G. W. Leibniz
Characteristica Geometrica

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Characteristica Geometrica

(1) *Characteres* sunt res quaedam quibus aliarum rerum inter se relationes exprimuntur, et quarum facilior est quam illarum tractatio. Itaque omni operationi quae fit in characteribus respondet enuntiatio quaedam in rebus: et possumus saepe ipsarum rerum considerationem differre usque ad exitum tractationis. Invento enim quod quaeritur in characteribus facile idem invenietur in rebus, per positum ab initio rerum characterumque consensum. Ita machinae exhiberi possunt modulis, corpora solida repraesentari possunt in plana tabula, ita ut nullum sit punctum corporis, cui non respondens aliud assignari possit in tabula secundum leges perspectivae. Itaque si quam operationem geometricam scenographica ratione in tabula plana super imagine rei peregerimus; poterit eventus illius operationis exhibere punctum aliquod in Tabula, cui facile sit invenire punctum respondens in re. Ac proinde solutio problematum stereometricorum in plano peragi poterit.

(2) Quanto autem characteres sunt exactiores, id est quo plures rerum relationes exhibent, eo majorem praestant utilitatem, et si quando exhibeant omnes rerum relationes inter

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(1) Characters are certain things that express the relations other things have among themselves, and which are easier to handle than those things. And so, to every operation taking place in the characters, there corresponds a certain proposition in the things, and we can often defer consideration of the things themselves until the very end. Once we have found in the characters what we sought, we will easily find the same among the things, through the agreement originally set up between the things and the characters. In this way machines could be exhibited by models, and solid bodies represented on a planar tablet so that no point of the body lacks a corresponding point assigned on the tablet according to the laws of perspective. And so if we perform some geometrical operation on the image of the thing by the scenographic method, this operation may yield some other point of the tablet from which we may easily find a corresponding point on the thing. And accordingly we may solve stereometric problems in the plane.

(2) Now, the more exact the characters are, that is, the more relations of things they exhibit, the more useful they can be, and if they exhibit all the relations among the things, as for instance the characters I use for

se, quemadmodum faciunt characteres Arithmetici a me adhibiti, nihil erit in re quod non per characteres deprehendi possit: Characteres autem Algebraici tantum praestant quantum Arithmetici, quia significant numeros indefinitos. Et quia nihil est in Geometria quod non possit exprimi numeris, cum Scala quaedam partium aequalium exposita est, hinc fit, ut quicquid Geometricae tractationis est, etiam calculo subjici possit.

arithmetic, there is nothing in a thing that could not be apprehended through the characters. The characters of algebra serve as well as arithmetic characters because they signify indefinite numbers. And since nothing in geometry cannot be expressed by numbers, once a certain scale of equal parts has been set, it turns out that whatever is treated in geometry is susceptible of calculation.

(3) Verum sciendum est, easdem res diversis modis in characteres referri posse, et alios aliis esse commodiores. Ita Tabula in qua corpus arte perspectiva delineatur potest et gibba esse, sed praestat tamen usus tabulae planae; et nemo non videt characteres numerorum hodiernos, quos Arabicos vel Indicos vocant, aptiores esse ad calculandum, quam veteres Graecos et Romanos; quanquam et his calculus peragi potuerit. Idem et in Geometria usu venit; nam Characteres Algebraici neque omnia quae in spatio considerari debent, exprimunt; (Elementa enim jam inventa et demonstrata supponunt;) neque situm ipsum punctorum directe significant; sed per magnitudines multa ambage investigant. Unde fit ut difficile sit admodum quae figura exhibentur exprimere calculo; et adhuc difficilius calculo inventa efficere in figura: itaque et constructiones quas calculus exhibet plerumque sunt mire detortae et incommodae. Quemadmodum alibi ostendi exemplo problematis hujus[:] data basi, altitudine et angulo ad verticem invenire Triangulum.

(3) But we know that the same things can be related in various ways in characters, and some ways are more convenient than others. Thus, the tablet on which a body is delineated by the art of perspective could also be gibbous, and yet a planar tablet serves better. And no one can fail to see that today's characters for numbers, which are called Arabic or Indian, are more suitable for calculation than the old Greek or Latin ones, even though those too can be used for calculation. The same thing happens also in geometry, namely, that neither can algebraic characters express everything that should be considered in space (indeed, they presuppose elements[1] discovered and demonstrated already), nor can they signify the situation of points itself directly, but rather they investigate it through magnitudes with many convolutions. Whence it happens that it is exceedingly difficult to express in calculus what is exhibited by a figure, and more difficult still to effect in a figure what is discovered by calculus. And so the constructions which calculus exhibits are for the most part amazingly distended and awkward, as I showed elsewhere in the case of the following problem: find the triangle with given base, altitude, and vertex angle.

(4) Equidem animadverto Geometras solere descriptiones quasdam figuris suis adjicere, quibus explicentur figurae, ut quae ex figura ipsa satis cognosci non possunt, ut linearum aequalitates ac proportionalitates saltem ex verbis adjectis intelligantur: plerumque et longius progrediuntur, et multa

(4) Now, I have noticed that geometers tend to add certain descriptions to their figures which explain them, so that what cannot be recognized from the figure alone, such as equality or proportionality of lines, may be understood at least from the added words. They often go further and express in many words even what is manifest from the figure

[1] Here 'elements' was capitalized. Although Leibniz's capitalization was a bit erratic, this capitalization may indicate a reference to the *Elements* of Euclid.

verbis exponunt, etiam quae ex figura ipsa sunt manifesta, tum ut ratiocinatio sit severior, nihilque a sensu atque imaginatione pendeat sed omnia rationibus transigantur; tum ut figurae ex descriptione delineari, aut si forte amissae sint, restitui possint.

itself, so that the argument is more rigorous, and nothing depends on sense or even imagination, but everything is carried out by logical reasoning; or else so that the figures can be drawn from the description or restored if by chance they are lost.

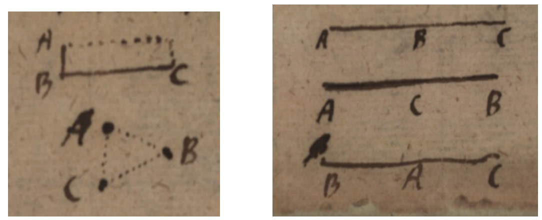


Figure 1

Hoc autem quamvis non satis exacte observent, praebuere tamen nobis Characteristicae Geometricae velut vestigia quaedam ut cum Geometrae dicunt rectang. ABC intelligunt factum ex ductu AB super BC ad angulos rectos. Cum dicunt AB aequ. BC aequ. AC exprimunt Triangulum aequilaterum. Cum dicunt ex tribus AB. BC. AC. duo quaedam aequari tertio designant omnia tria A. B. C. esse in eadem recta.

And even if they don't observe this sufficiently precisely, still they give us certain traces of a geometric characteristic, as when geometers say 'the rectangle ABC', they mean the one made by drawing [*ex ductu*] AB over BC at right angles. When they say AB equals BC equals AC, they express an equilateral triangle. When they say of the three, AB, BC, AC, some two equal the third, they signify that all three, A, B, C, lie on the same line.

(5) Ego vero cum animadverterem hoc solo literarum, puncta figurae designantium usu nonnullas figurae proprietates posse designari; cogitare porro coepi, an non omnes punctorum figurae cujusque relationes iisdem literis ita designari possint, ut tota figura characteristice exhibeatur, et quae crebris linearum ductibus, vix ac ne vix quidem praestantur sola harum literarum collocatione ac transpositione inveniantur. Nam plerumque confusio oritur in figura ex multiplicibus linearum ductibus praesertim cum adhuc tentandum est, cum contra tentamenta characteribus impune fiant. Sed subest aliquid majus nam poterimus characteribus istis veras definitiones omnium exprimere quae sunt Geometricae tractationis, et analysin ad principia usque nempe axiomata et postulata

(5) So when I noticed that some properties of a figure can be signified merely by this use of letters to signify points, I began to consider further whether we could in fact signify all relations of the points of any figure by the same letters, so that the whole figure might be exhibited in the characteristic, and we might discover by the mere arrangement and transposition of those letters what is barely or not at all apparent from drawing numerous lines. Usually confusion arises from the multiplicity of lines drawn in the figure, particularly when we are still trying things, whereas we can try things without risk by means of characters. But there is something greater behind this, for we can express with these characters the true definitions of everything that is treated in geometry, and we can continue analysis to the principles, namely the axioms and postulates, whereas algebra for

continuare cum Algebra sibi non sufficiat, sed propositionibus per Geometriam inventis uti cogatur; et dum omnia ad duas illas propositiones, quarum una duo quadrata in unum addit, altera vero triangula similia comparat, referre conatur pleraque a naturali ordine detorquere cogatur.

(6) Nos vero ubi semel Elementa characteribus nostris demonstraverimus, facile poterimus modum deprehendere inveniendi problematum solutiones quae statim eadem opera exhibeant constructiones et demonstrationes lineares; cum contra Algebraici inventis valoribus incognitarum de constructionibus adhuc soliciti esse debeant et constructionibus repertis demonstrationes lineares quaerant. Itaque miror homines non considerasse, si demonstrationes et constructiones esse possunt lineares omni calculo exutae, multoque breviores, profecto etiam inventionem dari debere linearem: nam in lineari non minus quam algebraica Synthesi regressum dari necesse est. Causa autem cur analysis linearis nondum deprehensa fuerit haud dubie nulla alia est, quam quod Characteres nondum inventi sunt quibus ipse situs punctorum directe repraesentaretur, nam in magna rerum multitudine et confusione sine characteribus expedire sese difficile est.

(7) Quod si jam semel figuras et corpora literis exacte repraesentare poterimus, non tantum Geometricam mirifice promovebimus, sed et opticen et phoronomicam, et mechanicam, in universum quicquid imaginationi subjectum est, certa methodo et velut analysi tractabimus, efficiemusque arte mirificia ut machinarum inventiones non sint futurae difficiliores quam constructiones problematum Geometriae. Ita etiam nullo negotio sumtuque machinae etiam valde compositae imo et res naturales delineari poterunt sine figuris, ita ut posteritati transmittantur, et quandocunque lubebit figurae ex

its part would not be enough, but would be forced to use propositions discovered through geometry; when they try to relate everything to those two propositions, the one which adds two squares in one, and the other which compares similar triangles, they are forced to distort most things from the natural order.

(6) Having once demonstrated the elements in our characters, we shall easily see how to find solutions to problems which exhibit immediately, by the same operation, the constructions and the demonstrations in terms of lines. On the contrary, after algebraists find the values of their unknowns, they still have to worry about the constructions, and after discovering the constructions, they still seek demonstrations in terms of lines. And so I am amazed that people haven't considered that, if there can be demonstrations and constructions in terms of lines, devoid of all calculation, and much shorter, surely there must also be discovery in terms of lines, for there must be a regression[2] in linear no less than in algebraic synthesis. Indeed, the reason we have not yet seized upon an analysis in terms of lines is doubtless nothing else than that characters haven't been contrived by which the situation of points is directly represented. For among a great multiplicity and confusion of things, it is difficult to move freely without characters.

(7) But if we can once represent figures and bodies exactly by letters, not only will geometry be marvellously improved, but also optics, kinematics [*phoronomicam*], mechanics, and in general whatever is subject to the imagination will be treated by a sure method and (as it were) analysis. And with a marvellous art we will render the invention of machines in the future no more difficult than the constructions of problems in geometry. Thus, without laborious effort or expense, even very complex machines, and indeed natural things, could be delineated without figures, so that they can be transmitted to posterity and figures can be formed, with the greatest precision, from the descriptions

[2] Latin '*regressum*' indicating a backwards-going stepwise logical sequence, moving from what is sought to what is known (giving a method of discovery); as opposed to *progression* which moves forward from known principles toward what is new.

descriptione summa cum exactitudine formari possint. Cum nunc quidem ob delineandarum figurarum difficultatem, sumtusque multa pereant, hominesque a rerum sibi exploratarum atque reipublicae utilium descriptione deterreantur, verba etiam neque satis exacta neque satis apta hactenus ad descriptiones concinnandas habeantur, quemadmodum vel ex botanicis et armorum insigniumque explicatoribus patet. Poterunt enim caeterae quoque qualitates quibus puncta quae in Geometria, ut similia considerantur inter se differunt facile sub characteres vocari: Ac profecto tum demum aliquando spes erit penetrandi in naturae arcana, cum id omne quod alius vi ingenii atque imaginationis ex datis extorquere potest, nos ex iidem datis certa arte securi et tranquilli educemus.

