G. W. Leibniz

De Calculo Situum

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De Calculo Situum.

§ 1. Ut in Calculo Magnitudinum cum ipsas Magnitudines formamus dum addimus, multiplicamus, in se ducimus et horum reciproca peragimus, tum etiam conferimus per rationes, aliasve relationes progressiones ac denique Majoritates, Minoritates et Aequationes. Ita in Situ formamus Extensa per Sectiones et Motus, deinde conferimus, spectamusque in eis praeter Magnitudines Similitudinem, Congruentiam (ubi concurrunt Aequalitas et Similitudo) Coincidentiam, adeoque Determinationem. Determinatum enim est cui aliquid, iisdem positis conditionibus, coincidere debet.

§ 2. Et ut doctrina Magnitudinis sua habet Axiomata, veluti Totum sua parte majus est. Quod majus est majore majus est minore. Si aequalibus aequalia addas proveniunt aequalia, aliaque id genus. Ita Doctrina Situs Axiomata propria habet qualia sunt:

Si Similitudo, Congruentia, Coincidentia sint in Determinantibus, esse etiam in determinatis, et vicissim, si ea sint in Determinatis erunt quoque in Determinantibus simplicissimis.

De Calculo Situum

(1) In the Calculus of Magnitudes, ⟨not only do we form those Magnitudes when⟩ we add, multiply, square [in se ducimus], and take their reciprocals, but we also convey them in ratios, and other relations, progressions, and finally Majorizations, Minorizations, and Equations. Just so in Situs we form Extensions by Sections and Motions, and then we compare and observe in them, besides Magnitudes, also Similarity, Congruence (when Equality and Similarity concur), Coincidence, and thus Determination. Indeed, a thing is determined to which some other must coincide when the same conditions are set.

(2) And the doctrine of Magnitude has its Axioms, for instance the Whole is greater than its part. What is greater than a greater is greater than a lesser. If you add equals to equals, they come out equals; and others of that kind. Just so the Doctrine of Situs has its own Axioms of this sort:

If Similarity, Congruence, Coincidence are in the Determiners, they are also in those determined, and conversely, if they are in those Determined, they will also be in the simplest Determiners.

Exempli causa. Ponamus non nisi unicam Rectam a puncto ad punctum duci posse, sequetur omnes Rectas esse inter se similes, quia ad determinandam Rectam ab A. ad B. nihil aliud opus est quam assumi A, B. et ad aliam LM, saltem assumi situm punctorum L, M. Situs vero duorum punctorum situi aliorum duorum semper similis est quia nihil differentiae praeter solam magnitudinem distantiae totius assignari potest, sed magnitudo jam est aliquid ad tertium relatum. Non tamen Situs punctorum duorum Situi punctorum aliorum duorum plane congruus erit nisi ita ponantur ut quodlibet Extensum continuum quod applicari potest inter Terminos unius situs possit etiam applicari inter Terminos situs alterius.

Similia vero sunt quae ambo seorsim spectata sunt indiscernibilia ita ut nihil sumi possit in uno cui simile sumi nequeat in altero, abstrahendo ubique ab aliqua determinata Magnitudine nisi excipias magnitudinem Angulorum, quae ad doctrinam situum, non vero ad doctrinam Magnitudinum referri debet.

Cum ergo probaverimus omnes situs binorum punctorum esse similes, etiam determinata, seu omnes Lineae Rectae erunt Similes.

§ 3. Contra non omnia Triangula per situm trium punctorum determinata sunt similia inter se.

For example, let us suppose that only a unique straight Line can be drawn from a point to a point; it follows that all Lines are similar to each other, since nothing is needed for the determination of the Line from A. to B. except for A, B to be taken, and for the other LM, just for the situs of the points L, M to be taken. The situs of two points, indeed, is always similar to the situs of another two, since no difference can be ascribed beyond the magnitude of the entire distance, but now the magnitude is something in relation to a third thing [aliquid ad tertium relatum]. ⟨Nonetheless, the Situs of two points clearly won't be the same as the Situs of another two points, unless indeed they are placed so that some continuous Extension that can be attached between the Termini of the one situs could in fact be attached between the Termini of the other situs.⟩

Things are similar, in fact, which are indiscernible when both are seen separately, so that nothing could be taken in one to which something similar could not be taken in the other, abstracting everywhere from some determined Magnitude, excepting the magnitude of Angles, which should be referred to the doctrine of situs, not in fact the doctrine of Magnitude.

Since, therefore, we have proved that every situs of two points is similar, then indeed those determined, or all Straight Lines, will be Similar.

(3) On the other hand, not all Triangles determined by the situs of three points are similar to each other. And indeed, ABC are not similarly related as LMN.

