G. W. Leibniz

De Determinatis et Congruis

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De Determinatis et Congruis

Extensum rigidum voco, in quo utcunque moto nulla quoad extensionem fit mutatio, id est non tantum ipsius, sed nec eorum quae in ipso sunt extensio non mutatur.

De Determinatis et Congruis

I call a *rigid extensum* that in which there is no change as regards extension during any kind of motion, that is, the extension not only of itself, but also of things in it, is not changed.

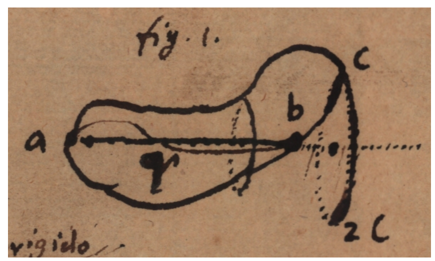

Figure 1

Si Extensum aliquod rigidum *abc* duobus ejus punctis *a* et *b* quiescentibus moveatur, tunc omnium ejus punctorum quiescentium, ut *r* locus erit recta *ab*, puncti autem alicujus moti *c* locus successivus *c* 2 *c* erit *arcus circuli*. Habemus ergo generationem rectae et circuli, nulla recta nulloque circulo praeexistente[,] rectae quidem per quietem, circuli per motum.

If some rigid extensum *abc* moves while two of its points *a* and *b* stay at rest, then the locus of all its points at rest, say *r*, will be the line *ab*, while the successive locus *c*2*c* of some point *c* that moves will be an *arc of a circle*. We therefore have the generation of a line and a circle without any line or circle pre-existing, indeed the line through rest, the circle through motion.

Ex hac generatione rectae sequitur punctum omne, ut r in eadem cum duobus datis punctis a. b recta esse, si cum ipsis rigido aliquo extenso ut arb connexum, ipsis quiescentibus moveri non potest. Et vicissim si punctum aliquod in eadem cum duobus datis punctis recta sit, cum quibus rigido extenso connexum est, quiescentibus illis moveri non posse. Satis autem extenso rigido connexa sunt puncta, ut $a.r.b$ quando omnia in eodem extenso rigido, ut abc reperiuntur.

From this generating process [*generatio*] of the line it follows that any point, say r, is on the same line with the two given points $a.b$ if, when it is connected to them by some rigid extensum arb, it cannot move while they stay at rest. And conversely, if some point is on the same line with two given points to which it is connected by a rigid extensum, it cannot move while they stay at rest. Now points such as $a.r.b$ are sufficiently connected by a rigid extensum when they are all found in the same rigid extensum such as abc.

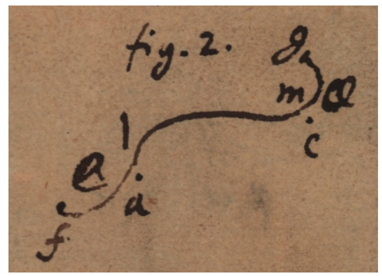

Figure 2

Si puncta ut a et c eundem semper servant situm inter se, tunc eadem semper ejusdem extensi rigidi $f\,l\,m\,g$ puncta $l.m$ eis applicari possunt; Et vicissim si eadem semper extensi rigidi puncta eis applicari possunt, eundem semper servant situm inter se.

If points such as a and c always preserve the same situation between themselves, then the same points $l.m$ of the same rigid extensum $f\,l\,m\,g$ can always be applied to them; and conversely if the same points of the same rigid extensum can always be applied to them, they always preserve the same situation between themselves.

Recta ergo definiri potest locus omnium punctorum, quorum unumquodque ut R eum habet situm, ad duo puncta A et B, ut quamdiu hunc ad ea situm servat (quod fit quamdiu tria puncta $A.R.B$. iisdem rigidi lmn punctis $l.m.n$ applicari possunt) ipsis A et B immotis, R moveri non possit, (id est si l et n manentibus immotis, tunc utcunque moveatur extensum lmn, necesse est et m esse immotum). Itaque locus ipsius R, est determinatus, determinato ejus situ ad puncta A et B. Hinc statim patet ipsa puncta A et B, utique cadere in hanc ipsam

Therefore a line can be defined as the locus of all points each one of which, say R, has such a situation to two points A and B that as long as it preserves this situation to them (which happens as long as the three points $A.R.B$. can be applied to the same points $l.m.n$ of the rigid [extensum] lmn), R cannot move while A and B are unmoved (that is, if l and n remain unmoved, then no matter how the extension lmn moves, it is necessary that m also is unmoved). And so the locus of this R is determined when its situation to the points A and B is determined. Hence it is immediately clear that the points A and B certainly fall on

