De Analysi Situs

On the Analysis of Situation

G. W. Leibniz, 1704

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[1] What is commonly lauded as *mathematical analysis* is analysis of *magnitude*, not of *situation*, and thus it is directly and immediately apposite to arithmetic, but to geometry it applies in a certain roundabout fashion. And so it happens that by considering situation, many things easily become clear which the algebraic calculus shows with greater difficulty. To reduce geometrical problems to algebra, that is, to reduce what is determined by figures to equations, is often a rather lengthy process; and yet another protracted and arduous effort is needed to return from equation to construction, from algebra to geometry; and often by this route we get constructions that are not quite suitable, unless we are fortunate enough to strike upon certain unforeseen suppositions or assumptions. Descartes tacitly admits this very thing when he solves a certain problem of Pappus in the third book of his Geometry. And indeed algebra, whether numeric or symbolic [*speciosa*], adds, subtracts, multiplies, divides, and extracts roots, which is, of course, arithmetic. For logistic itself, i.e. the science of magnitude and proportion in general, handles nothing but general or indeterminate number, and the symbols of operations on it, since *magnitude* is actually assessed by the multiplicity of determinate parts, which nevertheless varies according as one or another measure or unit is assumed, while the thing remains the same. Hence it is not surprising that the science of magnitudes in general is a kind of arithmetic, since it deals with uncertain numbers.

[2] The ancients had another kind of analysis, distinct from algebra, which comes nearer to the consideration of situation, treating of *givens* and *sites*[1] or *loci* of things sought. And Euclid's little book about givens, on which we have the commentary of Marinus, aims at this. Planar,

[1] Latin *sedibus*, literally 'seats', is not common in mathematical writing. It means the place occupied or inhabited by a thing.

solid, and linear loci were of course dealt with by others, and also by Apollonius, whose propositions were preserved by Pappus, from which more recently some have recovered the planar and solid loci, but so that they seem to have shown the truth more than the source of the ancient doctrine. This kind of analysis, however, neither reduces the matter to calculus, nor indeed is it carried all the way to the first principles and Elements of situation, which is necessary for a perfect analysis.

[3] Therefore, the true analysis of situation is yet to be supplied, and this is evident from the fact that all analysts either practise algebra in the new fashion or treat the givens and things sought according to the ancient form, and they must assume many things from elementary geometry which are not deduced from consideration of magnitude but of figure, and neither are they clear by any determinate path, even to this day. Euclid himself was compelled to assume certain quite obscure axioms without proof, that the rest might proceed. And the demonstration of theorems and the solution of problems in the *Elements* sometimes appears to be a work more of toil than of method and art, although now and then an art to the process seems to be concealed.

[4] Figure in general includes, besides quantity, also quality or form; and just as *equals* are things whose magnitude is the same, so *similars* are things whose form is the same. And indeed, the consideration of similarities or forms is clearly broader than mathesis and is learned from metaphysics, though it does also have many and varied uses in mathesis and is beneficial for the algebraic calculus itself; but similarity is seen most of all in situations, i.e. figures of geometry. And so a truly geometric analysis regards not only equalities and proportions, which actually reduce to equalities, but indeed similarities, and it should employ the congruences arising from equality and similarity combined.

[5] Now the reason why geometers have not made satisfactory use of the consideration of similarity I judge to be this: that they had no general notion of it sufficiently distinct or adapted to mathematical inquiries, a fault of philosophers who habitually content themselves with vague definitions equal in obscurity to what is defined, especially in