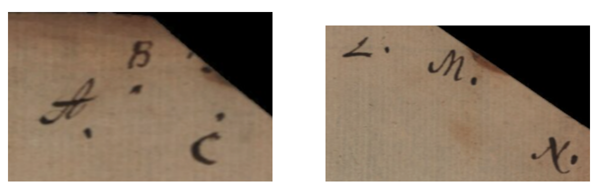

Figure 1

Neque enim ABC similiter se habent ut LMN. Potest enim Distantia AB ad Distantiam BC aliam rationem habere quam Distantia LM ad distantiam MN, ita ut in determinantibus sit dissimilitudo. Ex quo patet etiam in duabus Rectis lineis tria puncta tribus aliis dissimiliter sita eligi posse.

The Distance AB can, for instance, have another ratio to the Distance BC than the Distance LM to the distance MN, so that dissimilitude arises in those determining. From this it is also clear that in two Straight lines, three points could be chosen with situs dissimilar to three other points.

Nam similitudo a determinato reciproce tantum valet ad pure determinantia, non etiam ad ea quae sunt plus quam determinantia.

For, reciprocally, similarity from determination holds only for those purely determining, but indeed not for those that are more than determining.

Sic, etiamsi Circulus determinetur per tria puncta peripheriae data, et omnes Circulos inter se similes esse minime sit negandum, tamen hic Consequentia non valet a determinatorum similitudine ad determinantium similitudinem, quia Peripheriae tria puncta data plus determinant, quam ipsum Circulum, scilicet etiam certum Angulum in segmento, et tres partes peripheriae determinatam ad totum Circulum rationem habentes.

Likewise, although a Circle is determined by three given points of the periphery, and least of all should it be denied that all Circles are similar to each other, nonetheless here the Inference from the similarity of the determined to the similarity of the determiners is not valid, since the three given points of the Periphery determine more than the Circle itself, namely a fixed Angle in a segment, and three parts of the periphery having a determined ratio to the whole Circle.

At contra si Circuli duo determinentur per datas duas Chordas et per aequales Angulos in segmentis super Chordas factis, tum demum Circuli non solum similes erunt, sed etiam similiter determinati. Hic autem quaestio nec de tali quidem determinatione est, sed saltem de primis et simplicissimis determinantibus, quae ubi determinata fiunt similia, etiam similia esse debent.

But conversely, if two Circles are determined by two given Chords and by equal Angles in the segments created above the Chords, then finally the Circles are not only similar, but also similarly determined. This question, however, is not indeed about such determination, but merely about the first and simplest determiners, which, when the things determined come out similar, must also be similar.

Si vero contingeret, dissimilia determinantia nihilominus dare similia determinata, id ipsum certo indicio est hanc determinationem non esse simplicissimam, sed aliam dari simpliciorem.

If it actually happens that dissimilar determiners nevertheless give similar determined things, that fact itself is certain evidence that this determination is not the simplest, but there is another simpler one.

§ 4. Uti Magnitudinum Logisticam seu Mathesin generalem ad calculum reducimus, utimurque imprimis rationibus et

(4) As we reduce the general Logic or Mathesis of Magnitudes to calculation, making use especially of ratios and equations, just so

aequationibus, ita calculus quidam in situ institui potest per similitudines et congruentias. Literae autem in Calculo Magnitudinis designare solent ipsas Magnitudines. In Calculo Situs possunt designare puncta et loca. Hinc si $Y.A. \simeq B.A.$ locus omnium Y est superficies Sphaerae.

In hac Consignatione $B.A.$ significat situm puncti $B.$ ad punctum A, sed $\simeq$ est signum congruitatis. Sensus ergo illius Consignationis talis est. Quodlibet indeterminatum Y eum situm habere ad punctum determinatum A quem habet B ad $A.$ unde intelligitur ipsum B quoque inter ea Y seu in eadem superficie sphaerae esse. Sed si posuissem $Y.A. \simeq B.C.$ non opus fuerit B in superficie sphaerae poni. Sed jam maneat $Y.A. \simeq B.A.$

§ 5. Jam posita alia adhuc sphaera $ZL. \simeq ML.$ et considerando has duas superficies sphaericas se intersecare et loca communium concursuum vocari V; unumquodque $V.$ erit simul $Y.$ et $Z.$ ut scribere possim $V.A. \simeq BA$ et $V.L. \simeq ML.$ Potest autem B assumi coincidens ipsi M (quod ita signatur $B \infty M$) quod vocetur $F.$ determinatum ex ipsis $V.$ fietque $V.A. \simeq F.A.$ et $V.L. \simeq F.L.$ unde componendo fit $V.A.L. \simeq F.A.L.$ unde sequitur, Lineam in qua se secant duae superficies sphaericae ejus esse naturae ut quodvis ejus punctum V habeat ad duo data $A.L.$ situm eundem quem constans $F.$ (quae proinde una est ex ipsis V) ad eadem puncta $A.L.$

§ 6. Idem etiam sic enuntiari poterat:

Quodvis extensum $A.G.L.$ cujus duo puncta $A.$ et $L.$ quiescunt, motu suo talem lineam $V.V.V.$ describet qualem formant duae superficies sphaericae sua intersectione, id est Circularem, quia cum Extensum ponatur rigidum adeoque punctum quodvis ut $G.$ suum situm servet ad puncta $A.$ $L.$ durante motu extensi continuo quiescentia, inde quodlibet Vestigium ipsius

a certain calculus of situs can be established with similarities and congruences. The letters, further, in the Calculus of Magnitudes typically designate the Magnitudes themselves. In the Calculus of Situs they can designate points and loca. Hence if $YA \simeq B.A.$, the locus of all Y is a spherical surface.