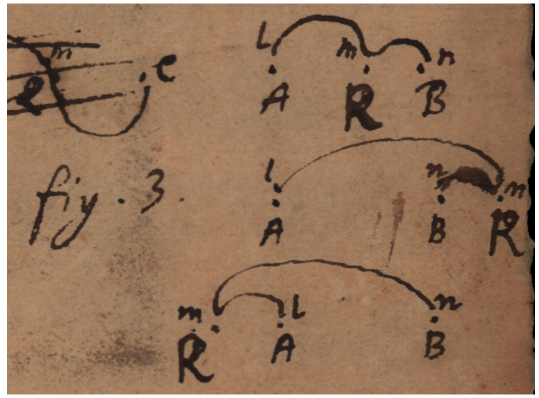

Figure 3

rectam seu locum omnium punctorum R. Utique enim ipsis A et B manentibus immotis ipsum R moveri non potest. Patet etiam in extenso quoad extensionem immutabili, (quale est ipsum spatium, item corpus aliquod rigidum) duobus punctis determinatis, determinatum esse rectam quae per ipsa transit.

this same line, the locus of all points R. For certainly R cannot move while A and B remain unmoved. It is also clear that in an extensum which is unchangeable as regards extension (such is space itself, as well as any rigid body), when two points are determined, the line passing through them is determined.

Figure 4

Hanc rectae proprietatem ut calculo exprimam, erit mihi ∞ signum *congruentiae,* exempli causa $F \infty G$ id est F congruum esse ipsi G.

In order to express this property of the line in the calculus, I will take ∞ as the symbol for *congruence*, for example $F \infty G$, that is, F is congruent to G.

At ɣ signum erit *coincidentiae*. Ut si in quadrato aequilatero $ACBD$, ducantur rectae AB, CD, angulos oppositos

Now ɣ will be the sign for *coincidence*. For instance, in an equilateral square $ACBD$, if the lines AB, CD are drawn connecting opposite

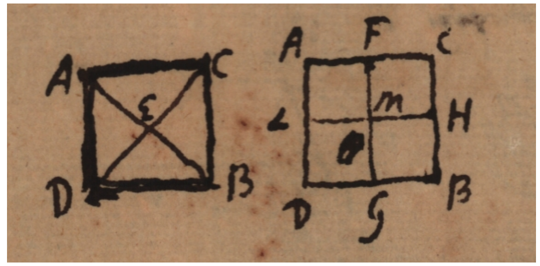

Figure 5

connectentes, et se secantes in puncto E, et rursus ducantur aliae rectae FG, HL, laterum oppositorum puncta media jungentes, et se secantes in puncto M, tunc revera puncta E et M coincidunt, poteritque scribi E ♉ M.

angles and intersecting each other at the point E, and again other lines FG, HL are drawn connecting the midpoints of the opposite sides and intersecting each other at the point M, then in fact the points E and M coincide, and one can write E ♉ M.

Situm punctorum quorundam *inter se* scribam simplici punctorum denominatione interpuncta, ut $A.R$ vel $R.A$. Significat situm inter puncta A et R. determinatum, vel rigidum quodcunque ipsa connectens. Similiter $A.B.R$ significat trium punctorum situm inter se. Et ita porro.

The *situation* of certain *points between themselves* I write with a simple denomination of points between points, as $A.R$ or $R.A$. This signifies the determined situation between the points A and R, or any rigid thing connecting them. Similarly $A.B.R$ signifies the situation of the three points among themselves. And so on.

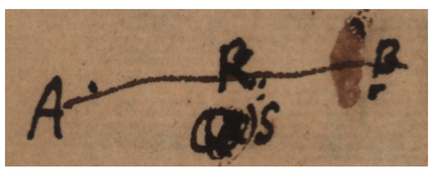

Figure 6

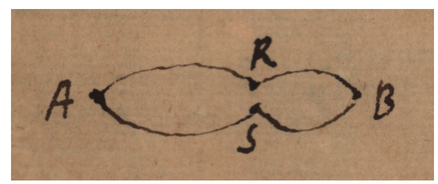

Figure 7

Cum itaque situs ille puncti R ad duo puncta $A.B$, qui locum puncti R determinat, sit situs puncti R in linea Recta cum punctis A et B; situsque ejus ad puncta A et B scribatur $A.R.B$. determinatus si placet rigido aliquo ARB tria puncta connectente; sitque praeterea punctum aliquod S, eundem habens situm ad puncta A et B quem habet punctum R, ita

And so, since the situation of the point R to two points $A.B$, which determines the locus of the point R, is the situation of a point R on a straight line with points $A.B$, let, therefore, point R be set in the same line with points A and B, and its situation to the points A and B is written $A.R.B$, determined if you like by some rigid thing ARB connecting the three points; and let there be moreover some point S

ut rigidum punctis $A.R.B$ applicatum eodem modo applicari possit ad $A.S.B$ puncta; sequetur puncta S et R coincidere, siquidem R cum ipsis A et B est in eadem recta.

having the same situation to the points A and B which the point R has, so that the rigid thing applied to the points $A.R.B$ could be applied in the same way to the points $A.S.B$; it follows that the points S and R coincide, if indeed R is on the same line with A and B.