first philosophy; whence it is no wonder that the teaching tends to be sterile and verbose. And so it is not enough to say that similars are things whose form is the same unless one has in turn a general notion of *form*. Now I learned, while working out an explanation of quality or form, that in the end the matter comes down to this, that similars are things which cannot be distinguished when observed individually. For quantity can only be apprehended by the co-presence of things, i.e. with an actual application[2] intervening. Quality presents something to the mind which you recognize in a thing separately and can use for comparison of two things, even with no actual application intervening by which one thing is compared[3] with another either immediately or by mediation of a third thing as a measure. Let us imagine that two temples or buildings have been constructed by the rule that nothing can be apprehended in the one that you would not also observe in the other: the material is of course everywhere the same, white marble of Paros if you like; the proportions of all the walls, the columns, and the rest are the same in both; the angles in both are the same, i.e. of the same ratio to a right angle; and so whoever is led into these two temples with eyes closed, and after entering opens them, and goes about now in the one and now in the other, will not find any token in them by which to distinguish the one from the other. And yet they can differ in magnitude, and thus can be distinguished if they are viewed simultaneously, from the same place, or even (letting them be far from each other) if some third thing is transported and compared now with the one and now with the other, just as some measure such as a cubit or foot or something else suitable for measuring is fitted now to the one and now to the other, for then at last a basis for distinguishing will be given by the observed inequality. It is the same if the viewer's body itself, or a

[2]Latin *applicatione actuali interveniente*; here *applicatio* is used as in the tradition of English geometry texts to render a term of Euclid that indicates a bringing of one thing to another by rigid motion; often the two are assumed to be congruent and *applicatio* results in superposition, though that is not the case here.

[3]Latin *confertur*. Although typically glossed as 'compared', it is worth pointing out that the Latin sense more evidently includes an element of 'bringing together' that is pertinent here.

limb, which certainly goes with oneself from place to place and serves the role of a measure, is applied to these temples; for then the different magnitude, and through it a method of distinguishing, will become apparent. But if the viewer is considered to be nothing but a sighted mind,[4] as though set in a point, not carrying with it any magnitudes either in reality or in the imagination, and considering in things only what may be grasped by the intellect, like numbers, proportions, and angles, then no distinction will present itself. Therefore, these temples will be called similar, since they cannot be distinguished except by this co-observation, either among themselves or with a third thing, and not at all when viewed individually and by themselves.

[6] This perspicuous and practical and general description of similarity will be beneficial to us for geometric demonstrations, as will soon become clear. We will say that two proposed figures are similar if nothing could be noted in the one, viewed individually, that could not equally be apprehended in the other. And so it follows that the ratio or proportion of the constituents must be the same in both, otherwise a distinction would appear by themselves individually, i.e. even if no co-observation of both of them were arranged. But geometers, since they lacked a general notion of similarity, defined similar figures from equal corresponding angles, which is specific and does not reveal the nature of similarity itself in general. And so they had to take a circuitous route to demonstrate what is clear at the first glance from our notion. But let us come to examples.

[7] It is shown in the *Elements* that similar, or equiangular, triangles have proportional sides, and vice versa, but Euclid finally completes this in the fifth book by many detours, when he could have shown it immediately in the first *Element*, if he had followed our notion. We will show first that *equiangular triangles are similar*.

[8] Let there be a triangle ABC and again another LMN, and let the angles A, B, C be equal to L, M, N respectively; I claim that the triangles are similar. Moreover, I will use this new axiom: *whatever cannot be distinguished through the determiners* (i.e. sufficient givens) *cannot*

[4] Latin *mentem oculatam*, literally ‘eye-ed mind’.

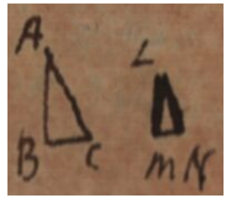

Figure 1

be distinguished at all, since everything else arises from the determiners. Now, given base BC and given angles B and C (and so also angle A), the triangle ABC is given; likewise, given base MN, and given angles M, N (and so also angle L), the triangle LMN is given. But the triangles cannot be distinguished through these sufficient givens individually. For in each one, both the base and the two angles to the base are given. Now the base cannot be compared with the angles, therefore nothing else remains from the determiners which could be evaluated in either triangle viewed individually, other than the ratio of each given angle to a right angle or to two right angles, that is, the magnitude of the angle itself. Since these things are found to be the same, it is necessary that the triangles cannot be distinguished individually, and so they are similar. I might add as a sort of scholium that even if the triangles could be distinguished in magnitude, nonetheless the magnitude cannot be recognized except through co-observation either of both triangles simultaneously, or of each one with some measure, but then they would not be viewed just individually as was required.