In this Co-signification, $B.A.$ signifies the situs of a point $B.$ to a point A, but $\simeq$ is the sign of congruence. The sense of this Co-signification is therefore that whatever indeterminate Y has the situs to the determined point A that B has to $A.$; from which B also is understood to be among those Y or i.e. in the same spherical surface. But if I had put $Y.A. \simeq B.C.$, then B would not need to be put in the spherical surface. ⟨But now $Y.A. \simeq B.A.$ remains.⟩

(5) By positing now yet another sphere $ZL. \simeq ML.$ and considering these two spherical surfaces to intersect, and the loca [sic] of mutual concurrence to be called $V.$, each $V.$ will be simultaneously $Y.$ and $Z.$, so I could write $V.A. \simeq BA$ and $V.L. \simeq ML.$ Moreover, B can be assumed coincident with M (which is then signified $B \infty M$), let us call it F, determined among these $V.$, and $V.A. \simeq F.A.$ and $V.L. \simeq F.L.$ will hold, hence by composing, $V.A.L. \simeq F.A.L.$ holds, from which it follows that the Curve in which two spherical surfaces intersect is of such a nature that any point V on it would have the same situs to two given ones $A.L.$ as the fixed one $F.$ (which is thus one of these V) to the same points $A.L.$

(6) We could state the same thing again thus: Some [extension] $A.G.L.$

, whose two points $A.$ and $L.$ are stationary, would describe by its motion the kind of curve $V.V.V.$ that two spherical surfaces form by their intersection, that is a Circle, since when a rigid Extension is placed so that some point such as G preserves its situs to the points $A.L.$ that are stationary during the continuous motion of the extension, thereupon any Trace of the revolved $G.$ retains the same situs to the

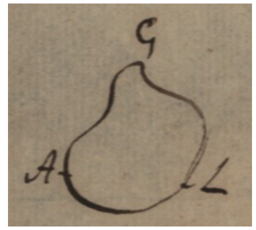

Figure 2

G. circumvoluti situm eundem ad duo puncta fixa A. et L. retinebit non aliter ac supra scripsimus $V.A.L \simeq F.A.L$.

two fixed points A. and L., not otherwise than what we wrote above: $V.A.L \simeq F.A.L$.

§ 7. Puncta vero quaevis quae dicto Motu durante una cum punctis A et L quiescunt, eo ipso quia quiescunt, oportet esse situs sui ad A. et L. unica. Nam si moverentur pluribus locis eundem situm ad A et L. exhibere possent, siquidem omnia eorum vestigia eundem situm ad A. et L. haberent. Jam vero ea puncta sunt sua ipsorum vestigia, id est describent Circulos indefinite parvos sive evanescentes in puncta. Ita prodit Linea Recta cujus Expressio haec erit. Posito puncto quovis ejus indeterminato R. dicetur $R.A.L$. Unicum seu si $R.AL \simeq (R)\,A.L$ erit $R \infty\ (R)$.

(7) Any points which are actually stationary during the stated Motion together with the points A and L, by that fact, since they are stationary, must be unique with their situs to A. and L. For if they are moved, the same situs to A and L. could be exhibited in multiple places, if in fact all of their traces have the same situs to A. and L. Now actually these points are their own traces; that is, they describe Circles indefinitely small, vanishing into points. So a Straight Line is produced, of which this is the Expression: Setting some indeterminate point R. on it, $R.A.L$. is said to be Unique, or i.e. if $R.AL \simeq (R)A.L.$, then $R \infty (R)$.

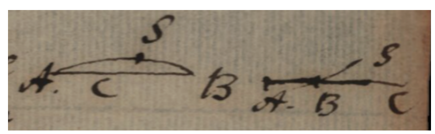

Figure 3

§ 8. Hinc patet duas Rectas non transire per eadem duo puncta ut ABC et ABS. Nam si in Rotatione Plani punctis A. et B. fixis totum planum moveatur, illa rotatio efficiet ut quicquid semel fuit altero superius seu propius externo initio rotationis id facie versa fiat postea inferius seu remotius ab initio rotationis externo. At, si tam Lineae ASB quam

(8) Hence clearly two Straight Lines do not pass through the same two points such as ABC and ABS, for if in a Rotation of the Plane, with the points A. and B. fixed, the whole plane is moved, the rotation will make whatever was once above or closer to another external thing at the beginning of the rotation, to become afterwards, with the face flipped around, lower or further from the external thing in the beginning of

ACB essent Rectae, facta rotatione ad Fixa puncta A. et B. oporteret ambas quiescere ex natura Lineae Rectae modo ostensa. Si ambae quiescerent, S semper maneret supra extensum ABC et nunquam caderet infra, quod est contra Naturam Rotationis.

the rotation. But, if both the Lines ASB and ACB were Straight, made by rotation on Fixed points A. and B., then both must be stationary, by the nature of a Straight Line just shown. If both are stationary, S always remains above the extension ACB and never falls lower, which is contrary to the Nature of Rotation.