(8) Cum vero nihil tale cuiquam hominum, quod sciam in mentem venerit, nec ulla uspiam praesidia apparerent, coactus sum rem a primis initiis repetere, quod quam difficile sit nemo credit nisi expertus. Itaque diversis temporibus plus decies rem aggressus sum diversis modis, qui omnes erant tolerabiles et praestabant aliquid, sed scrupulositati meae non satisfaciebant. Tandem multis resectis ad simplicissima me pervenisse agnovi, cum nihil aliunde supponerem, sed ex propriis characteribus omnia ipse demonstrare possem. Diu autem haesi etiam reperta vera characteristicae hujus ratione, quia ab Elementis per se facilibus atque aliunde notis incipiendum mihi videbam, quae tanta scrupulositate ordinare minime gratum esse poterat. Perrexi tamen et molestia hac superata denique ad majora sum eluctatus.

[Omnia in Geometria punctorum tantum consideratione absolvuntur. Nam et lineae natura ex eo constat, punctum eius utcunqve assumtum, certam qvandam ad data aliqvot puncta habet relationem, per qvam ex ipsis determinari sive inveniri

whenever we wish. Much is lost with the current difficulty and cost of delineating figures, and people are deterred from describing matters they have explored and that are of public utility. Up to now the terms we have had are indeed neither sufficiently exact nor sufficiently apt for the compilation of descriptions, which is evident, for instance, from the case of botanists, and also those who illustrate armaments and insignia. With characters we could easily denote the other qualities by which points, which are considered similar in geometry, do differ from each other. And then finally, at last, there will be hope of penetrating the secrets of nature,[3] when everything that another can wrest from the givens by sheer power of ingenuity and imagination, we, tranquil and secure, can draw from the same givens by a fixed technique.

(8) But since no such thing has occurred to anybody, as far as I know, and no helps appear anywhere, I am forced to undertake the matter again from the very beginning; how difficult that is, nobody believes without experience. And so I have attacked the thing on more than ten different occasions, in different ways, all of which were tolerable and had various advantages, but did not satisfy my scruples. Finally, after cutting out very much, I saw that I had arrived at the greatest simplicity, since I did not suppose anything from elsewhere, but I could demonstrate everything from my own characters. But I lingered for a long time even after finding the true method of this characteristic, since I saw that I should start from elements that are easy in themselves and familiar from elsewhere. Arranging them so carefully could be rather unpleasant, yet I persevered, overcoming the irritation, until finally I had struggled through to better things.

[[4]Everything in geometry can be resolved by considering mere points. For the nature of a curve also consists in the fact that any point assumed on it has a certain relation to some given points, through which it can be determined or discovered from them. And surfaces can

[3]Leibniz initially continued with 'since we are abler by technique to effect that which another ...', but later deleted this and continued as rendered above.
[4]Leibniz cancelled this entire paragraph. We have included it because the summary and emphasis it provides seem helpful for understanding the subsequent sections.

potest. Superficies autem vel per lineas, vel immediate per puncta cognosci possunt, et corpora per superficies vel lineas vel puncta. Punctorum autem duorum simul existentium consideratio nihil aliud qvam situm unius ad alterum sive distantiam continet, vel qvod idem est rectam interceptam.]

(9) Verum ut omnia ordine tractemus sciendum est primam esse considerationem ipsius *spatii,* id est Extensi puri absoluti. *Puri* inquam a materia et mutatione *absoluti* autem, id est illimitati atque omnem extensionem continentis. Itaque omnia puncta sunt in eodem spatio et ad se invicem referri possunt. An autem spatium hoc a materia distinctum res quaedam sit, an solum apparitio constans seu phaenomenon, nihil refert hoc loco.

(10) Proxima est consideratio *Puncti,* id est ejus quod inter omnia ad spatium sive extensionem pertinentia simplicissimum est quemadmodum enim Spatium continet extensionem absolutam, ita punctum exprimit id quod in extensione maxime limitatum est, nempe simplicem situm. Unde sequitur punctum esse minimum, et partibus carere et omnia puncta congruere inter se (sive coincidere posse), adeoque et similia atque si ita loqui licet aequalia esse.

(11) Si duo puncta simul existere sive percipi intelligantur, eo ipsa consideranda offertur relatio eorum ad se invicem quae in aliis atque aliis binis punctis diversa est, nempe relatio loci vel situs quem duo puncta ad se invicem habent, in quo intelligitur eorum distantia. Est autem *distantia* duorum, nihil aliud quam quantitas minimae unius ad alterum viae, et si bina puncta A. B. servato situ inter se binis aliis punctis C. D. etiam situm inter se servantibus simul congruant aut succedere possint utique situs sive distantia horum duorum eadem erit quae distantia illorum duorum. Nam congrua sunt quorum unum alteri coincidere potest, nulla intra alterutrum

be known either through curves or immediately through points, and bodies through surfaces or curves or points. But the consideration of two points existing simultaneously contains nothing but the situation of the one to the other, i.e. the distance, or what is the same thing, the intervening straight line.]

(9) But one should understand that to treat everything in order, we must first consider *space*, that is, pure absolute *extension*. *Pure*, I mean, from matter and change, and moreover *absolute*, that is, without limit and containing all extension. And so all points are in the same space and can be related to each other. Now whether this space is some thing [*res*], distinct from matter, or only a constant appearance or phenomenon, does not matter here.

(10) Next is consideration of the *point*, that is, of that which is simplest among all things pertaining to space or extension; in the way that *space* contains absolute extension, so the point expresses that which is maximally limited in extension, namely simple situation. It follows from this that a point is minimal and lacks parts, and that all points are congruent to each other (or are able to coincide), and so they are also similar and (if it is allowed to say this) equal.

(11) If two points are understood to exist or be perceived simultaneously, what is thereby presented for consideration is their relation to each other, which differs from one pair of points to another, namely the relation of place, or situation, that two points have to each other, in which we understand their distance. Now the *distance* of the two is nothing but the minimum quantity of a path from one to the other. And if a pair of points $A.B$, preserving the situation between themselves, is jointly congruent to, or able to displace, another pair of points $C.D$ also preserving the situation between themselves, then the situation, or distance, of the former two is the same as the distance of the latter two. For those things are congruent of which one can coincide with

mutatione facta. Coincidentium autem $A.B.$ itemque $C.D.$ eadem distantia est, ergo et congruorum, quippe quae sine distantiae intra $A.B.$ vel intra $C.D.$ mutatione facta, possunt coincidantia reddi.

(12) *Via* autem (qua et distantiam definivimus) nihil aliud est quam locus continuus successivus. Et via puncti dicitur *Linea*. Unde et intelligi potest extrema lineae esse puncta, et quamlibet lineae partem esse lineam, sive punctis terminari. Est autem via continuum quoddam, quia quaelibet ejus pars extrema habet cum alia anteriori atque posteriori parte communia. Unde consequitur, ut hoc obiter addam, si linea quaedam in aliqua superficie ducatur, non posse aliam lineam in eadem superficie continue progredientem inter duo prioris lineae extrema transire, quin priorem secet.

(13) Via lineae ejusmodi ut puncta ejus non semper sibi invicem succedant, *superficies* est; et via superficiei ut puncta ejus non semper sibi invicem succedant, est *corpus*. Corpus autem moveri non potest, quin omnia ejus puncta sibi succedant (quemadmodum demonstrandum est suo loco), et ideo novam dimensionem non producit. Hinc apparet nullam esse partem corporis cujus ambitus non sit superficies, nullamque esse partem superficiei cujus ambitus non sit linea. Patet etiam extremum superficiei pariter atque corporis in se redire sive esse *ambitum* quendam.

(14) Assumtis jam duobus punctis eo ipso determinata est via puncti per unum pariter atque alterum simplicissima possibilis: alioqui eorum distantia non esset determinata, adeoque nec situs. Haec autem linea quae a duobus solis punctis per quae transit determinata est nimirum, ut posito

the other without any change being made within either of them. Coincident things $A.B$ and $C.D$ have the same distance, therefore so do congruent things, since then, without any change of distance within $A.B$ or within $C.D$, they could be rendered coincident.

(12) Now a *path* (through which we also defined distance) is nothing but a continuous successive locus. And the path of a point is called a *curve*. It can also be understood from this that the extremes of a curve are points, and that any part of a curve is a curve, or terminates in points.[5] Now a path is a certain continuum, since any part of it has common extremes with its anterior and posterior parts. It follows from this, if I may add in passing, that if a certain curve is drawn on some surface, one cannot make another curve on the same surface pass continuously between the two extremes of the first curve without crossing the first one.

(13) The path of a curve of this type, such that its points do not always succeed one another, is a *surface*, and the path of a surface, such that its points do not always succeed one another, is a *body*. A body, however, cannot move without all of its points succeeding one another (as must be demonstrated in its place), and for that reason does not produce a new dimension. It is thus apparent that there is no part of a body whose ambit[6] is not a surface, and no part of a surface whose ambit is not a curve. It is also clear that the extreme of a surface (and equally of a body) returns on itself, i.e. it is an *ambit*.

(14) Now when two points are assumed, *eo ipso* the simplest possible path through both one and the other is determined: otherwise their distance would not be determined, and neither would their situation. But this curve, which is determined solely by the two points through which it passes, namely such that supposing it passes through the two

[5]Leibniz cancelled the following: 'Moreover, the minimum path from a point to a point is necessarily the path of a point, or is a curve; for certainly the path of a point is smaller, or simpler, than the path of another extensum'. Before 'is smaller', Leibniz had also written the phrase '*via rei alterius continui extensi*', and cancelled it in what seems to have been an earlier cancellation.

[6]I.e., boundary.

eam per duo data punta transire, ipsa sola hinc consideranda offeratur, ea inquam linea dicitur *recta,* et licet utcunque producatur dicitur una eademque recta. Ex quibus sequitur non posse duo eadem puncta duabus rectis communia esse, nisi ea duae rectae quantum satis est productae coincidant: ac proinde duas recta non habere segmentum commune (alioqui et duo segmenti hujus extrema haberent communia), nec spatium claudere sive componere ambitum in se redeuntem. Alioqui recta una ab altera digressa ad eam rediret, adeoque in binis punctis ei occurreret. Pars quoque rectae est recta nam et ipsa determinatur per duo illa puncta sola, per quae sola determinatur totum. Determinatur inquam, id est omnia ejus puncta consideranda seu percurrenda ex sola duorum punctorum consideratione offeruntur. Ex his patet si $A.B.C.$ et $A.B.D.$ congrua sint, et $A.B.C.$ in una recta esse dicantur, coincidere C et D. Seu si punctum tantum unicum sit quod eam habeat ad duo puncta relationem quam habet, erunt tria puncta in una recta. Contra si plura duobus sint puncta eodem modo se habentia ad tria vel plura puncta data erunt haec quidem in eadem recta, illa extra eam, cujus rei ratio est, quod quae ad determinantia eodem modo se habent, eo ipso ad determinata eodem modo se habent, itaque tria plurave puncta in eadem recta haberi possunt pro duobus. Puncta autem eodem modo se habentia requiro plura duobus. (Nam si sint duo tantum, res procedit modo tria ad quae unumquodque duorum eodem modo se habet, sint in eodem plano, licet non sint in eadem recta.)

