De Determinatis et Congruis

On Determined and Congruent Things

G. W. Leibniz, 1682

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I call a *rigid extensum* that in which there is no change as regards extension during any kind of motion, that is, the extension not only of itself, but also of things in it, is not changed.

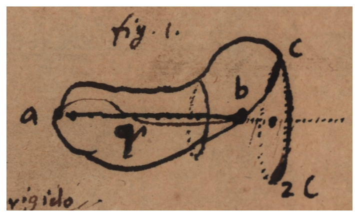

Figure 1

If some rigid extensum abc moves while two of its points a and b stay at rest, then the locus of all its points at rest, say r, will be the line ab, while the successive locus $c2c$ of some point c that moves will be an *arc of a circle*. We therefore have the generation of a line and a circle without any line or circle pre-existing, indeed the line through rest, the circle through motion.

From this generating process [*generatio*] of the line it follows that any point, say r, is on the same line with the two given points $a.b$ if, when it is connected to them by some rigid extensum arb, it cannot move while they stay at rest. And conversely, if some point is on the same line with two given points to which it is connected by a rigid extensum, it cannot move while they stay at rest. Now points such as $a.r.b$ are sufficiently connected by a rigid extensum when they are all found in the same rigid extensum such as abc.

If points such as a and c always preserve the same situation between themselves, then the same points $l.m$ of the same rigid extensum $f\,l\,m\,g$ can always be applied to them; and conversely if the same

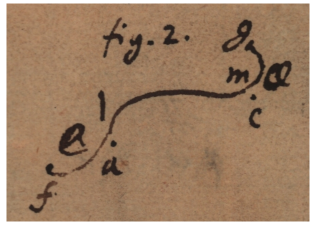

Figure 2

points of the same rigid extensum can always be applied to them, they always preserve the same situation between themselves.

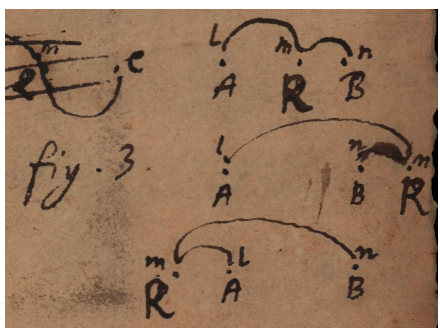

Figure 3

Therefore a line can be defined as the locus of all points each one of which, say R, has such a situation to two points A and B that as long as it preserves this situation to them (which happens as long as the three points $A.R.B.$ can be applied to the same points $l.m.n$ of the rigid [extensum] lmn), R cannot move while A and B are unmoved (that is, if l and n remain unmoved, then no matter how the extension lmn moves, it is necessary that m also is unmoved). And so the locus of this R is determined when its situation to the points A and B is determined. Hence it is immediately clear that the points A and B certainly fall on this same line, the locus of all points R. For certainly R cannot move while A and B remain unmoved. It is also clear that in an extensum which is unchangeable as regards extension (such is space itself, as well as any rigid body), when two points are determined, the line passing through them is determined.

Figure 4

In order to express this property of the line in the calculus, I will take ∞ as the symbol for *congruence*, for example $F \infty G$, that is, F is congruent to G.

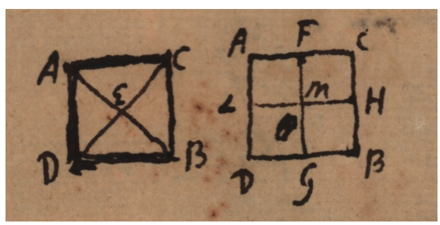

Figure 5

Now ɣ will be the sign for *coincidence*. For instance, in an equilateral square $ACBD$, if the lines AB, CD are drawn connecting opposite angles and intersecting each other at the point E, and again other lines FG, HL are drawn connecting the midpoints of the opposite sides and intersecting each other at the point M, then in fact the points E and M coincide, and one can write E ɣ M.

The *situation* of certain *points between themselves* I write with a simple denomination of points between points, as $A.R$ or $R.A$. This signifies the determined situation between the points A and R, or any rigid thing connecting them. Similarly $A.B.R$ signifies the situation of the three points among themselves. And so on.

And so, since the situation of the point R to two points $A.B$, which

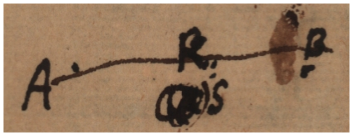

Figure 6

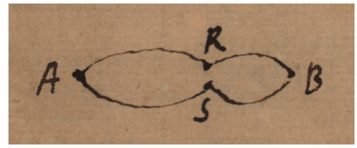

Figure 7

determines the locus of the point R, is the situation of a point R on a straight line with points $A.B$, let, therefore, point R be set in the same line with points A and B, and its situation to the points A and B is written $A.R.B$, determined if you like by some rigid thing ARB connecting the three points; and let there be moreover some point S having the same situation to the points A and B which the point R has, so that the rigid thing applied to the points $A.R.B$ could be applied in the same way to the points $A.S.B$; it follows that the points S and R coincide, if indeed R is on the same line with A and B.

