Characteristica Geometrica 1-72

Geometric Characteristic

G. W. Leibniz, 1679

Tr. David Jekel, Matthew McMillan

(1) Characters are certain things that express the relations other things have among themselves, and which are easier to handle than those things. And so, to every operation taking place in the characters, there corresponds a certain proposition in the things, and we can often defer consideration of the things themselves until the very end. Once we have found in the characters what we sought, we will easily find the same among the things, through the agreement originally set up between the things and the characters. In this way machines could be exhibited by models, and solid bodies represented on a planar tablet so that no point of the body lacks a corresponding point assigned on the tablet according to the laws of perspective. And so if we perform some geometrical operation on the image of the thing by the scenographic method, this operation may yield some other point of the tablet from which we may easily find a corresponding point on the thing. And accordingly we may solve stereometric problems in the plane.

(2) Now, the more exact the characters are, that is, the more relations of things they exhibit, the more useful they can be, and if they exhibit all the relations among the things, as for instance the characters I use for arithmetic, there is nothing in a thing that could not be apprehended through the characters. The characters of algebra serve as well as arithmetic characters because they signify indefinite numbers. And since nothing in geometry cannot be expressed by numbers, once a certain scale of equal parts has been set, it turns out that whatever is treated in geometry is susceptible of calculation.

(3) But we know that the same things can be related in various ways in characters, and some ways are more convenient than others. Thus, the tablet on which a body is delineated by the art of perspective could also be gibbous, and yet a planar tablet serves better. And no one can fail to see that today's characters for numbers, which are called

Arabic or Indian, are more suitable for calculation than the old Greek or Latin ones, even though those too can be used for calculation. The same thing happens also in geometry, namely, that neither can algebraic characters express everything that should be considered in space (indeed, they presuppose elements[1] discovered and demonstrated already), nor can they signify the situation of points itself directly, but rather they investigate it through magnitudes with many convolutions. Whence it happens that it is exceedingly difficult to express in calculus what is exhibited by a figure, and more difficult still to effect in a figure what is discovered by calculus. And so the constructions which calculus exhibits are for the most part amazingly distended and awkward, as I showed elsewhere in the case of the following problem: find the triangle with given base, altitude, and vertex angle.

(4) Now, I have noticed that geometers tend to add certain descriptions to their figures which explain them, so that what cannot be recognized from the figure alone, such as equality or proportionality of lines, may be understood at least from the added words. They often go further and express in many words even what is manifest from the figure itself, so that the argument is more rigorous, and nothing depends on sense or even imagination, but everything is carried out by logical reasoning; or else so that the figures can be drawn from the description or restored if by chance they are lost. And even if they don't observe this sufficiently precisely, still they give us certain traces of a geometric characteristic, as when geometers say 'the rectangle ABC', they mean the one made

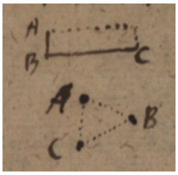


Figure 1

[1] Here 'elements' was capitalized. Although Leibniz's capitalization was a bit erratic, this capitalization may indicate a reference to the *Elements* of Euclid.

by drawing [*ex ductu*] AB over BC at right angles. When they say AB equals BC equals AC, they express an equilateral triangle. When they say of the three, AB, BC, AC, some two equal the third, they signify that all three, A, B, C, lie on the same line.

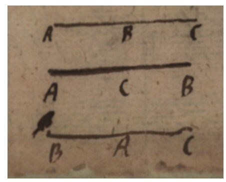


Figure 2

(5) So when I noticed that some properties of a figure can be signified merely by this use of letters to signify points, I began to consider further whether we could in fact signify all relations of the points of any figure by the same letters, so that the whole figure might be exhibited in the characteristic, and we might discover by the mere arrangement and transposition of those letters what is barely or not at all apparent from drawing numerous lines. Usually confusion arises from the multiplicity of lines drawn in the figure, particularly when we are still trying things, whereas with characters we can try things without risk. But there is something greater behind this, for we can express with these characters the true definitions of everything that is treated in geometry, and we can continue analysis down to the principles, namely the axioms and postulates, whereas algebra by itself would not be enough, but would be forced to use propositions discovered through geometry, and when it tries to relate everything to those two propositions, the one which adds two squares in one, and the other which compares similar triangles, it is forced to distort most things from the natural order.

(6) Having once demonstrated the elements in our characters, we shall easily see how to find solutions to problems which exhibit immediately, by the same operation, the constructions and the demonstrations in terms of lines. On the contrary, after algebraists find the values of their unknowns, they still have to worry about the constructions, and

after discovering the constructions, they still seek demonstrations in terms of lines. And so I am amazed that people haven't considered that, if there can be demonstrations and constructions in terms of lines, devoid of all calculation, and much shorter, surely there must also be discovery in terms of lines, for there must be a regression[2] in linear no less than in algebraic synthesis. Indeed, the reason we have not yet seized upon an analysis in terms of lines is doubtless nothing else than that we have not yet contrived characters by which the situation of points itself is directly represented. For among a great multiplicity and confusion of things, it is difficult to move freely without characters.

(7) But if we can once represent figures and bodies exactly by letters, not only will geometry be marvellously improved, but also optics, kinematics [*phoronomicam*], mechanics, and in general whatever is subject to the imagination will be treated by a sure method and (as it were) analysis. And with a marvellous art we will render the invention of machines in the future no more difficult than the constructions of problems in geometry. Thus, without laborious effort or expense, even very complex machines, and indeed natural things, could be delineated without figures, so that they can be transmitted to posterity and figures can be formed, with the greatest precision, from the descriptions whenever we wish. Much is lost with the current difficulty and cost of delineating figures, and people are deterred from describing matters they have explored and that are of public utility. Indeed, thus far we do not have either sufficiently exact nor sufficiently apt terms for the compilation of descriptions, which is evident, for instance, from the case of botanists, and also those who illustrate armaments and insignia. With characters we could easily denote the other qualities by which points, which are considered similar in geometry, do differ from each other. And then finally, at last, there will be hope of penetrating the secrets of nature,[3]

[2] Latin '*regressum*' indicating a backwards-going stepwise logical sequence, moving from what is sought to what is known (giving a method of discovery); as opposed to *progression* which moves forward from known principles toward what is new.

[3] Leibniz initially continued with 'since we are abler by technique to effect that which another...', but later deleted this and continued as rendered above.

when everything that another can wrest from the givens by sheer power of ingenuity and imagination, we can draw tranquilly and securely from the same givens by a sure technique.

(8) But since no such thing has occurred to anyone, as far as I know, and no helps appear anywhere, I am forced to undertake the matter again from the very beginning; how difficult that is, nobody believes without experience. And so I have attacked the thing on more than ten different occasions, in different ways, all of which were tolerable and had various advantages, but did not satisfy my scruples. Finally, after cutting out very much, I saw that I had arrived at the greatest simplicity, since I did not suppose anything from elsewhere, but I could demonstrate everything from my own characters. But I lingered for a long time even after finding the true method of this characteristic, since I saw that I should start from elements that are easy in themselves and familiar from elsewhere. Arranging them so carefully could be rather unpleasant, yet I persevered, overcoming the irritation, until finally I had struggled through to better things.

[[4]Everything in geometry can be resolved by the mere consideration of points. For the nature of a curve also consists in the fact that any point assumed on it has a certain relation to some given points, through which it can be determined or discovered from them. And surfaces can be known either through curves or immediately through points, and bodies through surfaces or curves or points. But the consideration of two points existing simultaneously contains nothing but the situation of the one to the other, i.e. the distance, or what is the same thing, the intervening straight line.]

(9) But one should understand that to treat everything in order, we must first consider *space*, that is, pure absolute *extension*. *Pure*, I mean, from matter and change, and moreover *absolute*, that is, without limit and containing all extension. And so all points are in the same space and can be related to each other. Now whether this space is some thing [*res*], distinct from matter, or only a constant appearance or phe-

[4]Leibniz cancelled this entire paragraph. We have included it because the summary and emphasis it provides seem helpful for understanding the subsequent sections.

nomenon, does not matter here.

(10) Next is the consideration of a *point*, that is, of what is simplest among all things pertaining to space or extension; in the way that *space* contains absolute extension, so a point expresses that which is maximally limited in extension, namely simple situation. It follows from this that a point is minimal and lacks parts, and that all points are congruent to each other (or are able to coincide), and so they are also similar and (if it is allowed to say this) equal.

(11) If two points are understood to exist or be perceived simultaneously, what is thereby presented for consideration is their relation to each other, which differs from one pair of points to another, namely the relation of place, or situation, that two points have to each other, in which we understand their distance. Now the *distance* of the two is nothing but the minimum quantity of a path from one to the other. And if a pair of points $A.B$, preserving the situation between themselves, is jointly congruent to, or able to displace, another pair of points $C.D$ also preserving the situation between themselves, then the situation, or distance, of the former two is the same as the distance of the latter two. For those things are congruent of which one can coincide with the other without any change being made within either of them. Coincident things $A.B$ and $C.D$ have the same distance, therefore so do congruent things, since then, without any change of distance within $A.B$ or within $C.D$, they could be rendered coincident.

(12) Now a *path* (through which we also defined distance) is nothing but a continuous successive locus. And the path of a point is called a *curve*. It can also be understood from this that the extremes of a curve are points, and that any part of a curve is a curve, or terminates in points.[5] Now a path is a certain continuum, since any part of it has common extremes with its anterior and posterior parts. It follows from

[5] Leibniz cancelled the following: 'Moreover, the minimum path from a point to a point is necessarily the path of a point, or is a curve; for certainly the path of a point is smaller, or simpler, than the path of another extensum'. Before 'is smaller', Leibniz had also written the phrase '*via rei alterius continui extensi*', and cancelled it in what seems to have been an earlier cancellation.

this, if I may add in passing, that if a certain curve is drawn on some surface, one cannot make another curve on the same surface pass continuously between the two extremes of the first curve without crossing the first one.

(13) The path of a curve of this kind, such that its points do not always succeed one another, is a *surface*, and the path of a surface, such that its points do not always succeed one another, is a *body*. A body, however, cannot move without all of its points succeeding one another (as must be demonstrated in its place), and for that reason does not produce a new dimension. It is thus apparent that there is no part of a body whose ambit[6] is not a surface, and no part of a surface whose ambit is not a curve. It is also clear that the extreme of a surface (and equally of a body) returns on itself, i.e. it is an *ambit*.

(14) Now when two points are assumed, *eo ipso* the simplest possible path of a point through both one and the other is determined: otherwise their distance would not be determined, and neither would their situation. But this curve, which is determined solely by the two points through which it passes, namely such that supposing it passes through the two given points, it alone is thereby presented for consideration, this curve, I say, is called a *straight line*. And however far it is extended, it will be called one and the same line. From this it follows that the same two points cannot have two distinct lines in common, unless the two lines coincide after being extended far enough. Consequently, two lines cannot have a common segment (otherwise they would also have in common the two extremes of this segment), neither can they enclose space, i.e. create an ambit that returns on itself, otherwise one line that diverges from the other would return to it and thus would meet it in two points.