Itaque ut rem paucis et calculo complectar, si posito esse $A.R.B \infty A.S.B$ sequitur esse $R \between S$ erit locus omnium R, *Recta*. Et vicissim sin compendii causa aliud praeterea adhibere velimus signum determinationis, poterimus scribendo $\overline{dt\,R}$ significare locum ipsius puncti R esse determinatum seu certum, et similiter scribendo $\overline{dt\,A.R}$ situm punctorum A et R esse determinatum seu certum. Ita $\overline{dt\,A.R.B}$ significabit situm horum trium punctorum esse certum. Itaque *Recta* transiens per A et B est locus punctorum R quorumcunque, si ex $\overline{dt\,A.R.B}$ sequitur $\overline{dt\,R}$.

And so, to summarize the matter in a few words and using the calculus, if supposing $A.R.B \infty A.S.B$ it follows that $R \between S$, the locus of all R will be a *straight line*. And again, if for brevity we want to add another sign for determination, [then] by writing $\overline{dt\,R}$[1] we can signify that the locus of the point R is determined or specified, and similarly by writing $\overline{dt\,A.R}$ that the situation of the points A and R is determined or specified. Thus $\overline{dt\,A.R.B}$ will signify that the situation of these three points is specified. And so the *line* passing through A and B is the locus of any points R whatever, if from $\overline{dt\,A.R.B}$ it follows that $\overline{dt\,R}$.

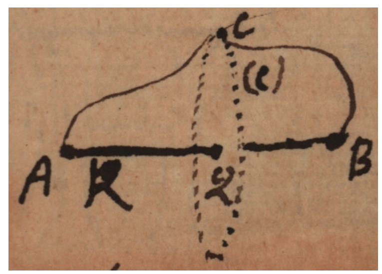

Figure 8

Omnia autem puncta extra hanc rectam moveri possunt servato suo situ ad duo puncta A et B. Ita licet punctum C ponatur ipsis A et B rigide connexum, si tamen non sit in recta per A et B transeunte, moveri poterit punctis A et B immotis, et motu suo describet circularem $C\,(C)$ quemadmodum supra definivimus; itaque cum durante motu eundem semper situm servet ad puncta A et B, habebimus $A.C.B \infty A.\,(C)\,.B$ eritque locus omnium C vel (C) circularis.

Now all points outside this line can move while preserving their situation to the two points A and B. Thus, although a point C connected rigidly to A and B may be set, nevertheless, if it is not on the line passing through A and B, it will be able to move while the points A and B are unmoved, and it will describe a circle $C(C)$ by its motion in the way we defined above; and so, since during the motion it always preserves the same situation to the points A and B, we will have $A.C.B \infty A.(C).B$ and the locus of all C or (C) will be circular.

[1]Cf. the 'unique situation' notation such as $\overline{a.b.c.d.F\;Un}$. in CGG (24) and (67).

Omne punctum, ut C. quod eundem situm servat ad aliqua puncta, ut A et B. servabit etiam eundem situm ad id punctum R. quod situ ad A et B determinato locum habet determinatum; (et proinde punctum C motum servato situ ad puncta A et B. eundem servabit situm etiam ad quodlibet punctum R. rectae per A et B transeuntis, seu ad ipsam hanc rectam) seu generaliter quod situm eundem servat ad determinantia, servat et ad determinatum. Haec propositio meretur demonstrari.

Every point such as C, that preserves during motion the same situation to some points such as A and B, will preserve the same situation to the point R which has a determined place when the situation to A and B is determined (and hence the point C, being moved while its situation to the points A and B is preserved, will preserve the same situation also to any point R of the line passing through A and B, i.e. to this line itself); or in general, whatever preserves its situation to the determiners also preserves it to the things determined. This proposition merits demonstration.

Poterit etiam ostendi, semper puncto uno sumto B in recta AB inveniri posse aliud R, ita ut sit $C.B \infty C.R$. excepto unico casu, ubi B et R ad se properantia uniformiter sibi occurrunt, scilicet in Q.

It can also be shown that when one point B is taken on the line AB, another point R can always be found such that $C.B \infty C.R$,[2] except for a single case, when B and R, speeding toward each other uniformly, meet each other at Q.

