De Calculo Situs

On the Calculus of Situation

G. W. Leibniz, 1682

Tr. David Jekel, Matthew McMillan

(1) The determined path from a point A to a point B[1] is called the *straight line* AB, that is, the path which passes through the points relating determinately to A and B and being less distant from A than A is from B.

(2) And so the line is the minimal path, i.e. if C is not on the line AB, a path from A to B through C will be greater than the line AB, or $AC + CB \sqcap AB$,[2] since the minimal path is determined by the posited extremes. Indeed it is certainly clear that a maximal path is not determined, because a path can go through any given point, and so whatever path is assumed, a greater can be taken, whereas a lesser path cannot be assumed to infinity, nevertheless there are points which are closer, and so there is a certain minimal path.

(3) Also, two lines between A and B coincide, i.e. if ACB is a straight path, and ADB a straight path, then $ACB \mathrel{\infty\mkern-13mu|} ADB$. Otherwise the path would certainly not be determined from the given points, since it would be uncertain which one ought to be chosen.

(4) Also, a point taken on the line is unique with its situation to the two extreme points; that is, it is determined by the given points and its situation to them. And thus if a point Y is on the line AB, I say that the situation $Y.A.B$ is determined. And thus if another point E were given and the situation $E.A.B \infty$ (is congruent to) the situation $Y.A.B$ and E and Y are on the line AB, then $E \mathrel{\infty\mkern-13mu|} Y$, or E and Y coincide.

(5) Moreover, we can determine the situation of a point to a point through the lines intervening between them by the very fact that the

[1] The manuscript has an extra 'AB' here. It seems he meant to cross it out along with two adjacent words that he crossed out.

[2] The manuscript has 'less than the line AC, or $AC + CB \sqcap AB$,' presumably a mistake.

line is determined, so the situation of E to A can be called $E.A$.[3]

(6) A line is similar to a line, or $AB \sim CD$, and conversely whatever is similar to a line is a line, because things are similar which are determined in the same way. But ABC is not immediately similar to LMN, unless it is previously established that they are determined in the same way, or there is no difference in the determiners, or $AB \sim LM$ and $BC \sim MN$ and $AC \sim LN$.

(7) If two lines are equal, they are congruent, i.e. if $AB = CD$ then $AB \infty CD$. For $AB = CD$ by hypothesis and $AB \sim CD$ by the preceding article. Therefore $AB \infty CD$ by the axiom of the subsequent article.

(8) Axiom: If two things are similar and equal, they are congruent, i.e. if $\odot = ☽$ and $\odot \sim ☽$, then $\odot \infty ☽$.

(9) If the situation $E.A.B \mathrel{\reflectbox{$\propto$}}$ the situation $Y.A.B$, then $E.A \mathrel{\reflectbox{$\propto$}} Y.A$.[4]

(10) If $E.A \mathrel{\reflectbox{$\propto$}} Y.A$, then $EA = YA$ by article (5), and conversely if $EA = YA$ then $E.A \mathrel{\reflectbox{$\propto$}} Y.A$. This is considered when E, A, Y are points.

(11) If $A.B \mathrel{\reflectbox{$\propto$}} L.M$ and $A.C \mathrel{\reflectbox{$\propto$}} L.N$ and $B.C \mathrel{\reflectbox{$\propto$}} M.N$, then $A.B.C \mathrel{\reflectbox{$\propto$}} L.M.N$.

(12) Hence if $AB = LM$ and $AC = LN$ and $BC = MN$, then $A.B.C \mathrel{\reflectbox{$\propto$}} L.M.N$ by (10) and (11).

(13) A part of a line is a line, for since a line is a determined path, it will be fully determined. And so if $AD + DB \mathrel{\ooalign{∞\cr\hfil$|$\hfil}} AB$, then AD is a line and DB is a line. Or if ADC is a line, then AD is a line.[5] [See Figure 1.]

(14) Every line can be extended. Let AB be a line; I claim it can be extended to C. Let a point D be taken in AB; certainly the line AD can be extended (since indeed it actually is extended to B); therefore

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[3] The dot in '$E.A$' is unclear in the manuscript and likely absent.

[4] Leibniz appears to use a different version of the congruence symbol, one which is partially open on the left, in (9) and (10), and reverts in (12). He does not use either symbol in this essay after (12), and it is not clear whether the change is deliberate.

[5] Article (13) was inserted after Leibniz wrote (14), which he then renumbered, and this explains the appearance of point C in (13).