A part of a line is also a line, for it is determined by those two points alone through which alone the whole is determined. Determined, I say, meaning that by the consideration of those two points alone, all of its points are presented to be considered or traversed. From this it is clear that if $A.B.C$ and $A.B.D$ are congruent, and $A.B.C$ are said to lie on a

[6] I.e., boundary.

line, then C and D coincide. I.e. if only a single point has that relation to the two points that it has, then these three points will be on one line. On the contrary, if there are more than two points related in the same way to three or more given points, then the latter will be on the same line and the former outside it; the reason is that things related in the same way to the determiners *eo ipso* are related in the same way to the determined things, and so three or more points on the same line can be supplied instead of two. However, I require more than two points having the same relation. (For if there are just two, each one of which has the same relation to the three, then we can conclude only that the three lie on the same plane, not that they lie on the same line.)

A line is also uniform on account of its simplicity, i.e. it has parts similar to the whole. And any line is similar to any other line, since a part of one is congruent to the other, and the part is similar to the whole. And concave and convex sides cannot be distinguished for a straight line, i.e. the line does not have two dissimilar sides, or what is the same thing: if two points are taken outside the line which relate in the same way to the line's extrema (or any two points on the line), then they will also relate in the same way to the whole line, i.e. to any point on the line, from whichever side of the line those two points outside the line are finally taken. The reason for this is that when things relate in the same way to the points that determine some extensum, they necessarily also relate in the same way to the whole extensum.

Finally, a line is the minimal [path] from point to point, and accordingly the distance of the points is nothing but the quantity of the intervening line. For the minimal line is certainly determined in magnitude solely by the two points; but it is also determined in position, for there cannot be multiple minimal paths from point to point in absolute space [*spatio absolute*] (as there are many minimal [paths] in a spherical surface from one pole to the other). For if it is absolutely minimal, the extremes could not be drawn apart while maintaining the quantity of the line, and therefore neither can the extremes of its parts (for it is also necessary that the parts are minimal between their own extremes) while preserving the quantity of the separate parts, and therefore neither while preserving the quantity of the whole.[7]

If, now, the line is transformed while its two extremes remain unmoved, it is necessary that some of its points will be drawn apart from each other. And so if the extremes of a line are fixed while the minimal quantity between the two points is preserved, it cannot be transformed into something else, and so one cannot give multiple incongruent, dissimilar minimal [paths] between two points. Consequently, if two [paths] between two points were minimal, they would be congruent to each other. Now a minimal one is a straight line (as shown above), therefore another minimal one must also be a straight line. And two lines between two points coincide. And so there is only a single minimal [path] between two points.

(15) Here is the simplest method of generating a straight line. Let there be some body, two points of which are unmoved and fixed, but the body itself nonetheless moves; then all stationary points of the body will fall on the line passing through the two fixed points. For it is evident that those points have the same place determined from the two given fixed points, i.e. they cannot move with the two points remaining fixed and the whole being solid; whereas other points outside the line can change their place while preserving the same relation to the two fixed points. The one inconvenience here is that the line described in this way is not permanent. A straight line can be generated in another way, when a flexible curve is given which cannot be extended to a greater length. For if its extremes are drawn apart as far as possible, the flexible curve will be transformed into a straight line.

In the same manner, the properties of a plane or a circle or a triangle can be deduced from the established definitions by a certain natural order of contemplation. For we have examined the straight line just as a sample.

(16) It is not difficult to follow all of this mentally, even though no figures were drawn (except by imagination), and neither did we make use of characters other than words. But because the words that geome-

[7]The last clause follows nine cancelled lines, so it is not surprising that it does not cohere perfectly with the preceding clauses.

ters have tended to adopt even to the present are not sufficiently exact for long, drawn-out argumentation, nor is the imagination sufficiently adept, they have until now used figures for that reason. And drawing figures is often difficult besides, and the delay allows the best thoughts to slip away, and sometimes the figure [*schemata*] gets muddled on account of the multiplicity of points and lines, particularly when we are still trying out and investigating things; for these reasons I believed characters could be beneficially employed in the manner of what follows.

(17) Space itself, i.e. an extensum (that is, a continuum whose parts exist simultaneously), cannot be designated conveniently in any other way here, that I can see, except by points. This is because we are proposing to express exactly the demarcations of figures, and in these we see nothing but *points* and *certain continuous tracts* from one point to another, in which infinitely many points could be taken arbitrarily. And so, then, we will express specified points by single letters, such as A and B (Figure 3).[8]

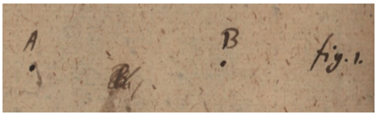


Figure 3

(18) Continuous tracts, on the other hand, we will express through certain unspecified or arbitrary points, assuming a certain order, so that it is nonetheless apparent that other points between them, as well as after and before them, could always be taken. Thus $3b6b9b$ (Figure 4) will signify for us an entire tract, any point of which will be called b, and in which we have chosen arbitrarily two parts, one whose extremes are the points $3b$, $6b$, and the other whose extremes are the

[8]Here Leibniz added and then cancelled the sentence: 'And points which are unspecified, and which could be assumed arbitrarily, we will express by adding numbers, so that it is apparent that these points as well as the others between them and after and before them could be assumed.'

points $6b$, $9b$.

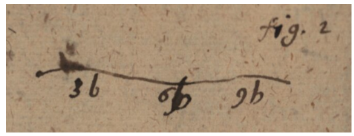


Figure 4

It is clear from this that these two parts are continuous, since they have a common point $6b$ and their division was made arbitrarily. This tract, in which the common extreme of two parts is nothing but a point, is called a *curve*, and it can also be represented by the motion of a point b which runs along a certain path, that is, it is understood to leave as many traces as there are distinct points $3b$, $6b$, $9b$, etc. Hence a curve may be called the path of a point. A path, moreover, is a continuous successive locus. It may also be signified in the following abbreviated fashion: Curve '$\overline{y}b$'. We designate by the letter $\overline{y}$ (or another letter) arbitrary ordinal numerals, taken collectively. But when we write 'yb' without the mark over y, we will understand each point of the curve $\overline{y}b$, taken distributively.

(18)[9] In the same way, one could make certain tracts whose parts are connected by curves, or which we understand to be described by the motion of a curve such that its points do not succeed one another but come to into new places. This tract, or path of a curve, is called a *surface*. Let us suppose, namely in Figure 5, that the curve above called $3b6b9b$ moves, and that one of its places is named $3{\scriptstyle 3b}3{\scriptstyle 6b}3{\scriptstyle 9b}$, and another, subsequent, place is $6{\scriptstyle 3b}6{\scriptstyle 6b}6{\scriptstyle 9b}$, and yet another is $9{\scriptstyle 3b}9{\scriptstyle 6b}9{\scriptstyle 9b}$; the surface $3{\scriptstyle 3b}3{\scriptstyle 6b}3{\scriptstyle 9b}$, $6{\scriptstyle 3b}6{\scriptstyle 6b}6{\scriptstyle 9b}$, $9{\scriptstyle 3b}9{\scriptstyle 6b}9{\scriptstyle 9b}$ will result, and we will signify it with the abbreviation $\overline{zy}b$.

(19) Here it is also clear that just as motion of the curve $\overline{y}b$ following the points $\overline{z}b$ describes the surface $\overline{zy}b$, so in turn, motion of the curve $\overline{z}b$ following the points $\overline{y}b$ describes the same surface $\overline{yz}b$. And yzb will signify each individual place of the point b, not collectively, but

[9] Leibniz added an additional, apparently spurious, paragraph number here.

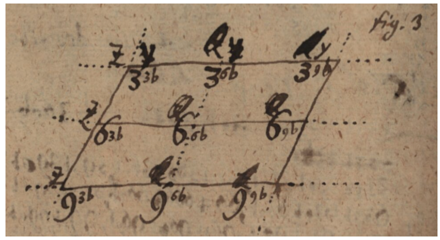


Figure 5

rather distributively; and $z\overline{y}b$ signifies any one curve $\overline{y}b$ on the surface $\overline{zy}b$, also taking each of them not collectively, but distributively.

(20) Neither does it matter what shape the moving curves themselves have; or even [those curves] according to which the motion happens, i.e. which are described by one of the points of the moving curve. See Figure 6.

Figure 6

It can also happen that during the motion, the moving curve itself changes shape, as the curve $\overline{z}b$ in the same Figure 6. This could be understood more clearly by thinking of the surface that would be described by a bow if the whole bow moved while firing, for example if it fell to the ground. The moving curve can even lose some parts during the motion, which are separated from it either actually or mentally, as is clear from Figure 7. It can also happen that one or more points in the moving curve (e.g. $3b$) remains stationary during the motion, and its places, expressed as though distinct (e.g. as $3_{3b}6_{3b}9_{3b}$), actually coincide with each other, as one realizes by considering Figure 8. But all these variations and many others can also be designated by characters,

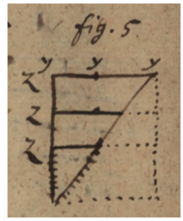


Figure 7

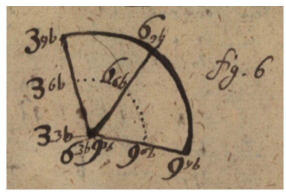


Figure 8

as will be shown in its own place.

(21) Now, as that tract which is called a surface is described by the motion of a curve, so also the tract that is called a solid, or body, is described by the motion of a surface (such that its parts, or points, do not succeed one another everywhere). This can be understood well enough from one example (Figure 9): If the (linear) curve $\overline{z}3b$ (namely $3{\scriptstyle 3b}6{\scriptstyle 3b}9{\scriptstyle 3b}$) remains fixed in the (rectangular) surface $\overline{zy}b$, namely:

$$\begin{Bmatrix} 3{\scriptstyle 3b} & 6{\scriptstyle 3b} & 9{\scriptstyle 3b} \\ 3{\scriptstyle 6b} & 6{\scriptstyle 6b} & 9{\scriptstyle 6b} \\ 3{\scriptstyle 9b} & 6{\scriptstyle 9b} & 9{\scriptstyle 9b} \end{Bmatrix},$$

and this same surface moves, then by its motion it will describe the solid:

$$\begin{Bmatrix} 33{\scriptstyle 3b} & 33{\scriptstyle 6b} & 33{\scriptstyle 9b} & 36{\scriptstyle 3b} & 36{\scriptstyle 6b} & 36{\scriptstyle 9b} & 39{\scriptstyle 3b} & 39{\scriptstyle 6b} & 39{\scriptstyle 9b} \\ 63{\scriptstyle 3b} & 63{\scriptstyle 6b} & 63{\scriptstyle 9b} & 66{\scriptstyle 3b} & 66{\scriptstyle 6b} & 66{\scriptstyle 9b} & 69{\scriptstyle 3b} & 69{\scriptstyle 6b} & 69{\scriptstyle 9b} \\ 93{\scriptstyle 3b} & 93{\scriptstyle 6b} & 93{\scriptstyle 9b} & 96{\scriptstyle 3b} & 96{\scriptstyle 6b} & 96{\scriptstyle 9b} & 99{\scriptstyle 3b} & 99{\scriptstyle 6b} & 99{\scriptstyle 9b} \end{Bmatrix}$$