Recta quoque uniformis est ob simplicitatem, seu partes habet toti similes. Et omnis recta rectae similis est quia pars unius alteri congrua est, pars autem rectae toti similis. Et in recta distingui non potest concavum a convexo, sive recta non habet duo latera dissimilia, vel quod idem est; si duo puncta sumantur extra rectam, quae eodem modo se habeant ad extrema rectae vel duo quaelibet puncta in recta, ea sese etiam eodem modo habebunt ad totam rectam, seu ad quodli-

given points, it alone is thereby presented for consideration, this line, I say, is called *straight*. And however far it is extended, it will be called one and the same line. From this it follows that the same two points cannot have two distinct lines in common, unless the two lines coincide after being extended far enough. Consequently, two lines cannot have a common segment (otherwise they would also have in common the two extremes of this segment), neither can they enclose space, i.e. create an ambit that returns on itself, otherwise one line that diverges from the other would return to it and thus would meet it in two points. A part of a line is also a line, for it is determined by those two points alone through which alone the whole is determined. Determined, I say, meaning that by the consideration of those two points alone, all of its points are presented to be considered or traversed. From this it is clear that if $A.B.C$ and $A.B.D$ are congruent, and $A.B.C$ are said to lie on a line, then C and D coincide. I.e. if the point were so unique [*tantum unicum*] that it would have that relation to the two points that it has, then they will be three points on one line. On the contrary, if there are more than two points related in the same way to three or more given points, then indeed these will be on the same line and those outside it; the reason is that things related in the same way to the determiners *eo ipso* are related in the same way to the determined things, and so three or more points on the same line can be supplied instead of two. However, I require more than two points having the same relation. (For if there are just two, each one of which has the same relation to the three, then we can conclude only that the three lie on the same plane, not that they lie on the same line.)

A line is also uniform on account of its simplicity, i.e. it has parts similar to the whole. And any line is similar to any other line, since a part of one is congruent to the other, and the part is similar to the whole. And concave and convex sides cannot be distinguished for a straight line, i.e. the line does not have two dissimilar sides, or what is the same thing: if two points are taken outside the line which relate in the same way to the line's extrema (or any two points on the line), then they will also relate in the same way to the whole line, i.e. to any point bet punctum in recta; a quocunque demum latera rectae illa duo extra rectam puncta sumantur. Cujus rei ratio est, quia quae ad puncta determinantia aliquod extensum eodem modo se habent modo, ea etiam ad totum extensum eodem modo se habere necesse est. Denique recta a puncto ad punctum minima est ac proinde distantia punctorum nihil aliud est quam quantitas rectae interceptae. Nam via minima utique magnitudine determinata est a solis duobus punctis; sed et positione determinata est neque enim in spatio absolute plures minimae a puncto ad punctum esse possunt (ut in sphaerica superficie plures sunt viae minimae a polo ad polum). Nam si minima est absolute, extrema non possunt diduci manente lineae quantitate ergo nec partium extrema (nam et partes inter sua extrema minimas esse necesse est) salva singularum partium quantitate, ergo nec salva totius quantitate. Jam si lineae duo extrema maneant immota et linea ipsa transformetur, necesse est puncta ejus aliqua a se invicem diduci. Itaque extremis rectae immotis, salva quantitate minima inter duo puncta, in aliam transformari non potest, itaque non dantur plures minimae incongruae dissimiles inter duo puncta. Quare si duae inter duo puncta essent minimae essent congruae inter se. Jam una aliqua minima est recta (ut supra ostendimus), ergo et alia minima erit recta. At duae rectae inter duo puncta coincidunt. Itaque minima inter duo puncta non nisi unica est.

on the line, from whichever side of the line those two points outside the line are finally taken. The reason for this is that when things relate in the same way to the points that determine some extensum, they necessarily also relate in the same way to the whole extensum. Finally, a line is the minimal [path] from point to point, and accordingly the distance of the points is nothing but the quantity of the intervening line. For the minimal line is certainly determined in magnitude solely by the two points; but it is also determined in position, for there cannot be multiple minimal paths from point to point in absolute space [*spatio absolute*] (as there are many minimal [paths] in a spherical surface from one pole to the other). For if it is absolutely minimal, the extremes could not be drawn apart while maintaining the quantity of the line, and therefore neither can the extremes of its parts (for it is also necessary that the parts are minimal between their own extremes) while preserving the quantity of the separate parts, and therefore neither while preserving the quantity of the whole.[7] If, now, the line is transformed while its two extremes remain unmoved, it is necessary that some of its points will be drawn apart from each other. And so if the extremes of a line are fixed while the minimal quantity between the two points is preserved, it cannot be transformed into something else, and so one cannot give multiple incongruent, dissimilar minimal [paths] between two points. Consequently, if two [paths] between two points were minimal, they would be congruent to each other. Now a minimal one is a straight line (as shown above), therefore another minimal one must also be a straight line. And two lines between two points coincide. And so a minimal [path] between two points can only be unique.

(15) Modus generandi lineam rectam simplicissimus hic est: Sit corpus aliquod cujus duo puncta sint immota et fixa, ipsum autem corpus nihilominus moveatur, tunc omnia puncta corporis quiescentia incident in rectam quae per duo puncta fixa transit. Manifestum enim est ea puncta locum habere ex datis duobus punctis fixis determinatum seu manentibus

(15) Here is the simplest method of generating a straight line. Let there be some body, two points of which are unmoved and fixed, but the body itself nonetheless moves; then all stationary points of the body will fall on the line passing through the two fixed points. For it is evident that those points have the same place determined from the two given fixed points, i.e. they cannot move with the two points remaining

[7] The last clause follows nine cancelled lines, so it is not surprising that it does not cohere perfectly with the preceding clauses.

duobus punctis fixis et toto solido existente, moveri non posse; cum caetera extra rectam eadem servata ad duo puncta fixa relatione, locum mutare possint. Unum hic incommodum est, quod ea recta hoc modo descripta non est permanens. Aliter generari potest linea recta, si qua detur linea flexilis, sed quae in majorem longitudinem extendi non possit. Nam si extrema ejus diducantur quousque id fieri potest; linea flexilis in rectam erit transmutata. Eodem modo et plani ac Circuli et Trianguli proprietates ex constitutis definitionibus duci possent. Nam de linea recta in speciem tantum disseruimus.

(16) Heac omnia animo consequi non difficile est, etsi neque figurae nisi imaginatione delineentur, neque characteres adhibeantur alii quam verba, sed quia in ratiocinationibus longe productis neque verba ut hactenus concipi solent satis exacta sunt, nec imaginatio satis prompta; ideo figuras hactenus adhibuere Geometrae. Sed praeterquam quod saepe delineantur difficulter, et cum mora quae cogitationes optimas interea effluere sinit; nonnumquam et ob multitudinem punctorum ac linearum schemata confunduntur, praesertim cum tentamus adhuc et inquirimus; ideo characteres sequenti modo cum fructu adhiberi posse putavi.

(17) Spatium ipsum seu extensum (id est continuum cujus partes simul existunt) non aliter hic quidem designari commode posse video quam punctis. Quoniam figurarum delineationes exacte exprimere propositum est, et in his non nisi *puncta* et *tractus quidam continui* ab uno puncto ad aliud spectantur, in quibus puncta infinita pro arbitrio sumi possunt. *AB* fig. 1.

Ideo puncta quidem certa exprimemus literis solis ut *A*, item

fixed and the whole being solid; whereas other points outside the line can change places while preserving the same relation to the two fixed points. The one inconvenience here is that the line described in this way is not permanent. A straight line can be generated in another way, whereby a flexible curve is given which cannot be extended to a greater length. For if its extremes are drawn apart as far as possible, the flexible curve will be transformed into a straight line. In the same manner, the properties of a plane or a circle or a triangle can be deduced from the established definitions by a certain natural order of contemplation. For we have examined the straight line just as a sample.

(16) It is not difficult to follow all of this mentally, even though no figures were drawn (except by imagination), and neither did we make use of characters other than words. But because the words that geometers have tended to consider (until now) are also insufficiently exact for long, drawn-out argumentation, and the imagination is not sufficiently adept, they have until now used figures for that reason. And drawing figures is often difficult besides, and the delay allows the best thoughts to float away, and sometimes the figure [*schemata*] gets muddled on account of the multiplicity of points and lines, particularly when we are still trying out and investigating various things; for all these reasons I believed that characters could be beneficially deployed in the way I describe below.

(17) Space itself, i.e. an extensum (that is, a continuum whose parts exist simultaneously), cannot be designated conveniently in any other way here, that I can see, except by points. This is because we are proposing to express exactly the demarcations of figures, and in these we see nothing but *points* and *certain continuous tracts* from one point to another, in which infinitely many points could be taken arbitrarily.

And so, then, we will express specified points by single letters, such as

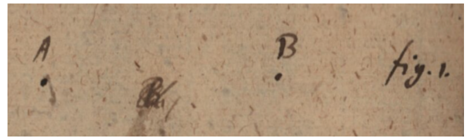


Figure 2

B fig. 1.

A and *B* (Figure 2).[8]

(18) Tractus autem continuos exprimemus per puncta quaedam incerta sive arbitraria, ordine quodam assumta, ita tamen, ut appareat semper alia intra ipsa tum ultra citraque semper posse sumi. fig. 2

(18) Continuous tracts, on the other hand, we will express through certain unspecified or arbitrary points, assuming a certain order, so that it is nonetheless apparent that other points between them, as well as after and before them, could always be taken.

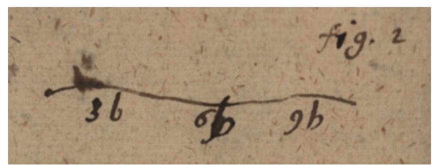


Figure 3

Ita 3 *b* 6 *b* 9 *b fig* 2. significabit nobis totum tractum, cujus quodlibet punctum appellatur *b*. et in quo pro arbitio assumsimus partes duas, unam cujus extrema sunt puncta 3 *b*. 6 *b*, alteram cujus extrema sunt puncta 6 *b*. 9 *b*. Unde patet illas duas partes continuas esse, cum habeant commune punctum 6 *b*. et divisio earum sit facta pro arbitrio. Hic tractus in quo duarum partium commune extremum nullum aliud est quam punctum, dicitur *Linea,* et repraesentari etiam potest motu puncti, *b*, quod viam quandam percurrit, sive vestigia tot quot puncta diversa 3 *b*. 6 *b*. 9 *b*. relinquere intelligitur. Hinc linea dici potest via puncti. Via autem est locus puncti continuus successivus. Potest et per compendium designari

Thus 3*b*6*b*9*b* (Figure 3) will signify for us an entire tract, any point of which will be called *b*, and in which we have chosen arbitrarily two parts, one whose extremes are the points 3*b*, 6*b*, and the other whose extremes are the points 6*b*, 9*b*. It is clear from this that these two parts are continuous, since they have a common point 6*b* and their division was made arbitrarily. This tract, in which the common extreme of two parts is nothing but a point, is called a *curve*, and it can also be represented by the motion of a point *b* which runs along a certain path, that is, it is understood to leave as many traces as there are distinct points 3*b*, 6*b*, 9*b*, etc. Hence a curve may be called the path of a point. A path, moreover, is a continuous successive locus. It may also be signified in the following abbreviated fashion: Curve ‘$\overline{y}b$’. We

[8]Here Leibniz added and then cancelled the sentence: ‘And points which are unspecified, and which could be assumed arbitrarily, we will express by adding numbers, so that it is apparent that these points as well as the others between them and after and before them could be assumed.’

hoc modo: Linea $\bar{y}b$ designando per literam $\bar{y}$. vel aliam numeros ordinales pro arbitrio sumtos collective. Cum vero scribemus: yb sine nota supra y intelligimus, quodcunque lineae $\bar{y}b$ punctum, distributive.

designate by the letter $\bar{y}$ (or another letter) arbitrary ordinal numerals, taken collectively. But when we write 'yb' without the mark over y, we will understand each point of the curve $\bar{y}b$, taken distributively.