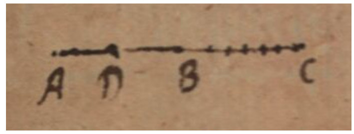

Figure 1

also the line AB, which is similar to the line AD by article (6), can be extended to somewhere, say C.

In symbols: Line AB is given. Therefore line ABC is given. The proof: Indeed, line AB is given. Therefore line ADB is given. Therefore (by article (13)) AD is a line. Therefore (by article (6)) $AD \sim AB$.

(15) Now similars can be treated similarly, and being treated similarly, they yield similars, which is an axiom we employ rather frequently.

Therefore since ADB is given, $ABC \sim ADB$ can be given as well. And since ADB is a line by hypothesis, ABC will also be a line (by 6). And so the line AB is extended. That is what we sought to show was possible.[6]

(16) A point taken on a line relates determinately to two other points situated on the same line, i.e. it is unique with its situation to them.

Let there be a line $ALMNB$; I claim that one of the middle points, say N, is unique with its situation to points L and M. For $N.A.B$ is a determined situation due to the nature of a line (or N is unique with its situation to A and B) by article (4). And in the same way, $M.A.B$ is determined. And $L.A.B$ is determined.

(17) Now let it be an axiom that, in any situation, determiner points can be substituted for the determined point.

And so $M.B$ can be substituted for A in the determined situation $N.A.B$ (since A is determined from $M.B$, or the situation $M.A.B$ is determined) and it will become $N.M.B.B$, or omitting the pointless duplication of B, it becomes $N.M.B$, which is also determined. Similarly

[6] Literally: 'That which was to be demonstrated to be able to be done', an extended variation on 'Q.E.D.'.

one can demonstrate that the situation $N.L.B$ is determined, and so, by substituting $N.L$ for B in the situation $N.M.B$, it finally becomes a determined situation $N.L.M$, Q.E.D.

(18) Every point unique with its situation to two points, and less distant from either one than they are distant from each other, falls on the line intervening between them. If $AC \sqcap AB$ and $BC \sqcap AB$ and $A.B.C$ is determined, then ACB is a line, as is clear from the definition of a line in article (1). But perhaps it can also be demonstrated another way from the intersections of circles, so that a definition of a line even stricter in kind can be assumed, which nonetheless achieves the same thing.

(19) Every point unique with its situation to two points falls on the line passing through those two points, extended if necessary.

Let there be three points, $A.B.C$, whose situation among themselves is determined; I claim that they fall on the same line. Indeed, one of them is the middle one, which is less, or at least not more, distant from the remaining ones than they are from each other. I show this as follows: let lines AB, BC, AC be joined and let AB be the greatest of these, or at least not less than any of the remaining ones, let AC be the next, or at least not less than the last one, and let BC be the third, i.e. not greater than any of the others. It is clear that C will be the middle

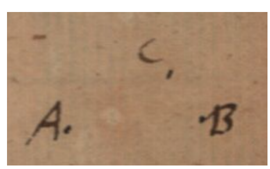

Figure 2

one, i.e. not more distant from A and B than they are from each other. Therefore, by the preceding, ACB is a line. And so for the line we can have such a local proposition:[7] Let $Y.A.B$ be determined, and $\overline{Y}$ (or the

[7] Latin '*propositionem localem*'; probably indicates a proposition regarding geometric loci as defined through situational properties. We have added the colon where the manuscript has a new line but no punctuation.

locus of all Y) will be a line. Hence if we suppose two lines have more than two points in common, they coincide when extended if necessary, and accordingly two lines do not have a common segment, nor do they enclose space.

(20) It also follows that the locus of all points unique with respect to two points is a curve, i.e. it falls on the determined path of a point passing through the two points.

(21) For a curve is the path of a point.

(22) A plane is the extensum determined from three points, or the locus of all points unique with their situation to three points. That is, if $Y.A.B.C$ is determined, $\overline{Y}$ will be a plane. Now, given any three points not falling on the same line, something is determined which I call a plane. And in general, assuming any things whatever, something is determined, i.e. relates determinately to them.

(23) Three points determining a plane fall on the plane itself. For the situation of A to A is determined, therefore the situation of A to $A.B.C$ is also.

(24) If two points of a line are on a plane, the line itself is on the plane. Let the line be $\overline{Z}.L.M$ and the plane $\overline{Y}.A.B.C$ and let L be Y and likewise let M be Y. I claim that every Z is Y. For first, $L.A.B.C$ is determined, and likewise second, $M.A.B.C$, and third, $Z.L.M$. Therefore, substituting the determiners from determinations 1 and 2 for L and M in the third determination yields that $Z.A.B.C$ is determined. Therefore every Z is Y.