Planum determinatum est, assumtis tribus punctis, seu omnia puncta quorum situ ad tria puncta dato determinatus est locus, cadunt in planum per illa tria puncta transiens; coincidit haec definitio cum generatione plani per rectam super duabus rectis immotis motam:

A *plane* is determined when three points are assumed; that is, all the points whose place is determined when their situation to three points is given fall on the plane passing through those three points; this definition coincides with the generation of a plane by a line moved over two unmoved lines.

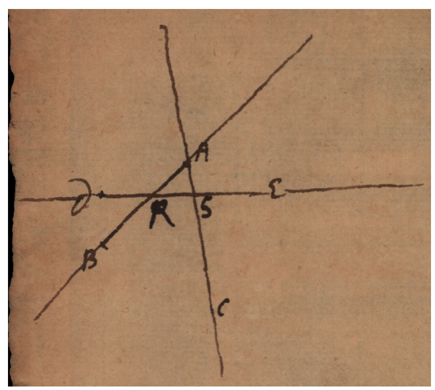

Figure 9

[2]It is '$C.B. \infty C.B.$' in the manuscript.

Sint enim duae rectae, una transiens per puncta A et B, altera transiens per puncta A et C. Et sit alia recta mobilis transiens per puncta $D.E$. Haec mota super his duabus rectis, et AB attingens in R, AC in S. dabit lineam rectam per $R.S$. transeuntem seu ex punctis R et S determinatam. Cumque omnia puncta ut R et S sint ex punctis $A.B.C$. determinata, et rectae ipsas jungentes determinatae ex ipsis, omniaque puncta harum rectarum (ex posita generatione plani per motum rectae) sint omnia puncta plani; patet omnia puncta plani ex tribus punctis $A.B.C$. esse determinata.

Indeed, let there be two lines, one passing through points A and B, the other passing through points A and C. And let there be another movable line passing through points $D.E$. This one, moved over those two lines, and meeting AB at R, AC at S, will give the straight line passing through $R.S$, or determined from the points R and S. And since all the points such as R and S are determined from the points $A.B.C$, and the lines connecting them are determined from them, and all points of these lines (from the proposed generation of the plane by the motion of lines) are all the points of the plane, it is clear that all the points of the plane are determined from these three points $A.B.C$.

At sumtis quatuor punctis determinatum est spatium ipsum infinitum, seu locus omnium punctorum quorum unumquodque est determinatum dato ejus situ ad quatuor puncta data; est spatium infinitum, seu locus omnium punctorum in universum, seu quod idem est omnis puncti locus est determinatus, si datus sit ejus situs ad quatuor puncta data, sed hoc debet demonstrari.

By taking now four points, infinite space itself is determined; that is, the locus of all points each one of which is determined given its situation to four given points is infinite space, or the locus of all points in general; or what is the same thing, the locus of every point is determined when its situation to four given points is given; but this should be demonstrated.

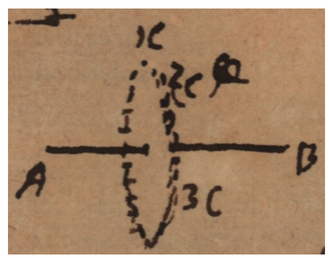

Figure 10

Cum supra dictum sit punctum C durante motu eodem modo se habere ad rectam AB ergo recta AB eodem modo se habebit ad tria puncta 1 C. 2 C. 3 C. ergo etiam eodem modo se habebit ad planum per haec tria puncta determinatum. Et eodem modo cum puncta tria planum determinantia eundem habeant situm ad duo quaedam puncta A et B, etiam ipsum

Since, above, it was said that the point C, during its motion, relates in the same way to the line AB, therefore the line AB will relate in the same way to the three points $1C.2C.3C$. Therefore it will also relate in the same way to the plane determined by these three points. And in the same way, when three points determining a plane have the same situation to some two points A and B, the determined plane itself will planum sit determinatum eundem ad ea situm habebit.

Demonstrandum est easdem propositiones esse reciprocas, seu omnia puncta eundem simul situm ad duo diversa puncta habentia, cadere in planum; et ad tria habentia cadere in rectam. His habitis conjungi poterunt demonstrata alibi de congruis, et demonstrata hic de determinatis, et habebitur[,] puncta duo eundem situm habentia ad quatuor puncta data, quae eundem situm ad tria aliqua puncta non habent, congruere.

also have the same situation to them.

It should be demonstrated that these same propositions are reciprocal, that is, that all points having simultaneously the same situation to two distinct points fall on a plane, and those having [the same situation] to three fall on a line. Having obtained these things, one may combine what was demonstrated elsewhere about congruents with what was demonstrated here[3] about determined things, and one will obtain that two points, having the same situation to four given points which do not have the same situation to three other points, will coincide [*congruere*].

[3]The manuscript shows that ‘elsewhere’ and ‘hic’ were inserted after the last paragraph was written.