(18 bis) Eodem modo tractus quidam fingi possunt, quorum partes cohaerent lineis, vel qui describi intelliguntur motu lineae tali ut puncta ejus non succedant sibi sed ad nova deveniant. fig. 3

(18)[9] In the same way, one could make certain tracts whose parts are connected by curves, or which we understand to be described by the motion of a curve such that its points do not succeed one another but come to into new places.

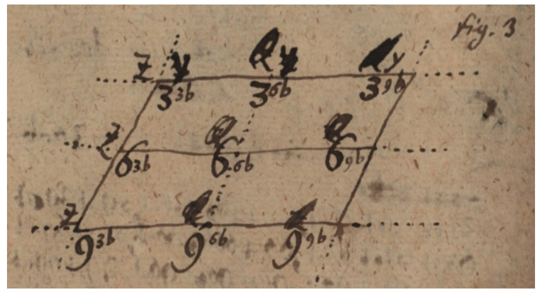


Figure 4

Hic tractus sive via lineae dicitur *superficies,* ponamus nimirum in *fig*. 3 lineam supradictam 3 b 6 b 9 b moveri, ejusque locum unum appellari 33 b 36 b 39 b, locum alium sequentem 63 b 66 b 69 b et rursus alium sequentem 93 b 96 b 99 b, fiet superficies 3 3 b 3 6 b 3 9 b, 6 3 b 6 6 b 6 9 b, 9 3 b 9 6 b 9 9 b quam et per compendium sic designabimus, $\overline{zy}b$.

This tract, or path of a curve, is called a *surface*. Let us suppose, namely in Figure 4, that the curve above called $3b6b9b$ moves, and that one of its places is named $33b36b39b$, and another, subsequent, place is $63b66b69b$, and yet another is $93b96b99b$; the surface $33b36b39b$, $63b66b69b$, $93b96b99b$ will result, and we will signify it with the abbreviation $\overline{zy}b$.

(19) Ubi patet etiam, quemadmodum motu lineae $\bar{y}b$ secundum puncta $\bar{z}b$ describitur superficies $\overline{zy}b$. Ita vicissim motu linea $\bar{z}b$ secundum puncta $\bar{y}b$ describi eandem superficiem $\overline{yz}b$. At yzb significabit unaquaeque loca puncti b, non collective, sed distributive, et $z\bar{y}b$ significat unam aliquam lineam $\bar{y}b$ in superficie $\overline{zy}b$ sumtam quamcunque etiam non collective

(19) Here it is also clear that just as motion of the curve $\bar{y}b$ following the points $\bar{z}b$ describes the surface $\overline{zy}b$, so in turn, motion of the curve $\bar{z}b$ following the points $\bar{y}b$ describes the same surface $\overline{yz}b$. And yzb will signify each individual place of the point b, not collectively, but rather distributively; and $z\bar{y}b$ signifies any one curve $\bar{y}b$ on the surface $\overline{zy}b$, also taking each of them not collectively, but distributively.

[9] Leibniz added an additional, apparently spurious, paragraph number here.

sed distributive.

(20) Neque refert cujus figurae sint ipsae lineae quae moventur; aut etiam secundum quas fit motus, sive quas unum ex lineae motae punctis, describit, inspiciatur *figura* 4. fig. 4

(20) Neither does it matter what shape the moving curves themselves have; or even [those curves] according to which the motion happens, i.e. which are described by one of the points of the moving curve. See Figure 5.

Figure 5

Potest etiam fieri ut durante motu, ipsa linea mota figuram mutet, ut linea $\bar{z}b$ in dicta fig. 4. Quod clarius intelligi potest, si quis cogitet quam superficiem descripturus esset arcus, qui durante explosione utcunque moveretur totus, exempli causa si caderet in terram. fig. 5.

It can also happen that during the motion, the moving curve itself changes shape, as the curve $\bar{z}b$ in the same Figure 5. This could be understood more clearly by thinking of the surface that would be described by a bow if the whole bow moved while firing, for example if it fell to the ground.

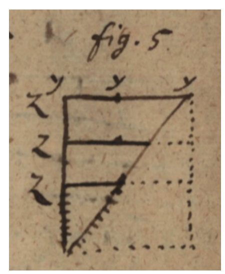


Figure 6

Potest etiam linea mota durante motu partes aliquas amittere, quae ab ea sive re sive animo separantur, ut patet ex *fig*. 5. fig. 6 99 *b*

The moving curve can even lose some parts during the motion, which are separated from it either actually or mentally, as is clear from Figure 6.

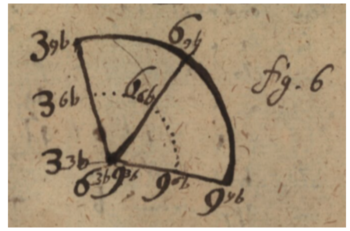


Figure 7

Fieri etiam potest, ut punctum unum plurave, exempli gratia $3\,b$ in linea mota durante motu quiescat, et loca ejus expressa velut plura, exempli gratia $33\,b$. $63\,b$. $93\,b$., inter se coincidant, ut intelligitur inspecta *fig*. 6. Sed hae varietates omnes multaeque aliae plures etiam characteribus designari poterunt, quamadmodum suo loco patebit.

It can also happen that one or more points in the moving curve (e.g. $3b$) remains stationary during the motion, and its places, expressed as though distinct (e.g. as $3_{3b}6_{3b}9_{3b}$), actually coincide with each other, as one realizes by considering Figure 7. But all these variations and many others can also be designated by characters, as will be shown in its own place.

(21) Quemadmodum autem lineae motu describitur Tractus ille quem vocant Superficiem, ita superficiei motu (tali ut partes ejus vel puncta sibi non ubique succedant) describitur Tractus quem vocant solidum sive corpus. fig. 7.

(21) Now, as that tract which is called a surface is described by the motion of a curve, so also the tract that is called a solid, or body, is described by the motion of a surface (such that its parts, or points, do not succeed one another everywhere).

Quod exemplo uno satis intelligi potest *fig*. 7., ut si immota manente linea (recta) $\overline{z}\,3\,b$ (nempe $33\,b\ 63\,b\ 93\,b$) in superficie (rectangulo) $\overline{yzb}$ (nempe

$$\begin{Bmatrix} 3_{3b} & 6_{3b} & 9_{3b} \\ 3_{6b} & 6_{6b} & 9_{6b} \\ 3_{9b} & 6_{9b} & 9_{9b} \end{Bmatrix},$$

) moveatur haec ipsa superficies, motu suo describet solidum

$$\begin{Bmatrix} 33_{3b} & 33_{6b} & 33_{9b} & 36_{3b} & 36_{6b} & 36_{9b} & 39_{3b} & 39_{6b} & 39_{9b} \\ 63_{3b} & 63_{6b} & 63_{9b} & 66_{3b} & 66_{6b} & 66_{9b} & 69_{3b} & 69_{6b} & 69_{9b} \\ 93_{3b} & 93_{6b} & 93_{9b} & 96_{3b} & 96_{6b} & 96_{9b} & 99_{3b} & 99_{6b} & 99_{9b} \end{Bmatrix}$$

ubi tamen notandum hoc loco ob rectam $\overline{z}\,3\,b$ immotam puncta

This can be understood well enough from one example (Figure 8): If the (linear) curve $\overline{z}3b$ (namely $3_{3b}6_{3b}9_{3b}$) remains fixed in the (rectangular) surface $\overline{zyb}$, namely:

$$\begin{Bmatrix} 3_{3b} & 6_{3b} & 9_{3b} \\ 3_{6b} & 6_{6b} & 9_{6b} \\ 3_{9b} & 6_{9b} & 9_{9b} \end{Bmatrix},$$

and this same surface moves, then by its motion it will describe the solid:

$$\begin{Bmatrix} 33_{3b} & 33_{6b} & 33_{9b} & 36_{3b} & 36_{6b} & 36_{9b} & 39_{3b} & 39_{6b} & 39_{9b} \\ 63_{3b} & 63_{6b} & 63_{9b} & 66_{3b} & 66_{6b} & 66_{9b} & 69_{3b} & 69_{6b} & 69_{9b} \\ 93_{3b} & 93_{6b} & 93_{9b} & 96_{3b} & 96_{6b} & 96_{9b} & 99_{3b} & 99_{6b} & 99_{9b} \end{Bmatrix}$$

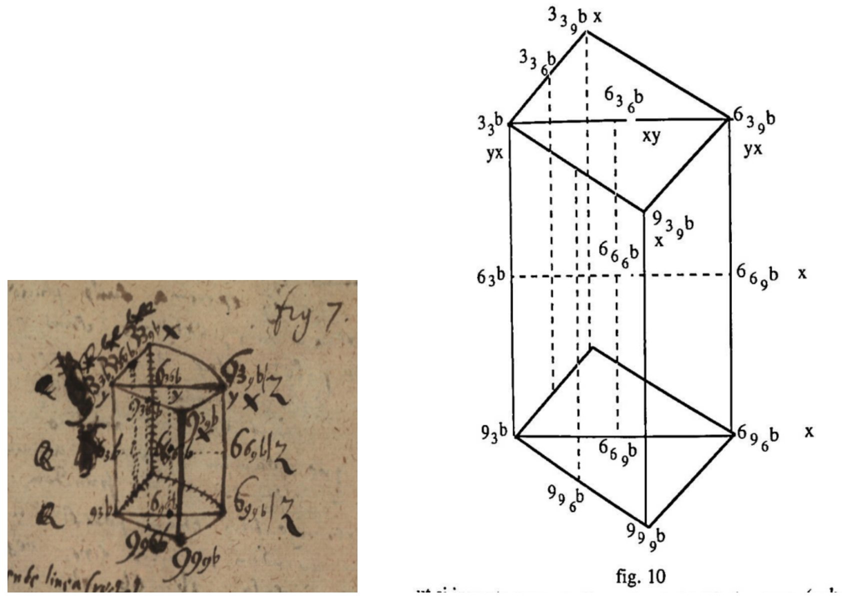


Figure 8

333 *b*, 633 *b*, 933 *b*, ideoque loco omnium in figura reperitur solum 33 *b* coincidere itemque puncta 363 *b*, 663 *b*, 963 *b*, unde etiam in figura habetur tantum 63 *b*; ac denique cum eodem modo hic coincidant puncta 393 *b*, 693 *b*, 993 *b*, tamen per 93 *b* expressa sint. Hoc solidum autem per compendium exprimemus hoc modo: $\overline{xyz}b$, et aliquam ejus superficiem seu locum aliquem ipsius $\overline{zy}b$ exprimemus hoc modo $x\overline{zy}b$ (ita exhibetur sectio cylindricae portionis seu solidi hujus facta plano per axem). Potest etiam aliqua ejus superficies assumi hic modo $z\overline{xy}b$ (ita exhibetur sectio hujus portionis cylindricae secundum basin seu plano basi parallelo); item hoc modo $y\overline{xz}b$ (ita exhibetur sectio hujus cylindricae portionis per alium cylindrum axem cum isto communem habentem). Aliae quoque sectiones ejusdem Figurae intelligi possunt, quia infiniti etiam fingi possunt modi, eam generandi per motum vel

One should note here that because the line $\bar{z}3b$ is fixed, the points 3_{33b}, 6_{33b}, and 9_{33b} coincide (and so the place of all of them is found in the figure to be 3_{3b} only), and likewise the points 3_{63b}, 6_{63b}, 9_{63b}, from which we also have just 6_{3b} in the figure; and finally, in the same way, when the points 3_{93b}, 6_{93b}, 9_{93b} coincide, they are expressed just by 9_{3b}. Now this solid we will express with an abbreviation in this way: $\overline{xzy}b$, and some surface of it, or locus of $\overline{zy}b$, we will express in this way: $x\overline{zy}b$ (thus exhibiting a section of a cylindrical region, or this solid, made by a plane through the axis). We can also take one of its surfaces $\overline{xy}b$ in this way: $z\overline{xy}b$ (thus exhibiting a section of this cylindrical region according to the base, i.e. a plane parallel to the base); likewise in this way: $y\overline{xz}b$ (thus exhibiting a section of this cylindrical region through another cylinder having a common axis with this one). Other sections of the same figure can also be understood, since one could fashion infinitely many ways of generating it by motion, or even of etiam resolvendi in partes secundum certam aliquam legem. Caeterum omnes varietates, quas in superficiei productione vel resolutione paulo ante indicavimus, multo magis in solido locum habere manifestum est. Denique dimensionem aliquam alteriorem solido, seu tractum ipsius solidi motu tali descriptum ut puncta ejus sibi ubique non succedant reperiri non posse, suo loco demonstrandum est.