One should note here that because the line $\overline{z}3b$ is fixed, the points $3{\scriptstyle 3b}$,

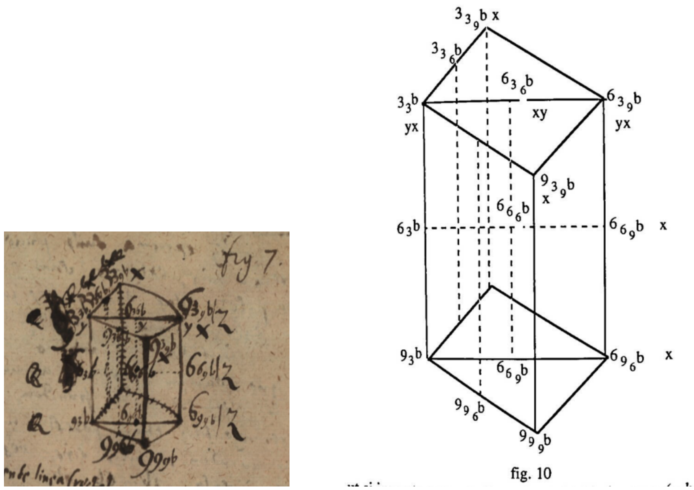


Figure 9

6_{33b}, and 9_{33b} coincide (and so the place of all of them is found in the figure to be 3_{3b} only), and likewise the points 3_{63b}, 6_{63b}, 9_{63b}, from which we also have just 6_{3b} in the figure; and finally, in the same way, when the points 3_{93b}, 6_{93b}, 9_{93b} coincide, they are expressed just by 9_{3b}. Now this solid we will express with an abbreviation in this way: $\overline{xzy}b$, and some surface of it, or locus of $\overline{zy}b$, we will express in this way: $x\overline{zy}b$ (thus exhibiting a section of a cylindrical region, or this solid, made by a plane through the axis). We can also take one of its surfaces $\overline{xy}b$ in this way: $z\overline{xy}b$ (thus exhibiting a section of this cylindrical region according to the base, i.e. a plane parallel to the base); likewise in this way: $y\overline{xz}b$ (thus exhibiting a section of this cylindrical region through another cylinder having a common axis with this one). Other sections of the same figure can also be understood, since one could fashion infinitely many ways of generating it by motion, or even of resolving it into parts according to some specified law. It is evident that all the other variations in production or resolution of a surface, which we indicated a little earlier, are even more relevant for solids. And finally we must demonstrate in its own place that a dimension higher than a solid, or the tract described by such a motion of the solid that its points do not succeed one another everywhere, could not be obtained.

(22) Furthermore, the tracts themselves, or the loci of certain indefinite points, are determined by certain specified points, and likewise by certain laws according to which the other indefinite points come into consideration, in order, from the few specified points, and the tract itself could be generated, i.e. described.

Before explaining this, let us clarify certain symbols that will be used in what follows. First, then, it can happen that two or more names, apparently distinct, really refer to just one thing or locus, that is, a point or line or another tract, and thus they are said to *be the same* or to *coincide*. So if there are two lines AB and CD, and the points A and C are one and the same, we will signify it thus:

$$A \infty C$$

that is, A and C coincide. [See Figure 10.] This will be especially useful for designating extremal points (and other extremes) shared by distinct tracts. For the same point, or rather extremum, has indeed its denominations according to one tract and according to another. But if we say $A.B \infty C.D$, the meaning will be $A \infty C$ and $B \infty D$ simultaneously. And likewise for several; and the same order should be observed on each side of the statement.

(23) But if the two do not in fact coincide, that is, they do not simultaneously occupy the same place, but nonetheless could be applied on each other, and without any change made in them viewed in themselves, the one is able to substitute in place of the other, then those two are said to be *congruent*, as $A.B$ and $C.D$ in Figure 11. And thus AB ɣ CD. Likewise $A.B$ ɣ $C.D$ in Figure 11, that is, while preserving the situation between A and B and likewise the situation between C and D, $C.D$ can be applied on $A.B$, that is, C can be applied on A and D on B simultaneously.

(24) If two extensa are not congruent but nonetheless could be rendered congruent without any change of mass or *quantity*, that is, retain-

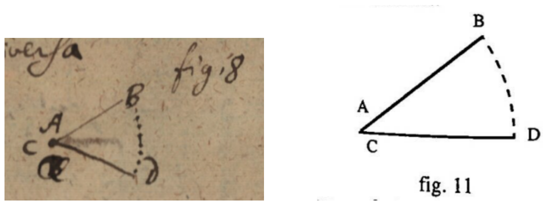


Figure 10

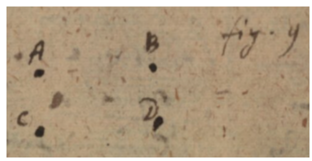


Figure 11

ing all of the same points, even if a certain transmutation, or transposition, is needed of the parts or points, then they are said to be *equal*. Thus in Figure 12 the square $ABCD$, and the isosceles triangle EFG

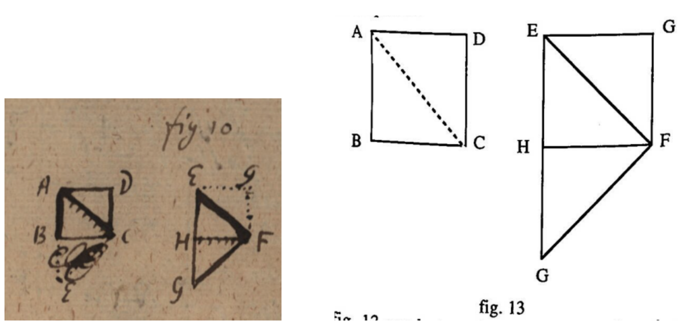


Figure 12

with base EG double the side AB of the square, are equal: for translate FHG to EGF, since $EGF \between FHG$. Then we have $EFG\ equ.\ EHFG$. Now $EHFG \between ABCD$, thus $EFG\ equ.\ ABCD$. Hence, more generally, if $a \between c$ and $b \between d$, then $a + b \sqcap$ *(or equ.)* $c + d$. Even further: if $a \between e$,

$b \between f$, $c \between g$, $d \between h$, we have: $a+b-c+d \sqcap e+f-g+h$. Or if two sums are made out of certain parts by one and the same way of addition or subtraction, and the parts of the one are congruent to the parts of the other coming together in the same way to comprise the whole, each part of the one sum congruent to the corresponding one of the other sum, in order, then the two resulting sums won't always be congruent but nevertheless will always be equal. And so the argument from congruence to equality is established by the definition of equality. Alternatively, things are *equal* when they have the same magnitude. Of course the magnitude is the multiplicity of parts congruent to a certain thing, i.e. a measure, such as (in Figure 13) if we have two magnitudes a and b,

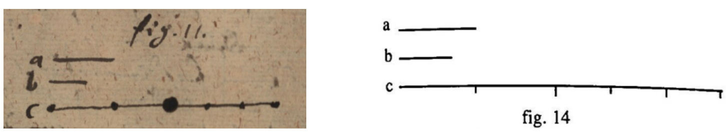


Figure 13

and we are given a third thing c which is twice $a+$thrice b, then clearly its magnitude is expressed by the multiplicity of parts congruent to a as well as to b, and so it is necessary that things which can be rendered congruent, with no addition or subtraction, are certainly equals.

[[10]Now since any point is congruent to any point, and the number of points in two congruent things is the same, that is, any point in the one has its own corresponding point in the other, certainly they are equal. But points cannot quite accurately be called parts or be said to have a number, and there can be equals which have no congruent parts, as a cylindrical surface and a circle, so for that reason we can understand magnitude more generally to be an attribute of a thing expressed through other things, homogeneous given things belonging to it, which remains even when the relation of these things among themselves is changed. *Homogeneous* things I understand as those which

[10]Leibniz cancelled this paragraph during revision. In no subsequent section of the essay does he underline *homogeneus* as a definition.

go[11] together in some common thing, as a point, a curve, a body, and their parts. *Given* things, that is, things congruent to certain specified things; *determined* things, namely, so that whichever of them is proposed, it is apparent from the expression itself whether or not it pertains to the thing. And so, that predicated of an extensum through which it can be determined whether some point belongs to it, and which remains the same when the situation of the points among themselves is changed, such that one or both could be generated by repetition of the other or the one, will be called the *magnitude* of the extensum. But if parts congruent to certain given things can be determined in the thing, surely also the points in them can be determined, and thus the number of parts congruent to certain specified things, when it exhibits a way of determining all the points of the thing such that it does not matter what the situation of the points is one to another, certainly will be the magnitude.]

(25) But to return to the matter more deeply, we should explain what is a part and a whole, what is a homogeneum, what is magnitude, and what ratio. A *part* is nothing but an immediate, distinct (i.e. such that one cannot be predicated of the other) requisite of the whole, with its corequisites directly.[12] Thus AB is a requisite of AC, that is, if it were not for AB there would be no AC; they are also distinct, for AC is not AB; in another fashion we have that 'rational' is a requisite of 'man', but since man is rational and thus 'rational' (which is a requisite of man) and 'man' is the same thing, even though they differ in expression, they nonetheless go together *in re*. A part is an immediate

[11] Latin *convenire* has a broad range of meaning, in Leibniz's usage, that it is hard to capture in a single English word. It can mean 'to meet' (as when two curves intersect), to 'agree' or 'be compatible', to 'fit together' or 'be appropriate to'. The somewhat colloquial English 'go' as in 'X goes with Y' or 'X and Y go together' (conveying both belonging and compatibility) covers much of the right ground. What matters is that X and Y are not only combined, but also have a certain kind of agreement of type. A part occurs 'with corequisites *directly*' because it *convenit* with them.

[12] Latin *in recto*. Cf. *in obliquo* ('obliquely') below. Although Leibniz usually used *in recto* in logico-grammatical writings for terms invoked in the nominative case, and *in obliquo* for terms in other cases, here it is applied with an apparently broader, and vaguer, sense.

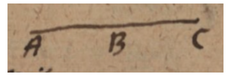


Figure 14

requisite; indeed, the connection between AB and BC does not depend upon any inference, or connection of causes, but rather it is apparent *per se* from the hypothesis of the assumed whole. It is, moreover, with its corequisites directly, for they should always go together according to a specified mode of consideration; indeed both things we view just as beings, and also things we view as objects of thought [*cogitabilia*], for example GOD, man, or virtue, we could consider as parts of one whole composed from them. Therefore, immediate and indeed distinct requisites are excluded such as rationality in the abstract, which is an immediate, distinct requisite of man; for man is not rationality, viewed here not as going[13] with man, but as an attribute: otherwise one cannot reasonably deny that even from these two, man and rationality, one whole could be formed, of which they are the parts. But the rationality of a man is not a part, since it is required for a man obliquely, i.e. it doesn't go together for a certain reason with other things that are additionally required for a man. But these are, rather, metaphysical matters, and are brought up only to oblige those desirous of an intimate understanding of concepts. We define it simply thus: Parts are things which are required for an individual [*ad unum*] insofar as they go with it.