§ 9. Hinc statim colligimus Rectas inter se similes esse, habere partem toti similem, quin etiam Rectam Lineam esse simplicissimam, cum nihil aliud quam extrema ad totam suam determinationem requirat, adeoque et minimam inter extrema, et pro distantia punctorum in posterum sumi posse. Pro distantia sumetur, quia Terminis immotis, distantiam Terminorum oportet esse immotam. Si ergo alia Linea inter A. et B. praeter Rectam assumeretur pro distantia, etiam illa punctis A. et B. Fixis in rotatione Plani maneret immota, preater Rectam AB. etiam immotam in eadem rotatione per § 7. Ergo darentur duae diversae Lineae simul immotae in hac rotatione, quod absurdum per § 7.

(9) Hence, we immediately deduce that Straight lines are similar to each other, and have part similar to the whole, and moreover that the Straight Line is the simplest, since it requires nothing other than the extremes for its whole determination, and so is also the minimum between extremes, and in what follows it can be taken for the distance of points. It will be taken for distance, since with the Termini unmoved, the distance of the Termini must be unmoved. If therefore another Curve between A. and B besides the Straight one is assumed for distance, then it too remains unmoved during rotation of the Plane with the points A. and B. being Fixed, besides, indeed, the Straight Line AB. [that is] unmoved during the same rotation by §7. Therefore, two distinct Curves simultaneously unmoved during this rotation would be given, which is absurd by §7.

Brevissima erit, quia si alia brevior ab A. ad B. pertingit, Linea seu extensum assequetur distantiam se ipso majorem quod absurdum.

It will be shortest, since if another shorter one reaches from A. to B., a Curve ⟨or extension⟩, it follows that the distance is greater than itself, which is absurd. If another, equal, is given, such as if ASB

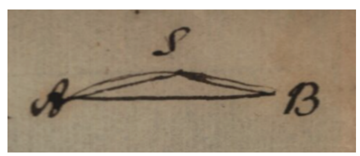

Figure 4

Si alia aequalis datur, ut si esset ASB non quidem Recta, aequalis tamen rectae ABC, oporteret distantias $AS + SB$. non esse majores quam AB. quia non possunt esse majores conterminis curvis $AS+SB$ (quae ponuntur ipsi AB aequales) ex natura brevissimi. Sed Euclides demonstravit esse $AS+SB$

were indeed not Straight, but nevertheless equal to the line ABC, then the distances $AS + SB$. must not be greater than $A.B.$, since they cannot be greater than the co-terminal curves $AS + SB$ (which are set equal to AB) by the nature of the shortest. But Euclid demonstrated that $AS + SB$ is greater than AB., relying on no implicitly assumed majores quam AB. nullis principiis huic (Brevissima duo inter eosdem terminos non dantur) innitentibus implicite assumtis, sed ex puris angulorum sitibus ratiocinando. Ergo patet quoque nostri asserti veritas, quod duo brevissima inter eosdem Terminos non dentur.

§ 10. Fortasse tamen illud Euclideum ex paucioribus etiam demonstrari potest. Scilicet[:]

Dissimiles Arcus in eodem Circulo a Chordis aequalibus abscindi nequeunt. Id quod ex natura similium per se constare censendum est.

principles [beyond] this (there are not two Shortest between the same termini), but by reasoning from the pure situs of angles. Therefore, clearly our assertion is also true, that there are not two shortest between the same Termini.

(10) Perhaps this thing from Euclid can still be demonstrated from fewer things, namely:

Dissimilar Arcs cannot be cut out by equal Chords in the same Circle. Something to be considered as established per se from the nature of similars. Thus, Diameter AB

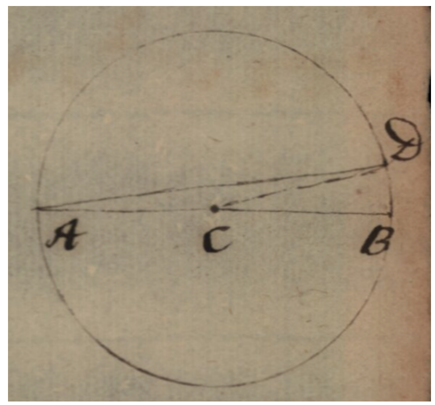

Figure 5

Itaque Diametrus AB major est Chorda AD. Nam Chorda AD abscindit Arcum dissimilem dimidio Circuli AB (alias ab A ad B rediret contra § 8). Ergo per positum principium non erit $AD = AB$. Sed nec $AD \Gamma AB$, quia $CA + CD = AB$ duplum Radii duplo Radii. Ergo hoc pacto esset $AD \Gamma CA + CD$ Brevissimum majus altero iisdem Terminis interjecto quod absurdum. Cum ergo Chorda AD nec aequalis sit Diametro nec major, patet Diametrum quavis Chorda majorem esse.