(22) Porro tractus ipsi seu loca punctorum quorundam indefinitorum, determinantur punctis quibusdam certis, itemque Legibus quibusdam secundum quas ex paucis illis punctis certis caetera puncta indefinita ordine in considerationem venire, et tractus ipsi generari sive describi possint.

Quod antequam exponamus, signa quaedam explicabimus quibus in sequentibus utendum erit. Primum itaque fieri potest ut duo vel plura nomina in speciem diversa non sint revera nisi unius rei sive loci, id est puncti vel lineae alteriusve tractus, atque ita *eadem esse* sive *coincidere* dicentur. fig. 8 *B*

resolving it into parts according to some specified law. It is evident that all the other variations in production or resolution of a surface, which we indicated a little earlier, are even more relevant for solids. And finally we must demonstrate in its own place that a dimension higher than a solid, or the tract described by such a motion of the solid that its points do not succeed one another everywhere, could not be obtained.

(22) Furthermore, the tracts themselves, or the loci of certain indefinite points, are determined by certain specified points, and likewise by certain laws according to which the other indefinite points come into consideration, in order, from the few specified points, and the tract itself could be generated, i.e. described.

Before explaining this, let us clarify certain symbols that will be used in what follows. First, then, it can happen that two or more names, apparently distinct, really refer to just one thing or locus, that is, a point or line or another tract, and thus they are said to *be the same* or to *coincide*.

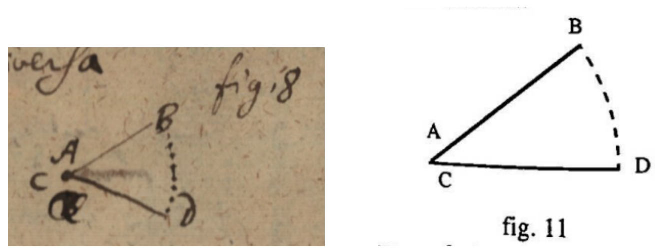


Figure 9

Ita si sint duae lineae *AB* et *CD*, sintque puncta *A* et *C* unum idemque hoc ita designabimus:

$$A \infty C$$

, id est *A* et *C* coincidunt. Hoc maxime usum habebit in designandis punctis aliisque extremis communibus diversorum Tractuum. Idem enim punctum sive extremum suas denomi-

So if there are two lines *AB* and *CD*, and the points *A* and *C* are one and the same, we will signify it thus:

$$A \infty C$$

that is, *A* and *C* coincide. [See Figure 9.] This will be especially useful for designating extremal points (and other extremes) shared by distinct tracts. For the same point, or rather extremum, has indeed nationes habebit, tam secundum unum tractum, quam secundum alterum. Quod si dicatur $A.B \infty C.D$ sensus erit simul esse $A \infty C$ et $B \infty D$. Idemque est in pluribus. Ab utraque enim enuntiationis parte, idem ordo est observandus.

its denominations according to one tract and according to another. But if we say $A.B \infty C.D$, the meaning will be $A \infty C$ and $B \infty D$ simultaneously. And likewise for several; and the same order should be observed on each side of the statement.

(23) Quod si duo non quidem coincidant, id est non quidem simul eundem locum occupent, possint tamen sibi applicari, et sine ulla in ipsis per se spectatis mutatione facta alterum in alterius locum substitui queat, tunc duo illa dicentur esse *congrua* fig. 9

(23) But if the two do not in fact coincide, that is, they do not simultaneously occupy the same place, but nonetheless could be applied on each other, and without any change made in them viewed in themselves, the one is able to substitute in place of the other, then those two are said to be *congruent*, as $A.B$ and $C.D$ in Figure 10.

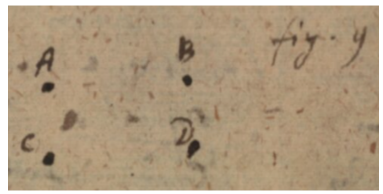


Figure 10

Itaque fiet: AB ɣ CD item $A.B$ ɣ $C.D$ *in fig.* 9. id est servato situ inter A et B, item servato situ inter C. et D. nihilominus $C.D$. applicari poterit ipsi $A.B$ id est simul applicari poterit C ipsi A et D ipsi B.

And thus AB ɣ CD. Likewise $A.B$ ɣ $C.D$ in Figure 10, that is, while preserving the situation between A and B and likewise the situation between C and D, $C.D$ can be applied on $A.B$, that is, C can be applied on A and D on B simultaneously.

(24) Si duo extensa non quidem congrua sint, possint tamen congrua reddi, sine ulla mutatione molis sive *quantitatis*, id est retentis omnibus iisdem punctis; facta tantum quadam si opus est transmutatione, sive transpositione partium vel punctorum; tunc dicentur esse *aequalia.* fig. 10

(24) If two extensa are not congruent but nonetheless could be rendered congruent without any change of mass or *quantity*, that is, retaining all of the same points, even if a certain transmutation, or transposition, is needed of the parts or points, then they are said to be *equal*. Thus in Figure 11

Ita in fig. 10 Quadratum $ABCD$ et Triangulum isosceles EFG basin habens EG lateri AB quadrati duplam; aequalia sunt: nam transferatur FHG in EGF quia EGF ɣ FHG. fiet EFG aequ. $EHFG$. Jam $EHFG$ ɣ $ABCD$. ergo EFG aequ. $ABCD$. Hinc generalius si a ɣ c et b ɣ d erit $a + b \sqcap$ (sive

the square $ABCD$, and the isosceles triangle EFG with base EG double the side AB of the square, are equal: for translate FHG to EGF, since EGF ɣ FHG. Then we have EFG *equ*. $EHFG$. Now $EHFG$ ɣ $ABCD$, thus EFG *equ*. $ABCD$. Hence, more generally, if a ɣ c and b ɣ d, then $a + b \sqcap$ *(or equ.)* $c + d$. Even further: if a ɣ e, b ɣ f, c ɣ g,

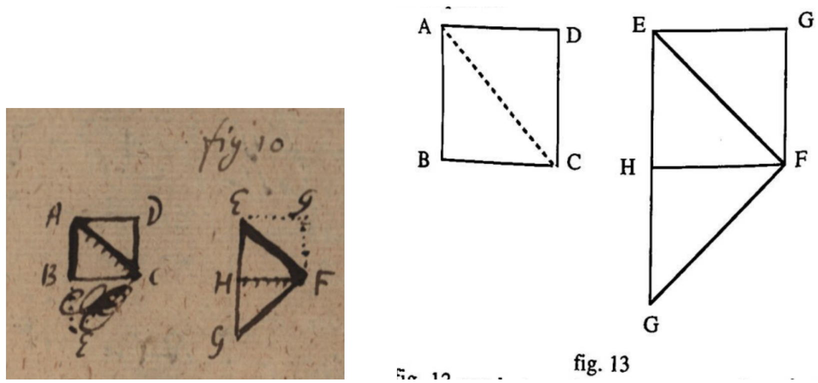


Figure 11

aequ.) $c + d$. Imo amplius: Si $a \between e$, $b \between f$, $c \between g$, $d \between h$ fiet: $a + b - c + d \sqcap e + f - g + h$. Sive si duae fiant summae ex quibusdam partibus uno eodemque modo addendo vel subtrahendo; partesque unius sint congruae partibus alterius eodem modo ad totum constituendum concurrentibus, quaelibet unius summae uni alterius summae sibi ordine respondenti; tunc duae summae quae inde fient, non quidem semper congruae erunt, erunt tamen semper aequales. Atque ita argumentatio a congruis ab aequalia ipsa aequalium definitione constituetur. Sunt quidem alias *aequalia*, quorum eadem est magnitudo. Verum ipsa partium congruentium cuidam rei sive mensurae, multitudo, est magnitudo. fig. 11.

$d \between h$, we have: $a + b - c + d \sqcap e + f - g + h$. Or if two sums are made out of certain parts by one and the same way of addition or subtraction, and the parts of the one are congruent to the parts of the other coming together in the same way to comprise the whole, each part of the one sum congruent to the corresponding one of the other sum, in order, then the two resulting sums won't always be congruent but nevertheless will always be equal. And so the argument from congruence to equality is established by the definition of equality. Alternatively, things are *equal* when they have the same magnitude. Of course the magnitude is the multiplicity of parts congruent to a certain thing, i.e. a measure, such as (in Figure 12)

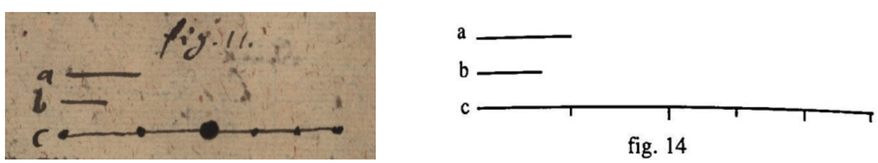


Figure 12

Ut si *in fig.* 1 1 sint duo magnitudinem habentia, a et b, et de-

if we have two magnitudes a and b, and we are given a third thing c

tur res tertia c quae sit bis a + ter b. Patet ejus magnitudinem multitudine partium tum ipsi a tum ipsi b congruentium exprimi. Itaque quae congruae reddi possunt nullo addito vel detracto utique aequalia esse necesse est.

[*Erster Ansatz, gestrichen*] Cum enim punctum quodvis cuivis congruat, et numerus punctorum in duobus rebus congruis sit idem, hoc est cuilibet puncto in uno respondeat suum proprium in altero, utique aequalia sunt sed quia non satis accurate puncta partes appellantur aut numerum habere dicuntur; et possunt aequalia esse quae nullas habent partes congruentes ut superficies cylindrica et circulus ideo magnitudinem poterimus intelligere generalius esse attributum rei expressum per alias res, ad ipsam pertinentes homogeneas datas manentem etiam harum rerum relatione inter se invicem mutata. *Homogeneas* intelligo quae in communi aliquo conveniunt, ut punctum, linea, corpus et horum partes. *Datas* id est congruentes certis quibusdam rebus; *Determinatas,* ita scilicet ut quaelibet earum proposita appareat ex ipsa expressione an ad rem pertineat vel non. Atque ita illud extensi praedicatum per quam determinari potest, an punctum aliquod ad ipsum pertineat, quodque eadem manet mutato punctorum situ inter se ita ut unum alterius vel ambo unius repetitione generari possint, *magnitudo* extensi appellabitur. Quodsi partes datis quibusdam rebus congruae determinari in re possint, utique et puncta in illis determinari poterunt, adeoque numerus partium rebus quibusdam certis congruentium, cum modum exhibeat omnia rei puncta ita determinandi, ut nihil referat quis sit situs punctorum invicem, utique erit magnitudo.

which is twice a + thrice b, then clearly its magnitude is expressed by the multiplicity of parts congruent to a as well as to b, and so it is necessary that things which can be rendered congruent, with no addition or subtraction, are certainly equals.