Hence it is clear that every line can be brought into [*applicari*] every plane; likewise a line can be moved in a plane.[8]

(25) Any three points taken in a plane determine the same plane, i.e. any point on a plane is unique with its situation to three points of the plane (not falling on the same line). This can be demonstrated in the manner of article 16 for a line. Thus, likewise: Let the three points be $L.M.N$; I claim that they determine the plane they are on; for certainly they can determine some plane if they do not fall on a line,

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[8]Leibniz originally began this statement as article (25), and then he crossed out the number.

and that plane is unique by article 22, and they are on the plane which they determine by article 23. Therefore they determine the plane they are in.

(26) The circumference of a circle is the locus in a plane of all points whose situation to some one point of the plane is the same, or is described by the extreme of a line moved in a plane (which can be done by article 24) with one point unmoved. And it is expressed thus with symbols: If $YA = BA$ in a plane, $\overline{Y}$ is the circumference of a circle.

(27) Every plane is similar to every plane when taken indefinitely (or if the ambits of two planes are congruent, the planes themselves are congruent); indeed, if a plane is determined by $A.B.C$, let $L.M.N$ be taken in another plane such that $L.M.N \sim A.B.C$, and what is determined by $L.M.N$ will also be similar to what is determined by $A.B.C$, in the sense of determination, that is, which we have explained.

(28) If a plane crosses a plane, the common section is a line. If two planes cross each other, they have at least more than one point in common; let us therefore take two common points A and B. And let us take a third C in one plane $\overline{Y}$, and a third L in the other plane $\overline{Z}$, and for the plane $\overline{Y}$, $Y.A.B.C$ will be determined, and for the plane $\overline{Z}$, $Z.A.B.L$ will be determined. Now let the line $\overline{V}$ be such that $V.A.B$ is determined. I claim that every V will be Z and every V will be Y. Indeed, if $V.A.B$ is determined, $V.A.B.C$ is therefore also determined and $V.A.B.L$ is determined. Therefore V is Y and V is Z, or V falls both in $\overline{Y}$ and in $\overline{Z}$. In fact, no additional point can be given, common to these two planes, which does not fall on this line. For if three points not falling on the same line were in common, the two planes would coincide, because three such points determine a plane.

(29) In a plane, the locus of all points relating in the same way to two points is a line, i.e. if $YL = YM$, then $\overline{Y}$ is a line. [See Figure 3.] Let there be two points in the proposed locus, namely A and B, each of which relates in the same way to L as to M, and now what is determined by them will also relate in the same way to L as to M, namely the line $\overline{Y}.A.B$. No other such point is given that is not on this line. Indeed, let H be such a point which does not fall on the line AB. Because the

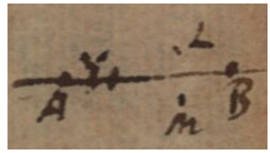

Figure 3

whole plane is determined from the points A, B, H, not falling on the same line of course, it would follow that the whole plane relates in the same way to L as to M, which is absurd, since[9] it would then follow that M itself relates in the same way to L as to M, that is, it coincides with L because it coincides with M, contrary to the hypothesis.

Now, more than two points are given relating in the same way to L as to M. For if circles are described with centres L and M and radii LM, they will cross each other, indeed in two points, since one is partly inside the other and partly outside. But more on this soon.

(30) It will be demonstrated in the same fashion that the locus of all points relating in the same way to two given points is a plane.

(31) Hence from article (31)[10] it follows that a plane extended indefinitely is cut by a line extended indefinitely into two parts not differing [*partes indifferentes*]. And it follows from (31) that space is cut similarly by a plane. But these things must be shown more distinctly.

(32) An indefinite line divides an indefinite line into two parts, each of which relates in the same way to the dividing one. Suppose $\overline{Y}$ is a line and $\overline{Z}$ is a line and B is Y and B is Z. Again let $\overline{Z} \,\phi\, \overline{V} + \overline{X}$ (or every Z is V or X) and some V is B and some X is B. I claim that $\overline{Y}.\overline{V} \,\phi\, \overline{Y}.\overline{X}$.[11]

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[9] Latin '*vel ideo quia*', 'or for the reason that because'; it seems Leibniz should have deleted the 'vel ideo'.

[10] It is not clear why Leibniz seems to refer to this article itself. He wrote and cancelled an initial draft of the previous article (30), but it is not clear how that explains this reference.

[11] The essay was not finished.