(26) The *number* is that whose relation to unity is that between part

[13] Leibniz first wrote *non ... ut homogeneum hominis*, and revised the phrase in a second draft to *conveniens cum homine*. A few lines later, *seu non convenienti quadam ratione* ('i.e. it doesn't go together for a certain reason') was *non ... id quod ex ipsis homogenea* ('i.e. not ... that which [arises] from them by reason of certain homogenea') in a prior draft. Possibly he made these changes when he also struck out the last section of (24) in which he had defined homogenea. He does not give a revised definition of homogenea, but he provides some explanatory comments about what homogenea are later in (27) and (30) in terms of parts and wholes. In any case, he seems to intend largely the same concept with 'conveniens'. Perhaps this is a 'deeper' concept.

and whole or whole and part; in this way fractions and even irrationals are included.

(27) The *magnitude* of a thing (distinctly known) is the number (or composite of numbers) of parts congruent to a certain specified thing (which is taken as a measure). So if I know there is a line segment that is equal to twice the side, thrice the diagonal of a certain rectangle I adequately know, so that I can return to the same thing at will, then I am said to know the magnitude of its line, which would be two parts congruent to the side + three parts congruent to the diagonal.

Although various distinct measures may be taken, by which the same thing is expressed in various ways, yet nonetheless the same magnitude always results, since when the measures themselves are resolved again, the same thing is always reached in the end; and thus the various measures already involve that very same number emerging in the resolution. For instance the number three quarters is one and the same with six eighths, if the quarter is resolved into two parts. Such is magnitude distinctly known.

In any case, *magnitude* is the attribute of a thing through which one can tell whether some proposed thing is a part of it, or another homogeneum belonging to the thing and indeed such that it remains the same if the configuration of the parts among themselves changes.

Or indeed magnitude is the attribute which remains the same, provided the homogenea belonging to it remain the same or congruent ones are substituted for them. By 'homogenea' belonging to some thing I mean not only parts, but also extrema and minima, or points. For a curve is made by a certain continuous repetition, or motion, of a point. Often, however, a thing is transformed such that there is not a single part of the new figure congruent to a part of the previous one.

I define magnitude in another way, below, as that by which two similar things could be distinguished, or that in things which is only discerned by co-perception. But these all come back to the same thing.

(28) The *ratio of A to B* is nothing but the number by which the magnitude of A is expressed if the magnitude of B is set as the unit. It is clear from this that magnitude differs from ratio as a concrete num-

ber from an abstract number; for magnitude is the number of things, namely of parts, but ratio is the number of unities. It is also clear that the magnitude of a thing remains the same, no matter what measure is taken through which we wish to express it; but the ratio becomes one or another thing for one or another assumed measure. And it is clear (from the definition of division) that if the number expressing the magnitude of A is divided[14] by the other number expressing the magnitude of B (provided the same measure or unit is used for both), the result will be the number that is the ratio of one to the other.

(29) *Equals* are things whose magnitude is the same, or which could be rendered congruent without loss or addition.

Something is called *less* which is equal to a part of the other, whereas that which has a part equal to the other is called *greater*.

Hence the part is less than the whole, since it is equal to a part of it, namely to itself. We will, now, use these symbols:

$a \sqcap b$	a equals b
$a \ulcorner b$	a greater than b
$a \urcorner b$	a lesser than b

If a part of one is equal to another whole, the magnitude of the remaining part of the greater is called the *difference*. The magnitude, moreover, of the whole is the *sum* of the magnitudes of the parts, or of others equal to its parts.

(30) If two things are homogeneous (or if any kind of parts could be assumed in the one equal to the parts of the other, and the same can always be done also with the leftover ones) and there is no difference between them, that is, if neither a is greater than b, nor b greater than a, then they are necessarily equal. Indeed, let them be transmuted into congruents to the extent possible; then certainly either something superfluous will remain in one, or they will be congruent, and thus they will be equal. And so this inference is valid here:

[14] Latin: '*si numerus … A et alius numerus … B dividatur*', i.e. 'if number … A and the other number … B is divided'. The syntactic infelicity can be explained as the result of Leibniz squeezing in this last sentence, between lines, after the next paragraph was written.

$$a \text{ not} \ulcorner b, \quad a \text{ not} \urcorner b. \quad \text{Therefore } a \sqcap b.$$

(31) *Similars* are things which cannot be distinguished when considered individually in themselves [*singula per se*]. Just as two equilateral triangles (in Figure 15). For there is no attribute, no property

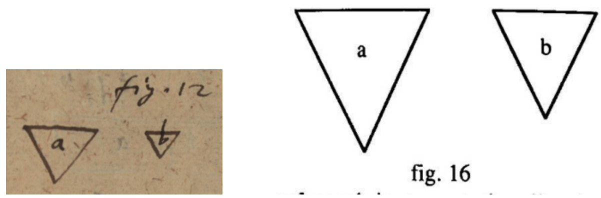


Figure 15

we could discover in one which we could not also find in the other. And calling one of them a, the other b, we notate *similarity* thus: $a \sim b$. If, however, they are perceived simultaneously, immediately there appears the difference that one is greater than the other. The same thing can happen even if they are not perceived simultaneously, only assuming something as an intermediary or a measure, which is applied first to the one, or to something in it, and noting how the first thing (or its part) is congruent to the measure (or its part), then later the same measure is also applied to the other. And so I usually say that *similars* are not distinguished except through co-perception.

Now, you might say, I could still properly distinguish two unequal equilateral triangles if I were to see them in succession. But of course I am speaking here about discernment [*intelligentia*], namely that the mind could note something in one that does not happen in the other, and not about sense and imagination.[15] The reason, more specifically, that the eyes distinguish two similar but unequal things is apparent: for we retain images of the things perceived previously, which, applied to the images of the newly perceived thing, show the difference by the very co-perception of these two images. And the back of the eye itself,

[15]Leibniz inserted *non de sensu et imaginatione* in a later draft, using the margin.

the greater or lesser part of which is occupied by the image, serves as something of a measure. Finally, we are accustomed to perceiving always together other things which we have also perceived with the previous ones; from this we note the difference without difficulty, relating the newest perceived thing back to them, just as we related the previous one to the same things.

But if we imagine that God were to shrink everything appearing in us and around us in some room, keeping the same proportions, everything would appear the same, and we could not distinguish the earlier state from the later one, unless of course we were to step out from the sphere of things proportionally diminished, that is to say, from our room; for then by that co-perception, shown with the undiminished things, the difference would become apparent. From this it is also manifest that *magnitude* is that which may be distinguished in things only by co-perception, that is, either by immediate application, whether by actual congruence or by coincidence, or else by mediate application, namely by the intervention of a measure, which is applied now to one and now to the other; whence it suffices for things to be congruent, that is, to be able to be made actually congruent.[16]

(32) From this one can understand, moreover, that things simultaneously similar and equal are congruent. And similarity we shall denote by this sign: $\sim$, namely $a \sim b$, that is, a is similar to b. See Figure 16. From this we have the inference:

$$a \sim b \text{ and } a \sqcap b. \text{ Therefore } a \between b.$$

[16] The Latin *actuali* in 'by actual congruence', and *actu* in 'made actually congruent', carry a complex sense that combines a side of the 'real/imaginary' distinction with a side of the 'actual/potential' distinction, the latter including the operational aspect of actuality, i.e. 'by action', stemming from Aquinas's use of *actus* to render Aristotle's Greek *energeia*. Actual congruence is real congruence, congruence in effect, and agreement through action. It is also noteworth that Leibniz makes use of the distinct forms 'congrua esse' and 'congruere', both meaning 'to be congruent'. English no longer allows the archaic verb 'to congrue'; sometimes we write 'make congruent' when he uses the second form. It is not a consistent, technical distinction.

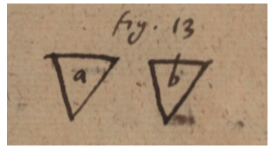


Figure 16

(33) We also have other inferences:

$$\begin{array}{ccc}
a \between b. & \text{Therefore} & a \sqcap b. \\
a \between b. & \text{Therefore} & a \sim b. \\
a \infty b. & \text{Therefore} & a \between b. \\
\ldots & \ldots & a \sqcap b. \\
\ldots & \ldots & a \sim b.
\end{array}$$

(34) For things which really coincide are certainly congruent; and congruent things are certainly similar, and likewise equal. From this we see that there are three modes, as it were degrees, of distinguishing things provided with extension and not otherwise differing in qualities.

The maximum is when they are dissimilar, for thus, when viewed individually in themselves, they are easily distinguished by the difference of the properties which are observed in them; thus an isosceles triangle is easily distinguished from a scalene one, even if they are not viewed simultaneously. Indeed if someone asked me to see whether some proffered triangle were isosceles or scalene, I would not need to assume anything from outside, I would simply compare its sides among themselves.

But on the other hand, if I were asked to choose the greater of two equilateral triangles, I would need to do so by bringing them together [*collatione*] of the triangles with other things, i.e. by co-perception, as I have explained, and neither could I attribute any mark of difference seen in them individually.

But if in fact two things are not only similar, but also equal, that is if they are congruent, then they cannot be distinguished even if perceived simultaneously, except by place; that is, unless I assume something extra outside of them, and I observe that they have a different situation to the third assumed thing.

Finally, if both are simultaneously in the same place, now I have nothing more by which to distinguish them.

And this is the true analysis of the thoughts [*cogitationum*] that we have about these matters, ignorance of which has led to the fact that a true geometric characteristic has not been established until now.

From these thoughts it may finally be understood that, as magnitude is assessed when things are understood to be congruent or reducible to congruity, so ratio is assessed by similarity, that is, when things are reduced to similarity, for then everything will necessarily become proportional.

(35) From these descriptions of coincidents, congruents, equals, and similars, certain inferences may be derived. Namely, things which are equal, similar, congruent, or coincident to the same thing, are also to each other, and therefore:

$$\begin{array}{cccc} a \infty b & \text{and} \quad b \infty c & \text{Therefore} & a \infty c \\ a \between b & b \between c & & a \between c \\ a \sim b & b \sim c & & a \sim c \\ a \sqcap b & b \sqcap c & & a \sqcap c \end{array}$$

However, these inferences are not valid:

$$a \text{ not } \infty\, b \quad \text{and} \quad b \text{ not } \infty\, c. \quad \text{Therefore} \quad a \text{ not } \infty\, c.$$

Just as in logic, nothing follows from pure negatives.

(36) If coincidents are joined to coincidents, or to the same thing, then coincidents will result:

$$a \infty c \quad \text{and} \quad b \infty d \quad \text{Therefore} \quad a.b \infty c.d.$$

But this does not follow with congruents because, for example, if $A.B.C.D.$ are points, it is always true that $A \between C$ and $B \between D$. Indeed any point is congruent to any point. But you cannot on that account say $A.B \between C.D$, that is, that A can be made congruent [*posse congruere*] with C and

B with D simultaneously, while preserving of course both the situation $A.B$ and the situation $C.D$. Conversely, though, upon supposing $A.B \between C.D$, it would follow that $A \between C$ and $B \between D$ from the meaning of our characters, and for that reason it is also true even if $A.B.C.D$ are not points, but magnitudes. But if congruents are joined together, equals result from this; so: *supposing* $a+b-c \sqcap d+e-f$*, then* $a \between d$ *and* $b \between e$ *and* $c \between f$ *since congruents are always equals*.