[[10]Now since any point is congruent to any point, and the number of points in two congruent things is the same, that is, any point in the one has its own corresponding point in the other, certainly they are equal. But points cannot quite accurately be called parts or be said to have a number, and there can be equals which have no congruent parts, as a cylindrical surface and a circle, so for that reason we can understand magnitude more generally to be an attribute of a thing expressed through other things, homogeneous given things belonging to it, which remains even when the relation of these things among themselves is changed. *Homogeneous* things I understand as those which go[11] together in some common thing, as a point, a curve, a body, and their parts. *Given* things, that is, things congruent to certain specified things; *determined* things, namely, so that whichever of them is proposed, it is apparent from the expression itself whether or not it pertains to the thing. And so, that predicated of an extensum through which it can be determined whether some point belongs to it, and which remains the same when the situation of the points among themselves is changed, such that one or both could be generated by repetition of the other or the one, will be called the *magnitude* of the extensum. But if parts congruent to certain given things can be determined in the thing, surely also the points in them can be determined, and thus the number of parts congruent to certain specified things, when it exhibits a way of determining all the points of the thing such that it does not matter what the situation of the points is one to another, certainly will be the magnitude.]

[10]Leibniz cancelled this paragraph during revision. In no subsequent section of the essay does he underline *homogeneus* as a definition.

[11]Latin *convenire* has a broad range of meaning, in Leibniz's usage, that it is hard to capture in a single English word. It can mean 'to meet' (as when two curves intersect), to 'agree' or 'be compatible', to 'fit together' or 'be appropriate to'. The somewhat colloquial English 'go' as in 'X goes with Y' or 'X and Y go together' (conveying both belonging and compatibility) covers much of the right ground. What matters is that X and Y are not only combined, but also have a certain kind of agreement of type. A part occurs 'with corequisites *directly*' because it *convenit* with them.

[*Zweiter Ansatz*] (25) Verum ut rem istam altius repetamus explicandum est, quid sit pars et totum, quid homogeneum, quid magnitudo, quid ratio. *Pars* nihil aliud est quam requisitum totius diversum (seu ita ut alterum de altero praedicari nequeat) immediatum, in recto cum correquisitis.

(25) But to return to the matter more deeply, we should explain what is a part and a whole, what is a homogeneum, what is magnitude, and what ratio. A *part* is nothing but an immediate, distinct (i.e. such that one cannot be predicated of the other) requisite of the whole, with its corequisites directly.[12]

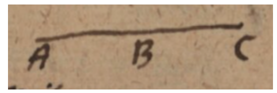


Figure 13

Ita AB. requisitum est ipsius AC. id est si non esset AB. neque foret AC. Diversum quoque est, neque enim AC est AB. alioqui enim rationalis est requisitum hominis, sed quia homo est rationalis, ideo rationalis (qui hominis requisitum est,) et homo, idem est, etsi enim expressione differant, re tamen conveniunt. Pars immediatum est requisitum, neque enim connexio inter AB et BC pendet a quadam consequentia sive connexione causarum, sed ipsa per se patet, ex hypothesi assumti totius. Est autem in recto cum correquisitis, semper enim convenire debent secundum certum quendam considerandi modum, nam et quae ut Entia tantum, imo et ut cogitabilia spectamus, verbi gratia DEUM, hominem, virtutem, possumus considerare velut partes unius totius ex ipsis compositi. Excluduntur ergo requisita immediata quidem et diversa ut rationalitas in abstracto quae requisitum est hominis immediatum diversum neque enim homo est rationalitas, attamen non hic spectatur ut conveniens cum homine, sed ut attributum: alioqui sane negari non potest, etiam ex hix

Thus AB is a requisite of AC, that is, if it were not for AB there would be no AC; they are also distinct, for AC is not AB; in another fashion we have that 'rational' is a requisite of 'man', but since man is rational and thus 'rational' (which is a requisite of man) and 'man' is the same thing, even though they differ in expression, they nonetheless go together *in re*. A part is an immediate requisite; indeed, the connection between AB and BC does not depend upon any inference, or connection of causes, but rather it is apparent *per se* from the hypothesis of the assumed whole. It is, moreover, with its corequisites directly, for they should always go together according to a specified mode of consideration; indeed both things we view just as beings, and also things we view as objects of thought [*cogitabilia*], for example GOD, man, or virtue, we could consider as parts of one whole composed from them. Therefore, immediate and indeed distinct requisites are excluded such as rationality in the abstract, which is an immediate, distinct requisite of man; for man is not rationality, viewed here not as going[13] with man, but as an attribute: otherwise one cannot reasonably deny that even from these two, man and rationality, one whole could be formed,

[12] Latin *in recto*. Cf. *in obliquo* ('obliquely') below. Although Leibniz usually used *in recto* in logico-grammatical writings for terms invoked in the nominative case, and *in obliquo* for terms in other cases, here it is applied with an apparently broader, and vaguer, sense.

[13] Leibniz first wrote *non …ut homogeneum hominis*, and revised the phrase in a second draft to *conveniens cum homine*. A few lines later, *seu non convenienti quadam ratione* ('i.e. it doesn't go together for a certain reason') was *non …id quod ex ipsis homogenea* ('i.e. not …that which [arises] from them by reason of certain homogenea') in a prior draft. Possibly he made these changes when he also struck out the last section of (24) in which he had defined homogenea. He does not give a revised definition of homogenea, but he provides some explanatory comments about what homogenea are later in (27) and (30) in terms of parts and wholes. In any case, he seems to intend largely the same concept with 'conveniens'. Perhaps this is a 'deeper' concept.

duobus: homo et rationalitas fingi posse unum totum, cujus hae partes. At rationalitas hominis pars non erit, requiritur enim ad hominem in obliquo, seu non convenienti quadam ratione, cum aliis quae ad hominem praeterea requiruntur. Sed haec sunt magis metaphysica, nec nisi in eorum gratiam adducuntur, qui notionum intima intelligere desiderant. Simplicus ita definiemus: *Partes* sunt quae requiruntur ad unum quatenus cum eo conveniunt.

of which they are the parts. But the rationality of a man is not a part, since it is required for a man obliquely, i.e. it doesn't go together for a certain reason with other things that are additionally required for a man. But these are, rather, metaphysical matters, and are brought up only to oblige those desirous of an intimate understanding of concepts. We define it simply thus: Parts are things which are required for an individual [*ad unum*] insofar as they go with it.

(26) *Numerus* est cujus ad unitatem relatio est quae inter partem et totum vel totum et partem. Quare fractos etiam et surdos comprehendo.

(26) The *number* is that whose relation to unity is that between part and whole or whole and part; in this way fractions and even irrationals are included.

(27) *Magnitudo* Rei distincte cognita est numerus (vel compositum ex numeris) partium rei cuidam certae (quae pro mensura assumitur) congruentium. Ut si sciam esse lineam quae bis aequetur lateri ter aequetur diagonali cujusdam quadrati certi mihique satis cogniti, ut ad ipsum cum lubet recurrere possim, ejus lineae magnitudo mihi cognita esse dicetur, quae erit binarius partium lateri congruentium, + ternarius partium diagonali congruentium. Diversis autem licet assumtis mensuris quibus eadem res diversimode exprimitur tamen semper eadem prodit magnitudo, quia ipsis mensuris iterum resolutis ad idem denique semper devenitur; adeoque mensurae diversae illum ipsum numerum eundem resolutione prodeuntem jam involvunt. Quemadmodum unus idemque est numerus tres quartae, ex sex octavae; si quartam iterum in duas partes resolvas. Atque talis est Magnitudo distincte cognita. Alioquin *magnitudo* est attributum rei, per quod cognosci potest, utrum aliqua res proposita sit illius pars, vel aliud homogeneum ad rem pertinens et quidem tale ut maneat licet partium habitudo inter se mutetur. Vel etiam Magnitudo est attributum quod iisdem manentibus homogeneis ad rem pertinentibus aut substitutis congruis, manet idem. Homogenea autem ad rem aliquam pertinentia intelligo non partes solum sed et extrema atque minima

(27) The *magnitude* of a thing (distinctly known) is the number (or composite of numbers) of parts congruent to a certain specified thing (which is taken as a measure). So if I know there is a line segment that is equal to twice the side, thrice the diagonal of a certain rectangle I adequately know, so that I can return to the same thing at will, then I am said to know the magnitude of its line, which would be two parts congruent to the side + three parts congruent to the diagonal. Although various distinct measures may be taken, by which the same thing is expressed in various ways, yet nonetheless the same magnitude always results, since when the measures themselves are resolved again, the same thing is always reached in the end; and thus the various measures already involve that very same number emerging in the resolution. For instance the number three quarters is one and the same with six eighths, if the quarter is resolved into two parts. Such is magnitude distinctly known. In any case, *magnitude* is the attribute of a thing through which one can tell whether some proposed thing is a part of it, or another homogeneum belonging to the thing and indeed such that it remains the same if the configuration of the parts among themselves changes. Or indeed magnitude is the attribute which remains the same, provided the homogenea belonging to it remain the same or congruent ones are substituted for them. By 'homogenea' belonging to some thing I mean not only parts, but also extrema and minima, or points. For a curve is made by a certain

sive puncta. Nam puncti repetitione quadam continua, sive motu, fit linea. Saepe autem res ita transmutantur, ut ne unica quidem pars figurae posterioris, prioris parti congruat. Aliter magnitudinem infra definio, ut sit id quo duae res similes discerni possunt, sive quod in rebus sola comperceptione discernitur. Sed omnia haec eodem recidunt.

(28) *Ratio ipsius A ad B* nihil aliud est quam numerus quo exprimitur magnitudo ipsius A, si magnitudo ipsius B ponatur esse unitas. Unde patet Magnitudinem a ratione differre ut numerum concretum a numero abstracto; est enim magnitudo numerus rerum, nempe partium; ratio vero est numerus unitatum. Patet etiam rei magnitudinem eandem manere, quacunque assumta mensura per quam eam exprimere volumus; rationem vero aliam atque aliam fieri pro alia atque alia mensura assumta. Patet etiam (ex definitione divisionis) si numerus magnitudinem exprimens ipsius A et alius numerus magnitudinem exprimens ipsius B (modo utrobique eadem mensura seu unitas adhibita sit) dividatur, provenire numerum qui est ratio unius ad alterum.

(29) *Aequalia* sunt quorum eadem est magnitudo, seu quae nullo amisso vel accepto congrua reddi possunt.

Minus dicitur quod alterius parti aequale est, id vero quod partem habet alteri aequalem dicitur *Majus.* Hinc pars minor toto, quia parti ipsius, nempe sibi, aequalis est. Signis autem his utemur:

$a \sqcap b$	a aequ. b
$a \ulcorner b$	a maj. quam b
$a \urcorner b$	a min. quam b

Si pars unius alteri toti aequalis est, reliquae partis in majore magnitudo dicitur *differentia.* Magnitudo autem totius est

continuous repetition, or motion, of a point. Often, however, a thing is transformed such that there is not a single part of the new figure congruent to a part of the previous one. I define magnitude in another way, below, as that by which two similar things could be distinguished, or that in things which is only discerned by co-perception. But these all come back to the same thing.

(28) The *ratio of A to B* is nothing but the number by which the magnitude of A is expressed if the magnitude of B is set as the unit. It is clear from this that magnitude differs from ratio as a concrete number from an abstract number; for magnitude is the number of things, namely of parts, but ratio is the number of unities. It is also clear that the magnitude of a thing remains the same, no matter what measure is taken through which we wish to express it; but the ratio becomes one or another thing for one or another assumed measure. And it is clear (from the definition of division) that if the number expressing the magnitude of A is divided[14] by the other number expressing the magnitude of B (provided the same measure or unit is used for both), the result will be the number that is the ratio of one to the other.

(29) *Equals* are things whose magnitude is the same, or which could be rendered congruent without loss or addition.