is greater than Chord AD, for Chord AD cuts out an Arc dissimilar to half of the Circle AB (otherwise it goes from A to B, contra §8). Therefore, by the principle proposed, $AD = AB$ will not hold. But neither will $AD \Gamma AB$ hold, since $CA + CD = AB$, [or] twice the Radius [equals] twice the Radius. Thus, by composing, we would have $AD \Gamma CA + CD$, the Shortest [path] being greater than another interposed between the same Termini, which is absurd. Since therefore the Chord AD would be neither equal to the Diameter, nor greater, clearly the Diameter is greater than any Chord. Hence follows [a third thing,] that the two sides of Isosceles Triangle AMN

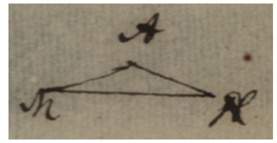

Figure 6

Hinc sequitur tertium Trianguli Isoscelis AMN duo latera tertio sunt majora. Nam Circulum Centro A, per M et N ducendo $AM + AN$ aequantur Diametro seu duplo Radii sed MN modo fiet Chorda ejus Circuli. Ergo ut paulo ante probatum $AM + AN \Gamma MN$.

are greater than the third. For, drawing a Circle with Center A passing through M and N, $AM + AN$ is equal to the Diameter, or twice the Radius, but MN will just become a Chord of the Circle. Therefore, as proven a little before, $AM + AN \Gamma MN$. Finally, I say, in any Triangle

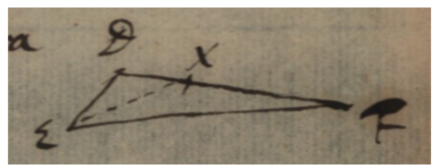

Figure 7

Denique Dico in quocunque Triangulo duo latera reliquo esse majora $DE + DF \Gamma EF$. Nam abscindo $DX = DE$, Ergo $DE + DX \Gamma EX$, ut de Triangulo Isoscele ostensum. Addo utrinque XF. Ergo $DE + DX + XF \Gamma EX + XF$. Id est $DE + DF \Gamma EX + XF$ (ℵ). Aut igitur $DE + DF$ minus erit brevissimo EF, quod absurdum per § 9. aut aequale (et sic per ea quae ad literam ℵ probavi erit $EF \Gamma EX + XF$ Brevissimum alio cointerjecto absurdum) aut denique $DE + DF$ majus erit quam EF quod erat demonstrandum.

at all, two sides [jointly] are greater than that remaining: $DE + DF \Gamma EF$. For by cutting out $DX = DE$, Therefore $DE + DX \Gamma EX$, as in the Isosceles Triangle shown. I add XF on either side. Therefore $DE + DX + XF \Gamma EX + XF$. That is, $DE + DF \Gamma EX + XF$ (ℵ). Then, either $DE + DF$ is smaller than the shortest EF, which is absurd by §9, or else equal (and likewise by what I showed at the letter ℵ, $EF \Gamma EX + XF$, the Shortest [greater than] another interposed, an absurdity) or else finally $DE + DF$ is greater than EF, what we sought to demonstrate.

§ 11. Ut Linea Recta est locus omnium punctorum sui situs ad duo puncta unicorum, ita Planum est locus omnium punc-

(11) As the Straight Line is the locus of all points unique with their situs to two points, so the Plane is the locus of all points unique

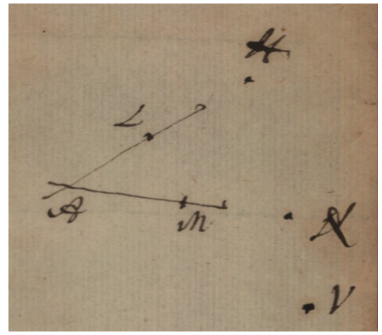

Figure 8

torum sui situs ad tria puncta unicorum, unde patet, etiam assumtis duabus rectis se intersecantibus haberi Planum. Esto enim Recta per A. L. et alia per A. M. Habemus tria puncta A. L. M. nec tantum determinata sunt puncta omnia Rectae per AL et omnia Rectae per AM sed et omnes distantiae a quovis puncto unius Rectae ad quodvis punctum alterius rectae, adeoque quodvis punctum in quavis harum distantiarum (quae etiam sunt Lineae Rectae) determinatum est seu sui situs ad $A.L.M$. unicum.

with their situs to three points, whence clearly a Plane is obtained by assuming two intersecting straight lines. Indeed, let there be a Straight Line through $A.L$. and another through $A.M$. We have three points $A.L.M$., and not only are all the points of the Line through AL determined and all [points] of the Line through AM, but also all distances from any point of the one Line to any point of the other line, and so any point on whatever of these distances (which indeed are Straight Lines) is determined, or i.e. is unique with its situs to $A.L.M$.