(37) Of course, if congruents are added to or subtracted from congruents in a similar way, congruents will result. The reason for this is that when congruents are added to congruents in a similar way, then similars will be added to similars in a similar way (since congruents are similars), therefore similars will result. And they will also be equals (for congruents added to congruents make equals). Now, things similar and equal are congruent. Therefore, *if congruents are added to congruents in a similar way, congruents will result*. Likewise if they are subtracted.

(38) Now, whether some things are treated similarly may be understood from our characteristic and the way in which each thing is described or determined. If no difference can be noted in each one considered individually, then it is certainly necessary that everything always comes out similar. The noteworthy point is that *if things are similar according to one way of determining (distinctly knowing, describing), the same things will also be similar according to another way*. For each way of determining involves the whole nature of the thing.

(39) The axioms used by Euclid, that if equals are added to equals then equals will result, and others of the sort, are easily demonstrated from the fact that equals have the same magnitude, that is, they can be substituted for each other while preserving the magnitude. For letting $a \sqcap c$ *and* $b \sqcap d$*, then* $+a+b \sqcap c+d$ *results*; for if we write $a+b$, and then substitute equals $c.d$ in place of $a.b$, this substitution will preserve magnitude. And accordingly, those yielding $+c+d$ will have the same magnitude as the prior ones [yielding] $+a+b$. But this belongs rather to the algebraic calculus, and has been explained sufficiently, so we will not delay over the rules of magnitude, ratio, and proportion. I will

undertake to explain chiefly the things that involve situation.

(40) I return now to what was interrupted in §22, and I will consider first to points, and from there, tracts. Every point is congruent, and consequently equal (if one may speak thus) and similar:

$$A \between B. \quad A \sqcap B. \quad A \sim B.$$

(41) $A.B \between C.D$ signifies simultaneously that $A \between C$ and $B \between D$, with the situation $A.B$ and $C.D$ remaining [the same].

[[17] $A.B \between A.Y$ is the proposition signifying that, positing two points $A.B$, some third Y can be found (which I denote by this letter since it is indefinite) such that, preserving the situation $A.Y$ and $A.B$, $A.Y$ and $A.B$ could be applied to each other, namely A to A and B to Y simultaneously. Then A remains on A (that is A remains where it is), and B on Y.]

(42) The meaning of the proposition $A.B \between B.A$ is that, supposing two points $A.B$, one could permute their locations, i.e. put A in the place of B and vice versa, while retaining the same situation between them. This is clear because the relation of situation that they have belongs to both in the same way, and without assuming external things, no difference can be noted after the permutation.

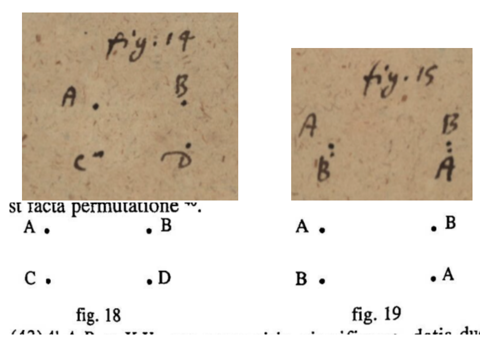


Figure 17

[17] Leibniz cancelled this portion during revision. Possibly he felt that both it and the cancelled portion in (42) were superseded by the material in (43).

[[18]The proposition $A.B \between C.Y$ signifies that, given three points $A.B.C$, one can find a fourth Y whose situation to one of them C is the same as the situation of the remaining two $A.B$ between themselves, i.e. such that A could be made congruent [*congruere possint*] to C and B to Y while simultaneously preserving the situation $A.B$ and $C.Y$. From this $A.B \between A.Y$ also follows, supposing $C \infty A$.

Now the reason for this arises from the nature of space, in which nothing can be taken which could not be taken again in exactly the same way, such that the only difference is in place. The same thing is shown by motion, thus: translate $A.B$ simultaneously, preserving situation, and when A falls in the place of C, then of course B will fall in the place of some point Y.]

(43) The proposition $A.B \between X.Y$ signifies that, given two points A and B, another two X and Y can be found having the same situation between themselves as those two, or such that they could simultaneously be made congruent to those two while preserving the situation in both pairs. It is demonstrated from the fact that $L.M$[19] could move preserving the situation between themselves, and they will be able to correspond at first to $A.B$ and later to $X.Y$, namely $3L.3M \between 6L.6M$. Let $A \infty 3L$, $B \infty 3M$, $X \infty 6L$, $Y \infty 6M$, then $A.B \between X.Y$. Furthermore, nothing prevents $X \infty A$, whence $A.B \between A.Y$. And $X \infty C$ can also be given, whence $A.B \between C.Y$.

(44) *If* $A.B \between D.E$ *and* $B.C \between E.F$ *and* $A.C \between D.F$, *then* $A.B.C \between D.E.F$. See Figure 18.

For $A.B \between D.E$ just signifies $A \between D$ and $B \between E$ simultaneously, preserving the situation $A.B$ and $D.E$. In the same way, from $B.C \between E.F$ follows $B \between E$ and $C \between F$, preserving the situation $B.C$ and $E.F$; and from $A.C \between D.F$ follows $A \between D$ and $C \between F$, preserving the situation $A.C$ and $D.F$. Therefore, we have $A.B.C \between D.E.F$ simultaneously, preserving the situation of $A.B$ and $B.C$ and $A.C$, and likewise the situation of $D.E$ and $E.F$ and $D.F$; whereas from $A.B \between D.E$ and $B.C \between E.F$ alone,

[18]Leibniz cancelled this portion during revision.

[19]The manuscript has '$L.M$ $A.B$' with the $L.M$ seemingly added at the end of a line. It seems Leibniz forgot to cancel the '$A.B$' starting the next line after adding $L.M$.

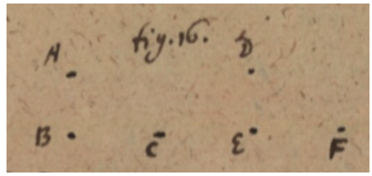


Figure 18

it would follow that $A \between D$ and $B \between E$ and $C \between F$ simultaneously. But if only the situations $A.B$ and $B.C$ and likewise $D.E$ and $E.F$ are preserved, it will not in fact be expressed that the situations $A.C$ and $D.F$ are also preserved unless $A.C \between D.F$ is added.

From this we now have a principle for extending the reasoning to even more points.

(45) *If* $A.B \between B.C \between A.C$, *then* $A.B.C \between B.A.C$ (or any other order). For if we join some congruents C and (C) to the congruents $A.B$

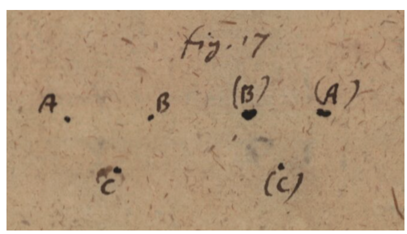


Figure 19

and $(B).(A)$ in a congruent way, then since $A.C \between (B).(C)$ and $B.C \between (A).(C)$, by the preceding we will get congruents $A.B.C \between (B).(A).(C)$, i.e. $A.B.C \between B.A.C$. Parentheses are added just to avoid confusion from the repetition. From this it is clear what it is to be joined in a congruent way, namely when all combinations on one side of the statement are congruent to all on the other side. It is clear from this that *if* $A.B \between B.C \between A.C$, *then:*

$$A.B.C \between A.C.B \between B.C.A \between B.A.C \between C.A.B \between C.B.A.$$

(46) *If* $A.B.C \between A.C.B$ *then* $A.B \between A.C$ *(only) follows* (i.e. the triangle is isosceles), for it follows that:

$$\begin{array}{c}\overbrace{A\overset{\frown}{.}B\overset{\frown}{.}C}\\[1em] \underbrace{A\underset{\smile}{.}C\underset{\smile}{.}B}\end{array} \qquad A.B \between A.C \quad B.C \between C.B \quad A.C \between A.B$$

of which $B.C \between C.B$ is clear *per se*, and the remaining two, $A.B \between A.C$ and $A.C \between A.B$, reduce to the same thing; from that we therefore deduce this one: $A.B \between A.C$.

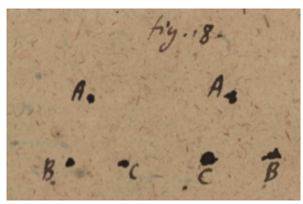


Figure 20

(47) *If* $A.B.C \between B.C.A$, *then* $A.B \between B.C \between A.C$ *follows* (i.e. the triangle is equilateral). For then:

$$A.B \between B.C \qquad B.C \between C.A.$$

(48) *If* $A.B.C \between A.C.B$ *and* $B.C.A \between B.A.C$ *then* $A.B \between B.C \between A.C$ *will result.* For from $A.B.C \between A.C.B$ we get $A.B \between A.C$, and in the same way, from $B.C.A \between B.A.C.$ we get $B.C \between B.A$, i.e. $A.B \between B.C$. (And so, whenever in a transposed order of the points, one of the three keeps the same place in both orders, and the latter situation is congruent to the former, from this one can prove only that the triangle is isosceles; but if none of the points keeps its place, and still the latter situation is congruent to the former, the triangle is equilateral.)

(49) *If we have* $A.B \gamma B.C \gamma C.D \gamma D.A$ *and* $A.C \gamma B.D$*, then*

$$A.B.C.D \gamma B.C.D.A \gamma C.D.A.B \gamma D.A.B.C$$
$$\gamma D.C.B.A \gamma A.D.C.B \gamma B.A.D.C \gamma C.B.A.D.$$

This is easy to show from the preceding, as are many others of the kind,

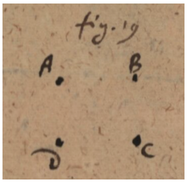


Figure 21

which it will suffice to demonstrate when we need them. Now we have adequately given the principle of discovering these things by calculus alone, without inspection of a figure.

(50) If three points $A.B.C$ are said to be *situated in alignment*, then setting $A.B.C \gamma A.B.Y$ will yield $C \infty Y$. This proposition is the definition of points which are said to be situated in alignment.

Consider Figure 22, where C has some situation to A and B. Take now some point Y having the same situation to $A.B$; if it can be taken distinct from C, then points $A.B.C$ are not situated in alignment, but if it necessarily coincides with C, they are said to be situated in alignment.

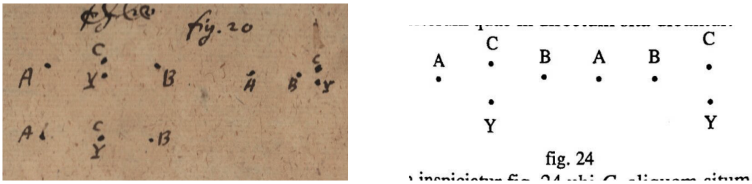


Figure 22

(51) With two points being given, we can always assume a third one situated in alignment with them, i.e. if $A.B.Y \gamma A.B.X$, then $Y \infty X$.

