the square of its diameter as also the area of the other circle is to the square of its diameter. Hence we could immediately infer that the peripheries are to each other as the diameters, and the circles as the squares of the diameters. And in the same way spheres are as the cubes of diameters; and the homologous sides of similar triangles would be proportional if permutation of ratios were already demonstrated. And thus, with the aid of this axiom, a great many geometrical theorems, proven by others with such great effort, are demonstrated with no trouble, and we have a new principle of discovery. Whereas Euclid was forced to demonstrate through deduction to absurdity[6] that circles are as the squares of the diameters, and spheres as the cubes, and Archimedes, on the other hand, assumed without proof that the centers of gravity of similar figures are similarly positioned. All of these emerge spontaneously from our definition of similarity, which, as far as I know, has not occurred to anyone until now. The same principle is valid not only in geometry, but also in everything else where quantity and quality are combined.

Heterogeneous things should not, however, be compared with each other, neither is there any ratio except between homogeneous things, otherwise absurdity would arise. For example, if the side AB were to be to the area of its square $ABCD$ as another side EF is to the area of its square $EFGH$, then (according to what will be shown in its own place), by permutation, sides AB and EF would be to each other as the squares $ABCD$ and $EFGH$. Which is absurd; for if, for example, EF

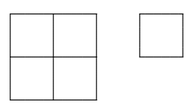

Figure 13

is the double of AB, then the square of EF is not the double but the quadruple of the square of AB. And the cube of something double the length is not the double but the octuple of the cube. The same for circles and spheres as for squares and cubes.

Equimultiples have the same ratio as the simples, which is clear from what we have said in defining proportionals. For six feet are the same ratio to three which two triple-feet are to one. Since the triple-foot

[6] Latin *'deductionem ad absurdum'*, a variant of *reductio* emphasizing a movement forwards which may not simplify things.

is the same as three feet. Moreover, their ratio is also the same when they are expressed in various ways according as one or another unit or primary measure is assumed.

Equidivisors have the same ratio as the whole quantities [*integrorum*]. For the wholes are equimultiples of equidivisors.

Hence simultaneous equimultiples and equidivisors have the same ratio. Let there be two, A and B; double A will have the same ratio to double B as A to B. Likewise the third part of A will have the same ratio to the third part of B as A to B, and two thirds part of A will have the same ratio to two thirds of B as A to B.

A ratio is composed with a ratio when the antecedent is multiplied by the antecedent and the consequent by the consequent; the ratio of products is called the composite of the simple ones; thus the ratio of one rectangular area to another is in the composite ratio of the lengths and widths. Hence composition of ratios is multiplication of ratio by ratio.

The *ratio* of A to C is said to be *composed* of the ratios of A to B and B to C. Hence the ratio of the product of compounding A by B to the product of compounding B by C is composed of the ratios of A to B and B to C. For the ratio of the product of A by B to the product of B by C is as A to C (since they are equimultiples, A by B and B by C, therefore they have the same ratio as the simples A and C by the preceding) and the ratio composed of A to B and B to C also is A to C.

Hence ratios composed of the same things are the same. Let ratios[7] A to B and B to C be given, and the ratios L to M and M to N. And let the ratio of L to M be the same as the ratio of A to B, and the ratio of M to N the same as the ratio of B to C. Therefore the ratio of A to C will be the same as the ratio of L to N, since things coincide which are the same or are determined in the same way by the same coincidents. Hence it follows that even if the order of the ratios of the coincidents in one composition were otherwise than in another, the composites would still coincide.

[7]This and another *ratio* in this paragraph were not plural as they should be, but in a third case Leibniz clearly pluralized to *rationes* later, while editing the sentence.

[Cancelled portion ends here.]

Two homogeneous things have an approximate common measure of any desired precision. We showed this above while explaining the scale. If one of two homogeneous things is neither greater nor less than the other, it is equal. In the scale supposed above [Figure 8], let AG and AE be compared, and AG applied to the scale with point A remaining fixed; point G will fall between B and A, supposing that AG is less than AB. We now suppose it demonstrable that with the line AG translated to AB, and point A remaining fixed, point G falls neither between E and B, nor between A and E, that is, that AG is neither greater nor less than AE. Then certainly point G will fall on point E itself, and thus AG will be equal to AE. What can be demonstrated of two lines, the same can be done of any homogeneous things, for they can all be rendered similar, and when they are rendered similar, if they don't differ in magnitude, there is no way in which they could be distinguished, but rather they will be congruent, and so since they can be rendered congruent, they are equal.