§ 12. Jam Rectae per $A.L$. omnia puncta vocentur Y et Rectae per $A.M$. omnia puncta appellentur Z. erit ita $A.L.Y$. unicum et $A.M.Z$. unicum. Ex ipsis Y unum sit H, et ex ipsis Z unum sit N erit $A.L.H$. unicum et $A.M.N$. unicum. Sumatur alius locus cujus quodvis punctum V sit unicum sui situs ad $H.N$. Sed ipsum H. est unicum ad $A.L$. et ipsum N. est unicum ad $A.M$. Ergo V. erit unicum ad $A.L.A.M$. Nam in Determinationibus pro Determinato substitui possunt Determinantia. Cum ergo sit V. ad $A.L.A.M$. unicum et repetitio ejusdem A. supervacanea sit, saltem inde inferetur esse V. ad $A.L.M$. unicum. Id est omnia puncta V. esse in eodem plano cum $A.L.M$. quia Planum est locus omnium punctorum sui situs ad tria puncta Fixa Unicorum.

(12) Now, let all points of the Line through $A.L$. be called Y, and all points of the Line through $A.M$. be termed Z., so $A.L.Y$. is unique and $A.M.Z$. unique. Letting one of the Y be H, and one of the Z be N, then $A.L.H$. will be unique and $A.M.N$. unique. Let another locus be taken, any point V of which is unique with its situs to $H.N$. But this H. is unique to $A.L$. and this N. is unique to $A.M$. Therefore V. will be unique to $A.L.A.M$. For the Determiners may be substituted for the Determined in Determinations. Since, therefore, V. is unique to $A.L.A.M$. and the repetition of its A. is redundant, we infer from this that V. is unique even to $A.L.M$., that is, all points V. are in the same plane with $A.L.M$. because the Plane is the locus of all points Unique with their situs to three Fixed points.

§ 13. Sequetur etiam Duo Plana sese secare in Linea Recta. Sit X. Unicum ad $A.B.C$. et Y. unicum ad $L.M.N$. Puncta vero utriusque Plani communia omnia vocentur Z. ita ut puncta Z sint unica sui situs tam ad $A.B.C$. quam ad $L.M.N$. Ergo omnia Z tam X. erunt quam Y. Producantur Distantiae $LM.LN$. et MN. dum Plano per $A.B.C$. occurrat in λ. μ. et ν. quod fieri necesse est quia planum quodvis secat totum spatium et sectio communis procedit in Infinitum. Item, omnis Recta procedet in infinitum. Necesse igitur est ut ad aliud Planum seu ad sectionem communem perveniat.

(13) It follows that two planes intersect in a Straight Line. Let X. be Unique to $A.B.C$. and Y. unique to $L.M.N$. All common points, actually of either Plane, are called Z. so that points Z are unique with their situs to $A.B.C$. as well as to $L.M.N$. Therefore all Z will be X. as well as Y. Draw out the distances LM., LN., and MN. until the plane through $A.B.C$. meets [them] in λ., μ., and ν., which must happen because every plane divides the whole of space, and a common section continues to Infinity. Likewise, every Straight Line continues to infinity. It is therefore necessary that it reaches to that Plane, or i.e. to a common section.

§ 14. Sed ne moveatur objectio, forsan unam inter Distantias $L.M.N$. esse sectioni Parallelam, duo nobis puncta λ. et ν. sufficiunt. Quodsi vero omnia tria in sectionem cadant nihilominus ex duobus eorum determinatis determinatum erit tertium, alioqui si tria essent indeterminata inter se determinarent Planum in ipsa intersectione Planorum, quod absurdum, quia sic ipsa quoque intersectio Planum foret. Itaque fiet $Z.\lambda.\nu$. unicum id est omnia puncta Z. cadent in Lineam Rectam. Hinc quia duae Rectae se mutuo non nisi in unico puncto secare possunt, trium Planorum Intersectio punctum erit.

(14) But lest an objection be raised that perhaps one of the Distances $L.M.N$. is Parallel to the section, two of the points λ. and ν. suffice for us. But if all three did fall on the section, then the third will nevertheless be determined from the two of them determined, otherwise if the three were mutually indeterminate, they would determine a Plane in the same intersection of Planes, which is absurd, since in this way the intersection would also be a Plane. Therefore $Z.\lambda.\nu$. becomes unique, that is, all points Z. fall in a Straight Line. Hence since two Lines cannot cut across each other except in a unique point, the Intersection of three Planes will be a point.

§ 15. Videndum etiam quid fiat, si tres superficies sphaericae se secent, ubi locus Intersectionis extensum esse nequit. Neque enim duarum Linearum sectio Extensum est. Facile autem ostendi potest, per duo puncta innumeros transire circulos, etsi possit etiam aliquando Ciculus circulum attingere saltem in uno puncto, etiam tum, quando non sunt in eodem Plano, etsi se non tangant. Circulum vero ex tribus punctis determinari manifestum est.