For, with two points $A.B$ being given, we can always assume a third one Y that can move, preserving its situation to them, while they remain unmoved. But the path which it describes by its motion can be ever smaller and smaller, as the point Y is assumed now one way and now another, until at last we can take one such that the space of the motion vanishes, and then the three points will be in alignment.

This is articulated significantly better as follows: $A.B.3Y \between A.B.6Y$ yields $3Y \infty 6Y$, that is, some Y can be assigned which cannot move or change its place while preserving its situation to $A.B$.

I seem to be able to show this differently, like this: suppose there is some extensum that moves while preserving the situation of its points among themselves and to two unmoved points taken in it. For if anyone denies that it can move, he thereby grants that its points cannot move while preserving their situation to the two assumed points, and so they are, by definition, situated in alignment with it. But there is no reason why only these points $A.B$ can be taken as unmoved during the same motion, and not others as well, i.e. there is no reason why the points of the extensum, which moves with these two unmoved, should preserve only the situation to these two unmoved points, and not also to others; for the situation that $A.B$ maintain between themselves makes no difference, and so one could have taken some Y in place of B that maintained another situation to A than B does. But whatever things can be taken as unmoved, those things are unmoved when the motion of the extensum remains the same. And because, with the two points being taken as unmoved, the motion of the extensum is determined, i.e. it is determined which of its points move and which do not move, hence with the two assumed points being unmoved, many others are determined which cannot move while preserving their situation to the same ones, i.e. which lie in alignment with them.

(52) *If* $E.A.B \between F.A.B$ *and* $E.B.C \between F.B.C$*, then* $E.A.C \between F.A.C$. See Figure 23. For by $E.A.B \between F.A.B$ we have $E.A \between F.A$, and by $E.B.C \between F.B.C$ we have $E.C \between F.C$. Now if $E.A \between F.A$ and $E.C \between F.C$,

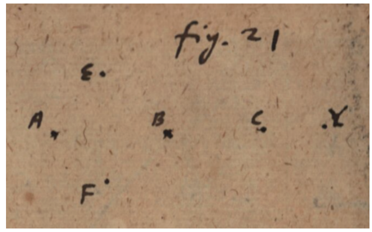


Figure 23

then

$$\overparen{E \,\overparen{.}\, A \,\overparen{.}\, C} \between \overparen{F \,\overparen{.}\, A \,\overparen{.}\, C}$$

by Prop. 44 (indeed, $E.A \between F.A$ and $E.C \between F.C$ and $A.C \between A.C$). Therefore if $E.A.B \between F.A.B$ and $E.B.C \between F.B.C$, then $E.A.C \between F.A.C$. *Quod erat demonstrandum.*

(53) Hence also *setting* $E.A.B \between F.A.B$ *and* $E.B.C \between F.B.C$ *yields* $E.A.B.C \between F.A.B.C$. For indeed $E.A.C \between F.A.C$ by the preceding; therefore we have: $E.A.B \between F.A.B$ and $E.A.C \between F.A.C$ and $E.B.C \between F.B.C$ and $A.B.C \between A.B.C$, that is, we have everything which may be derived from this: $E.A.B.C \between F.A.B.C$; therefore we now also have $E.A.B.C \between F.A.B.C$. (Evidently this is an excellent way of regressing[20] to the antecedent from all the consequences, exhausting the complete nature of the antecedent.)

(54) *If* $E.A \between F.A$, $E.B \between F.B$, $E.C \between F.C$, *then*

$$E.\overparen{A \,\overparen{.}\, B \,\overparen{.}\, C} \between F.\overparen{A \,\overparen{.}\, B \,\overparen{.}\, C},$$

since the combinations remaining on the two sides in need of comparison, $A.B$ and $B.C$ and $A.C$, coincide on the two sides.

(55) $A.B.X \between A.B.Y$, i.e. two points $A.B$ being given, two others $X.Y$ may be found such that X and Y have the same relation to $A.B$. See Figure 24. Indeed $A.X \between A.Y$ and $B.Z \between B.V$ can be obtained by

[20] Cf. the footnote to (6).

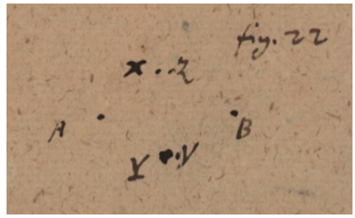


Figure 24

Prop. 43. Setting $Z \infty X$ (this can be done by Prop. 43, i.e. Z can be given, i.e. with X being now assumed, since $A.B \between A.V$) and likewise setting $A.X \between B.X$ (for also in $A.X \between B.Y$, X can be given since $A.C \between A.Y$ is given by Prop. 43), yields

$$V.B\,(\between B.Z \between B.X \between A.X) \between A.Y.$$

Therefore $V.B.X \between Y.A.X \between X.B.V$, in which everything determined up to now is contained. Therefore we can set $V \infty Y$; for nothing prevents it in what has been determined up to now. Then $Y.B.X \between Y.A.X$. Therefore $Y.B \between Y.A$, $B.X \between A.X$. Again, $Y.B.X \between X.B.Y$. Therefore $Y.B \between X.B$. Therefore $Y.B \between X.B \between Y.A \between A.X$ results. Therefore $A.B.X \between A.B.Y$.

(56) If three points $E.F.G$, taken distributively, have the same relation to three points $A.B.C$, taken collectively, the former three will be in the same arc of a circle, the latter three in the same line, i.e. will lie in alignment. It seems good to make note of this proposition; the reason will become clear from what follows.

(57) If $E.A.B.C \between F.A.B.C \between G.A.B.C$, and if E *not* ∞F, E *not* ∞G, F *not* ∞G, then we say *(however many) points* $A.B.C$ *are situated in alignment*, i.e. are on the same line.

(58) The point C may be omitted. If $E.A.B \between F.A.B \between G.A.B$, then the points $E.F.G$ are in *the same plane*.

(59) From the same things being supposed, the points $E.F.G$ will be in the same *arc of a circle*.

(60) Between any two congruent things, infinitely many other con-

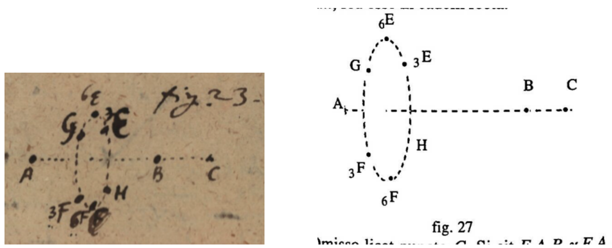


Figure 25

gruents can be taken, for it is only through congruents that the one can pass into the place of the other while preserving its form.

(61) Hence a curve can be drawn from any point to any other point. For a point is congruent to a point.

(62) Hence a curve can be drawn from any point through any point.

(63) A curve can be drawn which passes through any given point.

(64) In the same way it is shown that a surface could pass through any given curves. For if they are congruent, clearly the generating curve could be in all of them in succession. If they are not congruent, clearly the generating curve could be augmented, diminished, and transformed during its motion, such that when it arrives there it will be congruent.

(65) Each thing can be put in space, preserving its form; i.e. to anything existing in space, an infinity of other congruent things can be assigned.

(66) Each thing can move, preserving its form, in an infinity of ways.

(67) Each thing can so move, preserving its form, that it falls on a given point.

More generally: each thing can so move, preserving its form, that it falls on another thing to which something congruent can be specified in it. For one congruent thing can be translated into the place of another; and nothing prohibits that in which the congruent thing is, which needs to be translated, from being translated simultaneously with it, since there is no reason for a separation: what can be fitted to one of the congruents can also be fitted in a similar way to the other of the congruents.

(68) $A \between B$,[21] that is, to an assumed point, any other point is congruent.

(69) $A.B \between B.A$, as above.

(70) $A.B \between X.Y$. In the same way, $A.B.C \between X.Y.Z$, and $A.B.C.D \between X.Y.Z.\Omega$, and so on. Indeed, this is just the fact that however many points can move while preserving the situation among themselves. Now, we can understand that their situation among themselves is preserved if they are set as the extremes of a rigid curve of any kind.

(71) $A.B \between C.X$. $A.B.C \between D.X.Y$. Etc.

Indeed, this is just the fact that however many points, such as $A.B.C$, can move while preserving the situation among themselves, such that one of them, A, falls on some given point D, with the two remaining ones falling on any particular others, $X.Y$.

(72) If $A.B.C$ is not $\between A.B.Y$ unless $C \infty Y$, then the points $A.B.C$ are said to be *situated in alignment* (see Figure 26), i.e. C *is situated in alignment with* $A.B$ if what has that situation to $A.B$ is unique. Now whether such a situation of points can be obtained is something to be investigated later. A curve, all of whose points are situated in alignment, is called *straight*. Indeed, let $A.B.zY \between A.B.zX$ and thus $zY \infty zX$, then $\overline{z}Y$ ($\infty \overline{z}X$) will be *a straight line*; that is, if the point Y so moves that it always preserves the situation to the points $A.B$ which can belong only to itself, i.e. which is determined, and varies or wanders minimally, then it describes a straight line.

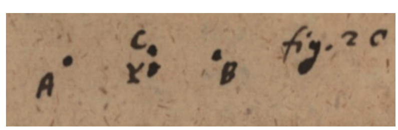
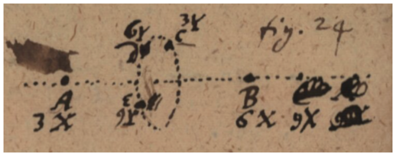


Figure 26

(73) If $A.B.C \between A.B.D$, then $\between A.B.zY$, see Figure 27, for then $C \infty$

[21]The manuscript shows Y and B written in the same place; probably the Y was amended into the B.



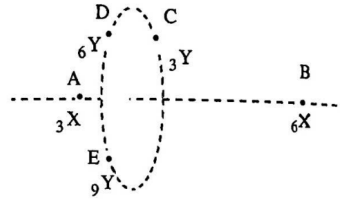


fig. 29

Figure 27

$3Y$ and $D \infty 6Y$, namely C and D are places of the motion of Y such that it preserves the same situation to $A.B$, between which there will necessarily be other indefinite ones, indicated of course through zY. Now the curve $\overline{z}Y$ is called *circular*. It should be noted, moreover, that this description of a circular curve is prior to the one Euclid gave. For Euclid requires the straight line and the plane. From ours it works whatever kind of rigid curve is assumed, just taking two points in it which are unmoved while the curve itself, or at least some point in it, does move, for this point keeps the same situation to the two assumed points as they are all in the rigid curve. Therefore this point describes a circular curve by this definition of ours.