Something is called *less* which is equal to a part of the other, whereas that which has a part equal to the other is called *greater*. Hence the part is less than the whole, since it is equal to a part of it, namely to itself. We will, now, use these symbols:

$a \sqcap b$	a equals b
$a \ulcorner b$	a greater than b
$a \urcorner b$	a lesser than b

If a part of one is equal to another whole, the magnitude of the remaining part of the greater is called the *difference*. The magnitude,

[14]Latin: '*si numerus … A et alius numerus … B dividatur*', i.e. 'if number … A and the other number … B is divided'. The syntactic infelicity can be explained as the result of Leibniz squeezing in this last sentence, between lines, after the next paragraph was written.

summa magnitudinum partium, vel aliorum partibus ejus aequalium.

moreover, of the whole is the *sum* of the magnitudes of the parts, or of others equal to its parts.

(30) Si duo sint homogenea (sive si in uno partes assumi possint utcunque, partibus alterius aequales, et idem fieri semper possit et in residuis) neque differentia ulla sit inter ipsa, id est si neque a sit majus quam b, neque b majus quam a, necesse est esse aequalia. Transmutentur enim ut congruant quoad licet utique aut in uno eorum supererit aliquid, aut congruent, adeoque erunt aequalia. Itaque in his consequentia haec valebit:

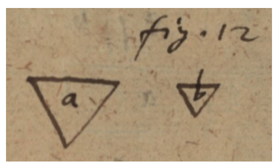
$$a \text{ non} \ulcorner b,\ a \text{ non} \urcorner b.\ \textit{Ergo}\ a \sqcap b.$$

(30) If two things are homogeneous (or if any kind of parts could be assumed in the one equal to the parts of the other, and the same can always be done also with the leftover ones) and there is no difference between them, that is, if neither a is greater than b, nor b greater than a, then they are necessarily equal. Indeed, let them be transmuted into congruents to the extent possible; then certainly either something superfluous will remain in one, or they will be congruent, and thus they will be equal. And so this inference is valid here:

$$a \text{ not} \ulcorner b, \quad a \text{ not} \urcorner b. \quad \text{Therefore } a \sqcap b.$$

(31) *Similia* sunt quae singula per se considerata discerni non possunt. fig. 12

(31) *Similars* are things which cannot be distinguished when considered individually in themselves [*singula per se*]. Just as two equilateral triangles (in Figure 14).



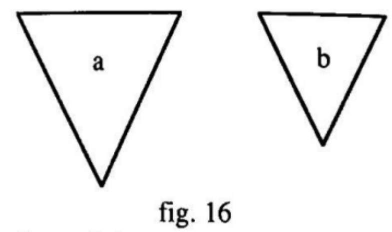


Figure 14

Velut duo triangula aequilatera (in fig. 12). Nullum enim attributum, nullam proprietatem in uno possumus invenire, quam non etiam possimus reperire in altero et unum ex ipsis appellando a, alterum b, *similitudinem* ita notabimus: $a \sim b$. Si tamen simul percipiantur, statim discrimen apparet, unum alio esse majus. Idem fieri potest etsi non simul percipiantur, modo aliquod velut medium assumatur sive mensura quae primum applicetur uni, aut aliquo in ipso, notatoque quomodo prius vel pars ejus cum mensura vel ejus parte congruat,

For there is no attribute, no property we could discover in one which we could not also find in the other. And calling one of them a, the other b, we notate *similarity* thus: $a \sim b$. If, however, they are perceived simultaneously, immediately there appears the difference that one is greater than the other. The same thing can happen even if they are not perceived simultaneously, only assuming something as an intermediary or a measure, which is applied first to the one, or to something in it, and noting how the first thing (or its part) is congruent to the measure (or its part), then later the same measure postea eadem mensura etiam applicetur alteri. Itaque dicere soleo *similia* non discerni nisi per comperceptionem. At, inquies, ego etsi successive videam duo triangula aequilatera inaequalia ea nihilominus probe discerno. Sed sciendum est me hoc loco loqui de intelligentia, ut nimirum mens aliquid notare possit in uno, quod non procedat in altero non de sensu et imaginatione. Ratio autem cur oculi duas res similes sed inaequales discernant, manifesta est; nam supersunt nobis rerum prius perceptarum imagines, quae rei nove perceptae imaginibus applicatae discrimen ipsa comperceptione harum duarum imaginum ostendunt. Et ipse fundus oculi, cujus partem majorem minoremque occupat imago, mensurae cujusdam officium facit. Denique alias res semper simul percipere solemus, quas etiam cum prioribus percepimus, unde rem novissime perceptam ad eas referendo, ut priorem ad easdem retulimus, discrimen non difficulter notamus. Si vero fingeremus, DEUM omnia in nobis ac circa nos in aliquo cubiculo apparentia proportione eadem servata minuere, omnia eodem modo apparerent neque a nobis prior status a posteriore posset discerni, nisi sphaera rerum proportionaliter imminutarum, cubiculo scilicet nostro egrederemur; tunc enim comperceptione illa cum rebus non imminutis oblata discrimen appareret. Hinc manifestum est etiam *Magnitudinem* esse illud ipsum quod in rebus distingui potest sola comperceptione, id est applicatione vel immediata, sive congruentia actuali sive coincidentia, vel mediata nempe interventu mensurae, quae nunc uni nunc alteri applicatur, unde sufficit res esse congruas, id est actu congruere posse.

is also applied to the other. And so I usually say that *similars* are not distinguished except through co-perception. Now, you might say, I could still properly distinguish two unequal equilateral triangles if I were to see them in succession. But of course I am speaking here about discernment [*intelligentia*], namely that the mind could note something in one that does not happen in the other, and not about sense and imagination.[15] The reason, more specifically, that the eyes distinguish two similar but unequal things is apparent: for we retain images of the things perceived previously, which, applied to the images of the newly perceived thing, show the difference by the very co-perception of these two images. And the back of the eye itself, the greater or lesser part of which is occupied by the image, serves as something of a measure. Finally, we are accustomed to perceiving always together other things which we have also perceived with the previous ones; from this we note the difference without difficulty, relating the newest perceived thing back to them, just as we related the previous one to the same things. But if we imagine that God were to shrink everything appearing in us and around us in some room, keeping the same proportions, everything would appear the same, and we could not distinguish the earlier state from the later one, unless of course we were to step out from the sphere of things proportionally diminished, that is to say, from our room; for then by that co-perception, shown with the undiminished things, the difference would become apparent. From this it is also manifest that *magnitude* is that which may be distinguished in things only by co-perception, that is, either by immediate application, whether by actual congruence or by coincidence, or else by mediate application, namely by the intervention of a measure, which is applied now to one and now to the other; whence it suffices for things to be congruent, that is, to be able to be made actually congruent.[16]

[15]Leibniz inserted *non de sensu et imaginatione* in a later draft, using the margin.

[16]The Latin *actuali* in 'by actual congruence', and *actu* in 'made actually congruent', carry a complex sense that combines a side of the 'real/imaginary' distinction with a side of the 'actual/potential' distinction, the latter including the operational aspect of actuality, i.e. 'by action', stemming from Aquinas's use of *actus* to render Aristotle's Greek *energeia*. Actual congruence is real congruence, congruence in effect, and agreement through action. It is also noteworth that Leibniz makes use of the distinct forms 'congrua esse' and 'congruere', both meaning 'to be congruent'. English no longer allows the archaic verb 'to congrue'; sometimes we write 'make congruent' when he uses the second form. It is not a consistent, technical distinction.

(32) Ex his autem intelligi potest similia et aequalia simul esse congrua. Et quia similitudinem hoc signo notare placet: $\sim$ nempe $a \sim b$ id est a est simile ipsi b. fig. 13

(32) From this one can understand, moreover, that things simultaneously similar and equal are congruent. And similarity we shall denote by this sign: $\sim$, namely $a \sim b$, that is, a is similar to b.

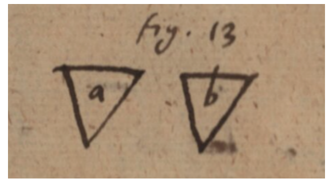


Figure 15

Vid. fig. 13. Hinc consequentia erit talis:
$a \sim b$ *et* $a \sqcap b$. E rgo $a \between b$.

See Figure 15. From this we have the inference:

$$a \sim b \text{ and } a \sqcap b. \text{ Therefore } a \between b.$$

(33) Sunt et aliae consequentiae:

$a \between b$.	*Ergo*	$a \sqcap b$.
$a \between b$.	*Ergo*	$a \sim b$.
$a \infty b$.	*Ergo*	$a \between b$.
....		$a \sqcap b$.
....		$a \sim b$.

(33) We also have other inferences:

$a \between b$.	Therefore	$a \sqcap b$.
$a \between b$.	Therefore	$a \sim b$.
$a \infty b$.	Therefore	$a \between b$.
...	...	$a \sqcap b$.
...	...	$a \sim b$.

(34) Nam quae reapse coincidunt, utique congrua sunt; quae congrua sunt utique similia, item aequalia sunt. Hinc videmus tres esse modos ac velut gradus res extensione praeditas, neque alias qualitatibus diversas discernendi. Maximus ille est, ut sint dissimiles, ita enim singulae per se spectatae ipsa proprietatum quae in ipsis observantur diversitate facile discernuntur, ita triangulum isosceles facile discernitur a scaleno, etsi non simul videantur. Si quis enim me jubeat videre an triangulum quod offertur sit isosceles an scalenum, nihil forinsecus assumere necesse habeo, sed sola latera ejus comparo inter se. At vero si jubear ex duobus Triangulis aequilateris eligere majus collatione Triangulorum cum aliis opus habeo sive comperceptione, ut explicui, neque notam ali-

(34) For things which really coincide are certainly congruent; and congruent things are certainly similar, and likewise equal. From this we see that there are three modes, as it were degrees, of distinguishing things provided with extension and not otherwise differing in qualities. The maximum is when they are dissimilar, for thus, when viewed individually in themselves, they are easily distinguished by the difference of the properties which are observed in them; thus an isosceles triangle is easily distinguished from a scalene one, even if they are not viewed simultaneously. Indeed if someone asked me to see whether some proffered triangle were isosceles or scalene, I would not need to assume anything from outside, I would simply compare its sides among themselves. But on the other hand, if I were asked to choose the greater of two equilateral triangles, I would need to do

quam discriminis in singulis spectabilem assignare possum. Si vero duae res non tantum sint similes, sed et aequales, id est si sint congruae, etiam simul perceptas non discernere possum, nisi loco, id est nisi adhuc aliud assumam extra ipsas, et observem ipsas diversum habere situm, ad tertium assumtum. Denique si ambo simul in eodem sint loco, jam nihil habere me amplius quo discriminentur. Atque haec est vera cogitationum quam de his rebus habemus Analysis cujus ignoratio fecit, ut characteristica geometriae vera hactenus non sit constituta. Ex his denique intelligitur, ut magnitudo aestimatur dum res congruere aut ad congruitatem reduci posse intelliguntur, ita rationem aestimari similitudine, seu dum res ad similitudinem reducuntur, tunc enim omnia fieri necesse est proportionalia.

so by bringing them together [*collatione*] of the triangles with other things, i.e. by co-perception, as I have explained, and neither could I attribute any mark of difference seen in them individually. But if in fact two things are not only similar, but also equal, that is if they are congruent, then they cannot be distinguished even if perceived simultaneously, except by place; that is, unless I assume something extra outside of them, and I observe that they have a different situation to the third assumed thing. Finally, if both are simultaneously in the same place, now I have nothing more by which to distinguish them. And this is the true analysis of the thoughts [*cogitationum*] that we have about these matters, ignorance of which has led to the fact that a true geometric characteristic has not been established until now. From these thoughts it may finally be understood that, as magnitude is assessed when things are understood to be congruent or reducible to congruity, so ratio is assessed by similarity, that is, when things are reduced to similarity, for then everything will necessarily become proportional.