(15) We should look at what happens if three spherical surfaces intersect each other, where the locus of intersection cannot be an extension. Indeed, neither is the intersection of two Curves an Extension. It is easily shown, moreover, that countless circles pass through two points, although sometimes a Circle could touch a circle in just one point, also when they are not even in the same plane, although then they would not be tangent. To be sure, it is clear that a circle is determined by three points. For from two points A. and B

Nam ex duobus punctis A. et B. determinatur Recta cujus omnia puncta ad duo puncta haec se habent eodem modo, inter

a Line is determined all points of which bear the same relation to these two points, among which indeed is the Center of the Circle. A

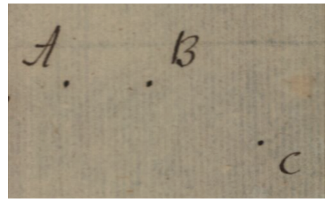

Figure 9

quae etiam est Centrum Circuli. Similis locus punctorum ad B et C eodem modo se habentium (inter quae idem Centrum esse debet) extat in Recta punctis B et C. determinata. Ergo Centrum Circuli est in ambabus iis Rectis, id est in earum Intersectione sive: Ergo intersectio ambarum Rectarum est punctum ejusdem relationis ad ($B.C.B.A$. et cum B repetere supervacaneum sit ad) $B.C.A$. quod punctum omnino debet esse Centrum Circuli per $A.B.C$.

similar locus of points bearing the same relation to B and C (among which the Center should likewise be) falls in a Line determined by points B and C. Therefore the Center of the Circle is on both of these Lines, or, that is, on their Intersection. Therefore the intersection of both Lines is a point of the same relation to ($B.C.B.A$. and since it is superfluous to repeat B, to) $B.C.A.$, which point assuredly should be the Center of the Circle through $A.B.C$.

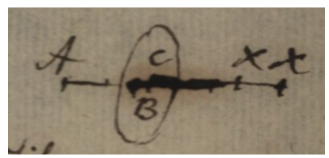

Figure 10

Sed nos supra definivimus Circumferentiam Circuli, locum punctorum eodem modo se habentium ad duo puncta Fixa. Hinc Circulus erit Locus punctorum eodem modo se habentium ad quodvis punctum X. Rectae per AB, determinatae substituendo pro Determinantibus.

But we defined above the Circumference of a Circle [to be] the locus of points bearing the same relation to two Fixed points. Hence the Circle will be the Locus of points bearing the same relation to any point X. of the Line through AB, determined things being substituted for Determiners.

§ 16. Sumantur tria puncta in Circumferentia hujus Circuli et Planum per ea transiens, cui occurrat Recta per AB. in Puncto quod sit C. Ergo Circumferentia est locus punctorum eodem modo se habentium ad C. ostendendumque erit omnia puncta Peripheriae cadere in hoc Planum per tria puncta Peripheriae

(16) Let us take three points in the Circumference of this Circle and the plane passing through them, which the Line through AB meets in a Point which is C. Therefore the Circumference is the locus of points bearing the same relation to C., and it ought to be shown that all points of the Periphery fall in this Plane drawn through the three

ipsius ductum. Quod fiet si ostendatur Planum esse locum omnium punctorum ad duo quaedam puncta eodem modo se habentium. Rectam vero esse locum omnium punctorum eodem modo se habentium ad tria quaedam puncta.

points of the Periphery itself. This will be accomplished if it is shown that the Plane is the locus of all points bearing the same relation to a certain two points. Certainly the line is the locus of all points bearing the same relation to a certain three points.

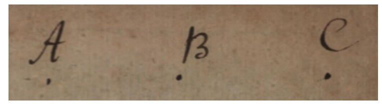

Figure 11

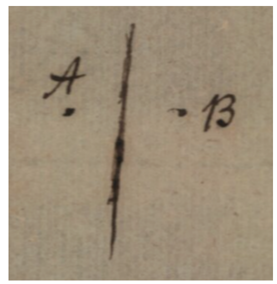

Figure 12

Sint puncta A. B. C. Duarum jam quarumcunque Sphaerarum circa A et circa B. intersectiones, cadent in Planum. Idem est de duabus quibuscunque sphaeris circa A et C. Inde, quia hoc sufficit ad determinandum, Consequens est, Planum ex intersectionibus sphaerarum circa A et B et Planum ex intersectionibus sphaerarum circa B. et C. aut circa A. et C. eandem determinare Rectam ad quaevis puncta hujus Plani eodem modo se habentem; ad quae illisio Rectae in illud planum eodem modo se habet.

Suppose there are points $A.B.C$. The intersections of any Two Spheres around A and B. will fall in a Plane. Likewise of any two spheres around A and C. From this, since here it suffices for determining, the Consequence is that the Plane from the intersections of spheres around A and B and the Plane from the intersections of Spheres around B. and C. or around A. and C. determine the same Line having the same relation to whatever points of this plane to which the impact of the line in this plane has the same relation.