If someone denies that such a point could be found on the rigid curve which moves while the two given ones remain unmoved, by the preceding definition (Prop. 72) it will be necessary that all the points of the rigid curve are situated in alignment, i.e., it will be necessary that a straight line is given. It must, therefore, be accepted in this way that either a straight line is possible, or a circle is. Once either one has been accepted, we then deduce the other one from it. It should be noted here by the way, as will be clear in its own place, that an arc of a circle can pass through any three given points, hence from three points being given, we can find one which relates in the same way to those three, namely $X.C \between X.D \between X.E$, and this can be done many times, i.e. various X can be found for the same $C.D.E$, and all X fall on one straight line which passes through the centre of the circle and is at right angles to its plane.

(74) Let there be some curve $\overline{z}Y$, see Figure 28, in which however many points $3Y.6Y.9Y.12Y$ etc. can be taken, such that

$$3Y.6Y \between 6Y.9Y \between 9Y.12Y \text{ etc.}$$

For in general, if there is any sufficiently small curve with one extreme in another curve, the former can so move, with its extreme point common to both curves being unmoved, such that it also meets the latter curve with its other extreme, and so one part will be cut off by this motion, and then again another one with a new point being found and remaining unmoved, and so on.

But now I observe that this is not actually necessary, and it suffices that one curve attaches to the same curve by its extremes in many ways, such that several parts of the same curve are assigned whose extremes are congruent to the extremes of the others. As in Figure 28, the rigid

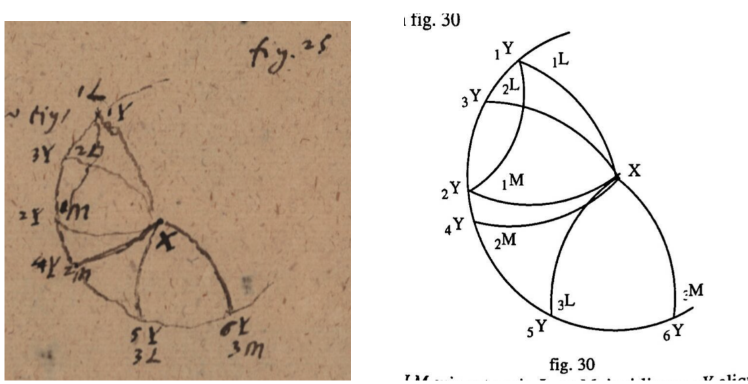


Figure 28

curve LM [is attached[22]] by its extremes L and M to the curve zY a number of times in $1Y.2Y$, or $3Y.4Y$, or $5Y.6Y$, which coincide with $1L.1M$, and $2L.2M$, and $3L.3M$. For if $L.M$ can be attached to $\overline{z}Y$ once, it can be attached in an infinite number of ways, provided the later ones

[22]Leibniz cancelled *applicatur* here, but he did not supply another verb.

were to be separated by however little from the [earlier] attachments.[23] Now let two congruent curves be drawn from L and M, with the one drawn from L relating in the same way to L as the one drawn from M relates to M, and which meet each other in X. And let $1LX1M$ ☋ $2LX2M$, and so on, that is, those drawn from $1L$ and $1M$ are extended so that they do not meet each other before the place where those (drawn in the same way) from $2L2M$ meet each other. It is clear from this that the point X relates in the same way to all those assigned Y, and indeed if the curve is such that it has a point of this kind, which relates in the same way to all of its points, it is found in this way. If, moreover, the curve is circular, as here, it suffices to find some point relating in the same way to three points of the circular curve, for then it will relate in the same way to all the others. The reason for this is that, from three given points $C.D.E$ (see Figure 27) with $C.D$ ☋ $D.E$ being set, some specific point is determined by the method written a little earlier for Figure 28, and accordingly, taking any other three points on the circle congruent to the first three, drawing congruent curves in the same way that meet each other, it is necessary that one will always arrive at the same X. Hence, since from three given points $D.C.E$ distinct points X can be found only according as the congruent curves are drawn in various distinct ways, i.e. converge faster or slower, clearly other points X could also be found, and they will all fall on one line.

(75) However, the same thing can be inferred more simply without a circle. Let there be three points, $A.B.C$, such that $A.B$ ☋ $B.C$ ☋ $A.C$, and suppose points X are found such that $A.X$ ☋ $B.X$ ☋ $C.X$, and this [happens] as many times as you like, or what is the same thing: let the point X move such that any of its places, say zX, relates in the same way to $A.B.C$, that is, such that $A.3X$ ☋ $B.3X$ ☋ $C.3X$; then the points zX will be set in alignment, i.e. $\overline{z}X$ will be *a straight line*. And thus it appears what Euclid intends when he says that a straight line lies equitably between its points,[24] that is, it doesn't swerve to either side,

[23]Latin: *si posteriores applicationibus quantumvis parum distent*; it would seem Leibniz meant *si posteriores prioribus applicationibus* or similar.

[24]Latin: *ex aequo sua interjacere puncta*.

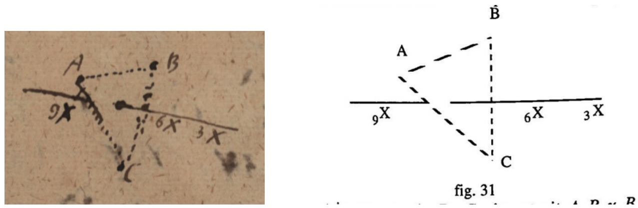


Figure 29

nor relate during its motion to point A in any other way than it does to B or C. Hence, moreover, one also has a method of finding the points X of the line $\overline{z}X$. Namely, if any curve whatever is drawn out from A relating in the same way to B and C, and likewise another through B that is congruent to the first one and congruently set, that is, so that a point of this corresponding to a point of that would relate in the same way to $B.A.C$ as the point of that does to $A.B.C$, and these curves are extended until they meet each other, then they will necessarily meet in a point X which relates in that same way to $A.B.C$. And if such a curve were also drawn through point C, congruent and congruently to the first ones, it would meet them in the same point X. Hence, however many points of this kind could be obtained, and so indeed a straight line can be described through the points.

(76) Let us revisit a few things: from any point to any point, one can understand a curve to be drawn, and that *rigid*.

(77) $A.B$ signifies the situation of A and B between themselves, that is, some rigid tract through A and B, which tract, it will suffice for us, is a curve. Thus $A.B.C$ signifies another rigid tract through $A.B.C$.

(78) No matter what may be put in space, it can move, whether a point, or a curve, or another tract, or anything to which another congruent thing can be assigned in the extensum. Hence $A \mathbin{\text{ɣ}} X$, $A.B \mathbin{\text{ɣ}} X.Y$, $A.B.C \mathbin{\text{ɣ}} X.Y.Z$, or $A.B.C \mathbin{\text{ɣ}} \omega\overline{X.Y.Z}$.

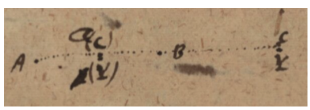
(79) Given two distinct things in an extensum, one can be set still, the other to move.

(81) If some of the things which are in a rigid tract move, the rigid tract itself moves.

(82) Every tract can so move that a given one of its points falls on another given one. $A.B.C \between D.X.Y$.

(83) Every tract can move with one of its points remaining unmoved. $A.B.C \between A.X.Y$.

(84) The *straight line* is a tract which cannot move while two points in it remain still, i.e. if a certain tract moves with two of its points remaining unmoved, and if moreover some other points in it are set to remain still, all these points are said to be situated in alignment, or to fall in a tract which is called a straight line. I.e. if $A.B.C \between A.B.Y$, then necessarily $C \infty Y$, that is, if some point C is found situated in



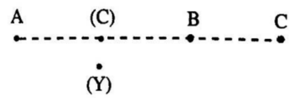


fig. 32

Figure 30

alignment with the points $A.B$, then the tract $A.B.C$ (or $A.C.B$) cannot so move, while $A.B$ remain unmoved, that C is translated to Y, and thus the tract $A.B.Y$ is congruent to the first one $A.B.C$. Or what is the same, another point Y cannot be found besides the one C, which has the same situation to the fixed points $A.B$ that C has, but rather it is necessary that if such a point Y is taken, it coincides with C, i.e. $Y \infty C$. From this we can say that the point C is the unique instance with its situation to the points $A.B$. And a point which so moves that it preserves a situation unique of its kind [*in sua species*] to two fixed points, moves in a straight line. Namely, if $A.B.Y \between A.B.X$ and for that reason $Y \infty X$, then $\overline{\omega}X$ ($\infty\, \overline{\omega}Y$) will be a straight line. Now whether two such points situated in alignment are given, and whether they form a tract, and whether this tract is a curve, should not be assumed but rather demonstrated. But the path of a point so moved will certainly be a *straight line* which, if indeed it passes through all points of this kind,[25] namely the locus of all points joined in alignment with the two

[25] At first Leibniz concluded the sentence with: 'certainly this line will be nothing

points, will not be any other tract than a curve.

(85) If two points $A.B$ remain unmoved in a tract $A.C.B$, and the tract itself moves, the curve which the point C describes by its motion is called *circular*. But whether some tract could move, with two points remaining unmoved, also should not be assumed but needs to be settled [*definiendum*] by a demonstration.

$A.C.B \between A.Y.B$: then $\overline{\omega Y}$ is called a *circular* curve, and if there are

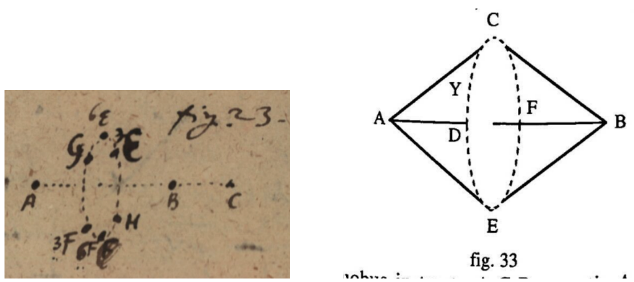


Figure 31

however many points $C.D.E.F$ and $A.B.C \between A.B.D \between A.B.E \between A.B.F$, they are said to be in one and the same circle. This definition of the circular curve does not presuppose a straight line and plane to be given, whereas Euclid's definition does.

(86) The locus of all points which relate in the same way to A as to B will be called a *plane*. I.e. if $A.Y \between B.Y$, then $\overline{Y}$ is a *plane*.

(87) Hence if $A.C.B \between A.Y.B$, and if $A.C \between C.B$ (and thus also $A.Y \between B.Y$), then the curve $\overline{\omega Y}$ will be circular in a plane. Whether every circle is in a plane needs to be settled later.

(88) If $A.B \between B.C \between A.C$ and $A.Y \between B.Y \between C.Y$, then $\overline{\omega Y}$ will be a straight line.

(89) If $A.Y \between A.(Y)$, then Y will be a *spherical* surface.

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but a curve' (*utique ipsa recta non nisi linea erit*). He cancelled that in favour of the presented version. This evidences what is reasonably apparent, namely that for Leibniz it is not patently clear that a tract's being a *curve* (i.e. a *linea*) is not logically prior to its being *straight* (i.e. *in directum*).

[[26] ♉ signifies congruence. ∞ coincidence. When I say $A.B$ ♉ $A.Y$, I could indeed say that the distance AB is equal to the distance AY. But since later on when three or more are brought, as in $A.B.C$ ♉ $A.BY$, by this we do not mean only that the triangle ABC is equal to the triangle ABY, but moreover that it is similar, that is, congruent, and for that reason it is better to use the sign ♉.]