[9] Conversely, it is clear that *similar triangles are also equiangular*; otherwise, if there were some angle, say A, in triangle ABC for which no equal angle could be found in triangle LMN, then there would certainly be an angle in ABC having a ratio to two right angles (or to the sum of all angles of the triangle) which none has in LMN, and this suffices to distinguish triangle ABC from triangle LMN individually. It is also established that *similar triangles have proportional sides*. For if some two sides were given, such as AB and BC, having a ratio to each other that no sides of triangle LMN have to each other, then the one

triangle could be distinguished from the other individually. Finally, *if the sides are proportional, the triangles will be similar*. Indeed, since triangles are given when the sides are given, it suffices (by our axiom) that a distinction could not be obtained from the ratio of the sides in order for us to judge that it could not be obtained from anything else in these triangles viewed individually. But from these things it is also clear that *equiangular triangles have proportional sides, and vice versa*.

In the same way, right at first mental glande, one shows directly from our notion of similarity that *circles are as the squares of the diameters*, something Euclid shows finally in book ten and indeed by means of inscribed and circumscribed [figures], reducing the matter to absurdity, when actually no detours were necessary.

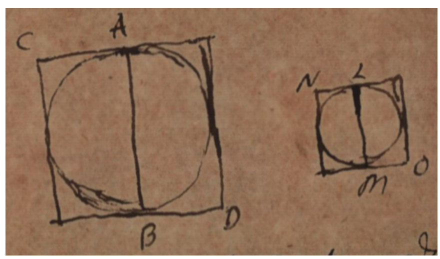

Figure 2

[10] Let a circle be described with diameter AB, and let the square CD[5] of the diameter be circumscribed on it; and in the same way let a circle be described with diameter LM, and let the square NO of the diameter be circumscribed on it. The determination of both is similar, the circle to the circle, the square to the square, and the fitting of the square to the circle, and so (by the aforementioned axiom) the figures $ABCD$ and $LMNO$ are similar. Therefore (by the definition of similarity) the circle AB is to the square CD as the circle LM is to the square NO; therefore also the circle AB is to the circle LM as the square CD is to the square NO, as we claimed. By comparable reasoning, *spheres* will be shown to be *as the cubes of the diameters*. And in similars in

[5] Meaning: the square whose side is the diameter CD.

general, homologous curves, surfaces, and solids will be as the lengths, squares, and cubes of homologous sides, respectively. Something which has until now generally been assumed rather than demonstrated.

[11] This consideration, furthermore, which affords such ease in demonstrating truths that would be difficult to demonstrate by another method, has also opened to us a new kind of calculus, completely different from the algebraic calculus, equally new in its signs and in its use of signs, or its operations. And so I like to call it the analysis of situation, because it unfolds [*explicat*] situation directly and immediately, so that figures might be depicted in the mind through the signs even without being drawn, and whatever the imagination understands empirically from figures, the calculus will derive from the signs by sure demonstration, and even obtain everything else which the power of imagining cannot reach. Therefore a supplement to, and if I may say so the perfection of, the imagination is contained in this calculus of situation I have proposed, and it will have uses never known until now, not only in geometry, but also for the invention of machines and the descriptions of nature's own machines.[6]

[6]The Latin *ipsasque machinarum naturae descriptiones* can be parsed either as 'and the descriptions themselves of the nature of machines', or as we have done. Note that a prior, cancelled draft had 'explicationesque', apparently instead of this phrase.