§ 17. In Plano quoque possumus concipere Rectam ut locum omnium punctorum eodem modo se habentium ad duo tantum puncta A. et B. Adeoque omnes Circumferentiae aequales circa A. et B. se secabunt in hoc loco seu in hac Linea Recta.

(17) In a Plane we can also understand the Line as the locus of all points having the same relation to just two points A. and B.. And then all equal Circumferences around A. and B. will cross in this locus or i.e. in this Straight Line. Here the manner of determining

Hic modus locum determinandi diversus est a priore. Aliud enim est dicere, locum omnium punctorum eodem modo se habentium ad duo puncta $A.$ et $B.$ esse Rectam. Aliud locum omnium punctorum eodem modo se habentium ad $A.$ ut ad $B.$ esse Planum. Nam prior proprietas sic exprimitur: $A.B.C. \simeq A.B.Y.$ in solido. Locus omnium $Y.$ Recta sed posterior proprietas sic exprimitur: $A.Y. \simeq B.Y.$ erit locus omnium $Y.$ Planum. Sed, si omnia $Y.$ sint in eodem Plano cum AB et inter se posito $A.Y \simeq B.Y.$ erit locus omnium $Y.$ Linea Recta.

a locus is different than earlier. It is one thing, that is, to say the locus of all points having the same relation to two points $A.$ and $B.$ is a Line, another thing to say that the locus of all points having the same relation to $A.$ as to $B.$ is a Plane. For in the former, the property is expressed thus: $A.B.C. \simeq A.B.Y.$ in a solid. The locus of all [such] $Y.$ [is] a Line; but in the latter the property is expressed thus: $A.Y. \simeq B.Y.$; the locus of all [such] $Y.$ will be a Plane. However, if all $Y.$ are in the same plane as AB and among them $A.Y \simeq B.Y.$ is supposed, the locus of all $Y.$ will be a Straight Line.

Ex $A.B.C. \simeq A.B.Y.$ sequitur $A.C. \simeq A.Y.$ et $B.C. \simeq B.Y.$ unde constat $Y.$ cadere in Sphaeram Centro $A.$ Radio $AC.$ et in Sphaeram centro $B.$ radio $B.C.$

From $A.B.C. \simeq A.B.Y.$ it follows that $A.C. \simeq A.Y.$ and $B.C. \simeq B.Y.$, whence it is established that $Y.$ falls in a Sphere with Center $A.$ and Radius $AC.$ and in a Sphere with center $B.$ and radius $B.C.$

§ 18. Ex Contactibus etiam Sphaerarum in uno puncto sequitur dari locum Unicorum ad duo puncta, vel vicissim ex hoc sequitur Contactus Sphaerarum in uno puncto. Idem est in Plano de Contactibus Circulorum.

(18) Yet again, from the Contacts of Spheres [being] in one point, it follows that there is a locus of those Unique [with situs] to two points; or conversely, from this follows the Contact of Spheres [being] in one point. Likewise for the Contacts of Circles in a Plane.

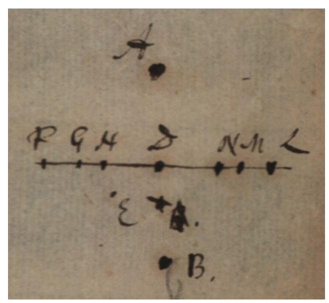

Figure 13

$FA \simeq FB \simeq LA \simeq LB.$ sic $GA \simeq GB \simeq MA \simeq MB.$ Nempe circulus centro A radio AE descriptus cum sit E infra Rectam et $A.$ supra Rectam, secabit eam bis in F et L, quae sectionum puncta sibi continuo appropinquant, $F.$ transeundo in $G.$ $H.$ etc.: et L in $M.$ $N.$ etc.: Ubi autem sibi occurrent, ibi in unum

$FA \simeq FB \simeq LA \simeq LB.$ thus $GA \simeq GB \simeq MA \simeq MB.$; certainly the circle described with center A and radius AE, when E is below the Line and $A.$ above the Line, will cross it twice, in F and L; these points of the crossings approach each other continuously, with $F.$ passing to $G.H.$ etc. and L to $M.N.$ etc. Where they meet each other, moreover, coalescent in D. eritque ibi duorum Circulorum Contactus. Hinc si A et B sint ea ad quae omne punctum rectae FL eodem modo se habet, erit D sui situs ad ea unicum et in Rectam per $A.B$. cadet. Videtur etiam sequi has Rectas se non nisi in uno puncto secare.

there they will coalesce to one in D., and there it will be the Contact of two Circles. Hence if A and B are those to which every point of the line FL has the same relation, then D will be unique with its situs to them and will fall in the Line through $A.B$. It also appears [videtur] to follow that these Lines do not cross each other except in one point.