(90) If Y ♉ (Y), then the locus of all Y, or $\overline{Y}$, is absolute extension [*extensum absolutum*], or *space*. For the locus of all points congruent to each other is the locus of all points in general. Indeed, all points are congruent.

(91) The same if Y ♉ A, for (by the signification of the characters) if Y ♉ A, then (Y) ♉ A. Therefore Y ♉ (Y). Namely the locus of all points Y congruent to a given point A is certainly also unbounded space itself, for all points are congruent to any given one.

(92) Next: $A.Y$ ♉ $A.(Y)$. The locus of all Y, i.e. $\overline{Y}$, is called a *sphere*. This is the locus of all points being of the same situation to a given point. And the given point is called the *center*.

(93) It is the same if $A.B$ ♉ $A.Y$. For then also $A.B$ ♉ $A.(Y)$ and accordingly $A.Y$ ♉ $A.Y$, where we note that B is from the number of Y, or is bY. If indeed Y covers all points having the situation to A that B has, then certainly it covers B, which certainly has the situation to A that it has. The sphere is the locus of all points being of a given situation to a given point (that is, of a situation congruent to the situation of a given point).

(94) If $A.Y$ ♉ $B.Y$, the locus of all Y, or $\overline{Y}$, is called a *plane*, i.e. the locus of points Y, each of which is situated in the same way to one point A of two given points as it is to the other B, is a *plane*. It should be noted that this expression of a locus could not be converted to the other in which there are Y and (Y) together.

(95) If $A.B$ ♉ $C.Y$, then $\overline{Y}$ will be a *sphere*. For then:

$$A.B \text{ ♉ } C.Y \text{ ♉ } C.dY.$$

[26] Leibniz added this portion, in his own brackets, in a page header. It is not clear when he added it.

Let $dY \infty D$, then $C.D \between C.Y$, and thus the locus will be a sphere by Prop. 92.

(95) Y and (Y) signify any point of some locus and any other besides the first. zY signifies whatever point of a locus, i.e. all points of a locus distributively. The same is signified by Y put absolutely. Then dY signifies some one particular point of the locus. $\overline{Y}$ signifies all points of the locus collectively, or the whole locus. If the locus is a curve, I signify this by $\overline{\omega Y}$. If it is a surface, by $\overline{\omega\psi Y}$. If a solid, by: $\overline{\omega\psi\varphi Y}$.

(96) If $A.B.C \between A.B.Y$ (i.e. if $A.B.Y \between A.B.(Y)$), then the locus of all Y, or $\overline{Y}$, is called a *circle*. That is, if the situation to two given points is the same for many points (or it is given), the locus will be a *circle*.

(97) If $A.Y \between B.Y \between C.Y$, then the locus of all Y, or $\overline{Y}$, is called a *straight line*.

(98) If $A.B.C.Y \between A.B.D.Y$, then $\overline{Y}$ will be a *plane*, i.e. if $C.D$ are two points relating in the same way to the three $A.B.Y$, these three will be in the same plane, and if from two of these given ones, $A.B$, a third Y is sought, the locus of all Y will be a plane. It is clear from this that $A.B$ are included under Y. It needs to be demonstrated that this locus coincides with the other one which is in Prop. 94. This will be done as follows: $C.Y \overset{(1)}{\between} D.Y$ is the locus for a plane by Prop. 94.[27] Let $3Y \infty A$ and $7Y \infty B$, then $C.A \overset{(2)}{\between} D.A$ and $C.B \overset{(3)}{\between} D.B$. Therefore:

Namely 1 is clear *per se*, and 2 through (3), and 3 through (1), and 4 through (2), and 5 *per se*, and 6 *per se*.

(99) If $A.Y \between B.Y \between C.Y$, the locus $\overline{Y}$ will be a *point*, or the Y satisfying it will be not but unique, i.e. $Y \infty (Y)$. This proposition should be demonstrated.

[27] Manuscript: 92.

(100) We therefore have loci for a point, a line, a circle, a plane, and a sphere, expressed with wonderful simplicity only with congruences; but some of these are real, some possible, and it needs to be demonstrated that our definitions coincide with others.

(101) If any tract or extensum whatever moves with one point being unmoved, any other point of it moves on a sphere. Now let the extensum be rigid, i.e. let its parts preserve the same situation. We will then have a way of obtaining however many points of a sphere. For $A.B$ ☋ $A.Y$ can also be given if the tract passes through two points $A.B$. Certainly it is in our power to draw some tract (whether it be a curve, a surface, or a solid) through two given points, and to move the tract with one point being fixed.

(101)[28] If two congruent tracts pass *congruently* through two given points, that is, such that corresponding points in the two tracts have congruent situations to the two given points, each one to its own, and they move, or even lengthen if necessary, also congruently, until they meet each other, then the places at which their corresponding points meet each other will be points of the plane which we have defined to relate in the same way at any of its points to the two given points. I claim, moreover, that they can move congruently, or be extended congruously[29] and congruently, until they meet.

(102) If we now section a sphere by a sphere, or by a plane, we will have a circle; if a plane by a plane, we will have a line; if a line by a line, a point. But it needs to be shown that these sections can be made, and that [the locus] of points in common to a sphere and a sphere, or a plane and a plane, or a plane and a sphere, is a tract. If a sphere is tangent to a plane or a sphere, the locus is a point, namely when the circle is made minimal, i.e. vanishes.

(103)[30] As for the rest, all the common definitions of a straight line

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[28] The manuscript has two paragraphs '(101)'.

[29] Latin: *congruè ac congruenter*. It is not clear whether the two adverbs are meant to carry different meanings.

[30] Paragraphs 103 through the end of the essay were circled and a note provided: 'let the included portion be omitted or moved back somewhere else'.

are not yet sufficiently perfect, since a demonstration is still always needed that such a line is possible. But it seems that this should be regarded among the easiest things. And so we need such a definition that it is immediately apparent that the straight line is possible. If you define it as the minimal one, this too can be doubted, whether there is a minimal one from a point to a point. If you define it as that whose points cannot be drawn farther apart, you presuppose distance, or a minimal path, for to be drawn apart is to acquire a greater distance. Of course, it is one thing to exhibit a material straight line, i.e. to draw it, and another thing to comprehend it in thought.

(104) When two points are perceived simultaneously, *eo ipso* their situation to each other is perceived.[31] But any two situations between two points are similar, and thus are distinguished only by coperception, i.e. by magnitude. The difference of situation is therefore the magnitude of a certain extensum, when two points are perceived simultaneously and *eo ipso* a certain extensum is perceived.

(105) The *straight line* is the extensum that, when two points are simultaneously perceived, is perceived *eo ipso*.

(106) When one point is perceived, and then again another is perceived separately, no diversity can be noted.

(107) If $A.B$ are simultaneously perceived, and then again $C.D$ simultaneously perceived, a difference is one of situation. Nevertheless, that which is perceived when $A.B$ are perceived and that which is perceived when $C.D$ are perceived, are similar: for clearly nothing can be noted in each individual which cannot be noted in both of them, and so that which is perceived when $A.B$ are perceived and that which is perceived when $C.D$ are perceived, if they can be distinguished from each other, will be distinguished only by magnitude, and so $A.B$ being simultaneously perceived, something having magnitude is simultaneously perceived.

[31] This sentence underwent a few revisions that may be of interest. It was initially '...*eo ipso* a certain curve is perceived', then '...*eo ipso* their distance, i.e. situs, is perceived', then '...*eo ipso* their situs, i.e. distance, is perceived', then the version presented.

When two things are simultaneously perceived to be in space, *eo ipso* a path from one to the other is perceived. And when they are congruent, *eo ipso* a path of one into the place[32] of the other is conceived. But two points are congruent. And so what is perceived, two points being simultaneously perceived, is a curve, i.e. the path of a point. It is also clear that this path is such that, whether you give the line from A to B, or from B to A, the path is the same. And it will be the same if A tends toward B and simultaneously B toward A, and the locus in which A will be is congruent to the locus in which B will be, and congruently set. That is, as the locus of a point coming from A will be to A, so the locus of a point coming from[33] B will be to B, and also, as the locus of a point coming from B will be to A, so the corresponding locus of a point coming from A will be to B.

When we conceive of two points as simultaneously existing, and we examine the reason why we say they simultaneously exist, we will think that they are simultaneously perceived, or at least are able to be simultaneously perceived. When we perceive something as though existing, *eo ipso* we perceive it to be in space; that is, indefinitely many others can exist which could in no way be distinguished from it. Or what is the same, it can move, i.e. it can be in one place as well as in another, and since it cannot be simultaneously in several places, nor move in an instant, we therefore perceive the locus as a continuum. But since it is still indefinite to where it moves, for indeed it can move in many ways which cannot be distinguished from each other, hence, if we suppose another thing besides, congruent to the first, and indeed *eo ipso* the one is thought to be able to pass through into the place of the other, then the mind is determined to some specified motion. And although it could be done in several ways, nonetheless a unique one is determined, for considering which there is no need to assume anything else besides these two being supposed. That is, from supposing two congruent ones to be set in space, a path from one to the other is supposed,

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[32] We use either 'place' or 'locus' where the Latin is just 'locus' in this vicinity.

[33] *ad* ('to') in the manuscript. Leibniz corrected at least three other instances of *ad* to *ab* in this paragraph, and likely this one was missed.

connecting them. But the simplest position is that of a point. For even if you suppose otherwise, nonetheless, since one of the various motions of one to the other would be determined according to which the corresponding points move determinately towards each other, clearly the mind eventually settles upon the consideration of two points, for they are established to be congruent *per se*.

[[34]And so I think first: a point A can exist *per se*. A point exists in space. A point coexists with other points. A point can move. A point can coexist with infinitely many other points. A *point* is the simplest in extension.[35] Two points cannot be distinguished *per se* in turns. Therefore another point can also exist *per se*. I.e. any point whatever can exist *per se*. Some point exists. Whatever exists can exist. Therefore some point [can] exist. An *extensum* exists.]

An *extensum* is a continuum. Parts can be made in an extensum. Parts can be made in an extensum which simultaneously exist. Parts can be made in an extensum in infinitely many ways. A part of an extensum is extended. Many [parts] can exist in one extensum. Infinitely many [parts] can exist in one extensum. Infinitely many similars can exist in one extensum. Infinitely many congruents can exist in one extensum. If something exists in an extensum, and it is congruent to it, it coincides with it. If something exists in an extensum and it is not congruent to it, there can exist infinitely many in the same extensum which do not coincide with the first, but nonetheless they are congruent. Two moveable things that are taken in an extensum can be congruent to each other, i.e. can be located at different times such that the earlier state could not be distinguished from the later one. The locus itself of an extensum is extended. The locus of an extensum is congruent to the extensum. The locus is immovable.

[34]From here to the end, the script style is different, suggesting that Leibniz wrote this last part in a much later sitting. He also cancelled in bulk the part in brackets, which we include because of its potential philosophical interest.

[35]Latin *in extenso simplicissimum*; this could be 'the simplest in terms of extension' or 'the simplest thing in an extensum'.