And so, to summarize the matter in a few words and using the calculus, if supposing $A.R.B \infty A.S.B$ it follows that R ỽ S, the locus of all R will be a *straight line*. And again, if for brevity we want to add another sign for determination, [then] by writing $\overline{dt\,R}$ we can signify that the locus of the point R is determined or specified, and similarly by writing $\overline{dt\,A.R}$ that the situation of the points A and R is determined or specified. Thus $\overline{dt\,A.R.B}$ will signify that the situation of these three points is specified. And so the *line* passing through A and B is the locus of any points R whatever, if from $\overline{dt\,A.R.B}$ it follows that $\overline{dt\,R}$.

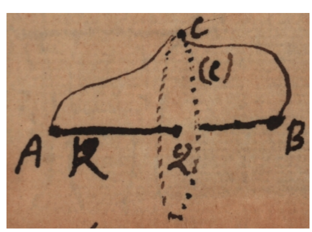

Figure 8

Now all points outside this line can move while preserving their situation to the two points A and B. Thus, although a point C connected rigidly to A and B may be set, nevertheless, if it is not on the line passing through A and B, it will be able to move while the points A and B are unmoved, and it will describe a circle $C(C)$ by its motion in the way we defined above; and so, since during the motion it always preserves the same situation to the points A and B, we will have $A.C.B \infty A.(C).B$ and the locus of all C or (C) will be circular.

Every point such as C, that preserves during motion the same situation to some points such as A and B, will preserve the same situation to the point R which has a determined place when the situation to A and B is determined (and hence the point C, being moved while its situation to the points A and B is preserved, will preserve the same situation also to any point R of the line passing through A and B, i.e. to this line itself); or in general, whatever preserves its situation to the determiners also preserves it to the things determined. This proposition merits demonstration.

It can also be shown that when one point B is taken on the line AB, another point R can always be found such that $C.B \infty C.R$,[1] except for a single case, when B and R, speeding toward each other uniformly, meet each other at Q.

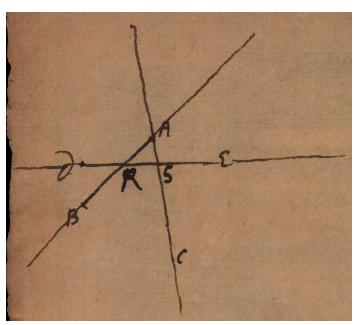

Figure 9

[1]It is '$C.B. \infty C.B.$' in the manuscript.

A *plane* is determined when three points are assumed; that is, all the points whose place is determined when their situation to three points is given fall on the plane passing through those three points; this definition coincides with the generation of a plane by a line moved over two unmoved lines. Indeed, let there be two lines, one passing through points A and B, the other passing through points A and C. And let there be another movable line passing through points $D.E$. This one, moved over those two lines, and meeting AB at R, AC at S, will give the straight line passing through $R.S$, or determined from the points R and S. And since all the points such as R and S are determined from the points $A.B.C$, and the lines connecting them are determined from them, and all points of these lines (from the proposed generation of the plane by the motion of lines) are all the points of the plane, it is clear that all the points of the plane are determined from these three points $A.B.C$.

By taking now four points, infinite space itself is determined; that is, the locus of all points each one of which is determined given its situation to four given points is infinite space, or the locus of all points in general; or what is the same thing, the locus of every point is determined when its situation to four given points is given; but this should be demonstrated.

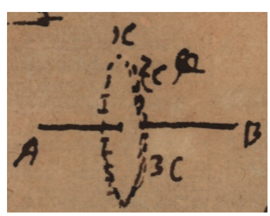

Figure 10

Since, above, it was said that the point C, during its motion, relates in the same way to the line AB, therefore the line AB will relate in the same way to the three points $1C.2C.3C$. Therefore it will also relate in the same way to the plane determined by these three points. And in the same way, when three points determining a plane have the same situation to some two points A and B, the determined plane itself will also have the same situation to them.

It should be demonstrated that these same propositions are reciprocal, that is, that all points having simultaneously the same situation to two distinct points fall on a plane, and those having [the same situation] to three fall on a line. Having obtained these things, one may combine what was demonstrated elsewhere about congruents with what was demonstrated here[2] about determined things, and one will obtain that two points, having the same situation to four given points which do not have the same situation to three other points, will coincide [*congruere*].

[2]The manuscript shows that 'elsewhere' and 'hic' were inserted after the last paragraph was written.