(35) Ex his explicationibus coincidentium, congruorum, aequalium ac similium consequentiae quaedam duci possunt. Nempe quae sunt eidem aequalia, similia, congrua, coincidentia, sunt etiam inter se, ideoque

$$\begin{array}{cccccc} a \infty b & \text{Et} & b \infty c & \text{Ergo} & a \infty c \\ a \between b & & b \between c & & a \between c \\ a \sim b & & b \sim c & & a \sim c \\ a \prod b & & b \prod c & & a \prod c \end{array}$$

Non tamen consequentia haec valet:

$$a \text{ non } \infty b \quad \text{et} \quad b \text{ non } \infty c. \quad \text{Ergo} \quad a \text{ non } \infty c,$$

prorsus ut in Logica ex puris negativis nihil sequitur.

(35) From these descriptions of coincidents, congruents, equals, and similars, certain inferences may be derived. Namely, things which are equal, similar, congruent, or coincident to the same thing, are also to each other, and therefore:

$$\begin{array}{cccccc} a \infty b & \text{and} & b \infty c & \text{Therefore} & a \infty c \\ a \between b & & b \between c & & a \between c \\ a \sim b & & b \sim c & & a \sim c \\ a \prod b & & b \prod c & & a \prod c \end{array}$$

However, these inferences are not valid:

$$a \text{ not } \infty b \quad \text{and} \quad b \text{ not } \infty c. \quad \text{Therefore} \quad a \text{ not } \infty c.$$

Just as in logic, nothing follows from pure negatives.

(36) Si coincidentibus sive iisdem ascribas coincidentia prodeunt coincidentia, ut

$a \infty c$ et $b \infty d$. Ergo $a.b \infty c.d$. Sed in congruis hoc non se-

(36) If coincidents are joined to coincidents, or to the same thing, then coincidents will result:

$$a \infty c \quad \text{and} \quad b \infty d \quad \text{Therefore} \quad a.b \infty c.d.$$

quitur, exempli causa si $A.B.C.D.$ sint puncta, semper verum est esse $A \between C$ et $B \between D$. Quodlibet enim punctum cuilibet congruum est. Sed non ideo dici potest $A.B \between C.D.$ id est simul congruere posse A ipsi C et B ipsi D. servato nimirum tum situ $A.B$ tum situ $C.D$. Quanquam viceversa ex positis $A.B \between C.D$ sequatur $A \between C$. et $B \between D$. ex significatione characterum nostrorum, idque etiam verum est, licet $A.B.C.D.$ non sint puncta sed magnitudines. At si congrua sibi ascribantur, inde oriuntur aequalia, ita: $a + b - c \sqcap d + e - f$, *posito esse* $a \between d$ *et* $b \between e$ *et* $c \between f$, *quia congrua semper aequalia sunt*.

(37) Verum si congrua congruis similiter addantur, adimanturque, fient congrua. Cujus rei ratio est quia si congrua congruis similiter addantur, similia similibus similiter addentur (quia congrua sunt similia) ergo fient similia, fiunt autem etiam aequalia (nam congrua congruis addita faciunt aequalia). Jam similia et aequalia sunt congrua. Ergo *si congrua congruis similiter addantur fient congrua.* Idem est si adimantur.

(38) An autem similiter aliqua tractentur intelligi potest ex characteristicis nostris modoque unumquodque describendi aut determinandi. In quo si sigillatim nullum discrimen notari potest, utique semper omnia similia prodire necesse est. Illud quod notandum est *si qua sint similia secundum unum determinandi* (distincte cognoscendi, describendi) *modum eadem fore similia etiam secundum alium modum.* Nam unusquisque modus totam rei naturam involvit.

(39) Axiomata autem illa quibus Euclides utitur, si aequalibus addas aequalia fient aequalia, aliaque id genus facile ex eo demonstrantur, quod aequalium eadem est magnitudo, id est quod substitui sibi possunt salva magnitudine. Nam sint $a \sqcap c$

But this does not follow with congruents because, for example, if $A.B.C.D.$ are points, it is always true that $A \between C$ and $B \between D$. Indeed any point is congruent to any point. But you cannot on that account say $A.B \between C.D$, that is, that A can be made congruent [*posse congruere*] with C and B with D simultaneously, while preserving of course both the situation $A.B$ and the situation $C.D$. Conversely, though, upon supposing $A.B \between C.D$, it would follow that $A \between C$ and $B \between D$ from the meaning of our characters, and for that reason it is also true even if $A.B.C.D$ are not points, but magnitudes. But if congruents are joined together, equals result from this; so: *supposing* $a + b - c \sqcap d + e - f$, *then* $a \between d$ *and* $b \between e$ *and* $c \between f$ *since congruents are always equals*.

(37) Of course, if congruents are added to or subtracted from congruents in a similar way, congruents will result. The reason for this is that when congruents are added to congruents in a similar way, then similars will be added to similars in a similar way (since congruents are similars), therefore similars will result. And they will also be equals (for congruents added to congruents make equals). Now, things similar and equal are congruent. Therefore, *if congruents are added to congruents in a similar way, congruents will result*. Likewise if they are subtracted.

(38) Now, whether some things are treated similarly may be understood from our characteristic and the way in which each thing is described or determined. If no difference can be noted in each one considered individually, then it is certainly necessary that everything always comes out similar. The noteworthy point is that *if things are similar according to one way of determining (distinctly knowing, describing), the same things will also be similar according to another way*. For each way of determining involves the whole nature of the thing.

(39) The axioms used by Euclid, that if equals are added to equals then equals will result, and others of the sort, are easily demonstrated from the fact that equals have the same magnitude, that is, they can be substituted for each other while preserving the magnitude. For

et $b \sqcap d$, *fiet* $+ a + b \sqcap c + d$, nam si scribatur $a + b$ et in locum ipsorum $a.b.$ substituantur aequalia $c.d$, ea substitutio fiet salva magnitudine, ac proinde eorum quae prodibunt $+ c + d$ eadem erit magnitudo quae priorum $+ a + b$. Sed haec ad calculum Algebraicum potius pertinent, satisque explicata habentur, itaque regulis magnitudinum ac rationum atque proportionum non immorabor; sed ea Maxime explicare nitar, quae situm involvunt.

(40) Redeo nunc ad ea quae §. 22 interrupta sunt, et primum de punctis, inde de Tractibus agam. Omne punctum congruum adeoque aequale (si ita loqui licet) et simile est:

$$A \between B, A \sqcap B, A \sim B.$$

(41) $A.B \between C.D$ significat simul esse $A \between C$ et $B \between D$. manente situ $A.B$ et $C.D$. fig. 14

[$A.B \between A.Y$. est propositio qvae significat positis duobus punctis $A.B$. posse reperiri tertium aliqvod Y (qvod ideo qvia indefinitum, hac litera significavi) tale ut servato situ $A.Y$. et $A.B$. ipsa $A.Y$ et $A.B$ sibi applicari possint, nempe simul A ipsi A et B ipsi Y. A ipsi A (id est A manente ubi erat) et B ipsi Y.]

letting $a \sqcap c$ *and* $b \sqcap d$*, then* $+a + b \sqcap c + d$ *results*; for if we write $a + b$, and then substitute equals $c.d$ in place of $a.b$, this substitution will preserve magnitude. And accordingly, those yielding $+c + d$ will have the same magnitude as the prior ones [yielding] $+a + b$. But this belongs rather to the algebraic calculus, and has been explained sufficiently, so we will not delay over the rules of magnitude, ratio, and proportion. I will undertake to explain chiefly the things that involve situation.

(40) I return now to what was interrupted in §22, and I will consider first to points, and from there, tracts. Every point is congruent, and consequently equal (if one may speak thus) and similar:

$$A \between B. \quad A \sqcap B. \quad A \sim B.$$

(41) $A.B \between C.D$ signifies simultaneously that $A \between C$ and $B \between D$, with the situation $A.B$ and $C.D$ remaining [the same].

[[17] $A.B \between A.Y$ is the proposition signifying that, positing two points $A.B$, some third Y can be found (which I denote by this letter since it is indefinite) such that, preserving the situation $A.Y$ and $A.B$, $A.Y$ and $A.B$ could be applied to each other, namely A to A and B to Y simultaneously. Then A remains on A (that is A remains where it is), and B on Y.]

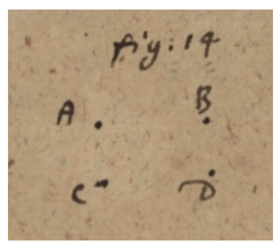


Figure 16

[17] Leibniz cancelled this portion during revision. Possibly he felt that both it and the cancelled portion in (42) were superseded by the material in (43).

(42) $A.B \mathbin{\gamma} B.A$ est Proposito cujus est sensus, positis duobus punctis $A.B.$ ac situm eundem inter se retinentibus posse loca eorum permutari, seu poni A in locum B et contra. fig. 15

(42) The meaning of the proposition $A.B \mathbin{\gamma} B.A$ is that, supposing two points $A.B$, one could permute their locations, i.e. put A in the place of B and vice versa, while retaining the same situation between them.

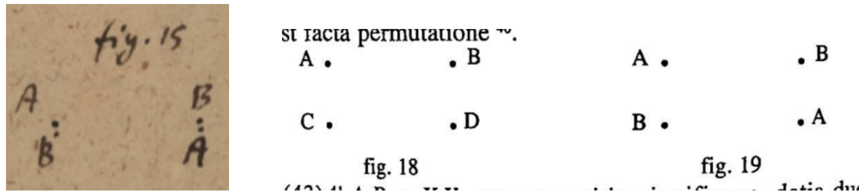


Figure 17

Quod ex eo manifestum est, quia relatio situs quam habent ad ambo eodem modo pertinet, nec nisi externis assumtis discrimen ullum notari potest facta permutatione.

This is clear because the relation of situation that they have belongs to both in the same way, and without assuming external things, no difference can be noted after the permutation.

[$A.B \mathbin{\gamma} C.Y$. est propositio significans datis tribus punctis $A.B.C.$ inveniri posse qvartum Y. cuius situs ad unum ex ipsis C. idem sit qvi situs reliqvorum duorum $A.B.$, inter se sive ut A ipsi C et B ipsi Y simul servato situ $A.B.$ et $C.Y$. congruere possint. Hinc seqvitur et $A.B \mathbin{\gamma} A.Y$. posito $C \infty A$. Ratio autem horum oritur ex natura spatii in qvo nihil sumi potest, qvod non iterum sumi possit eodem plane modo, ita ut solum discrimen sit in loco. Idem per motum sic demonstratur, transferantur simul $A.B.$ servato situ, et A qvidem incidat in locum ipsius C, utiqve B in cuiusdam puncti Y locum incidet.]

[[18]The proposition $A.B \mathbin{\gamma} C.Y$ signifies that, given three points $A.B.C$, one can find a fourth Y whose situation to one of them C is the same as the situation of the remaining two $A.B$ between themselves, i.e. such that A could be made congruent [*congruere possint*] to C and B to Y while simultaneously preserving the situation $A.B$ and $C.Y$. From this $A.B \mathbin{\gamma} A.Y$ also follows, supposing $C \infty A$. Now the reason for this arises from the nature of space, in which nothing can be taken which could not be taken again in exactly the same way, such that the only difference is in place. The same thing is shown by motion, thus: translate $A.B$ simultaneously, preserving situation, and when A falls in the place of C, then of course B will fall in the place of some point Y.]

(43) $A.B \mathbin{\gamma} X.Y$ est propositio significans datis duobus punctis A et B. posse reperiri alia duo X et Y. quae eundem inter se situm habeant quem illa duo, sive ut haec simul illis duobus servato situ utrobique, congruere possint. Quod ex

(43) The proposition $A.B \mathbin{\gamma} X.Y$ signifies that, given two points A and B, another two X and Y can be found having the same situation between themselves as those two, or such that they could simultaneously be made congruent to those two while preserving the situation in

[18]Leibniz cancelled this portion during revision.

eo demonstratur, quia $L.M$, $A.B$ moveri possunt servato situ inter se, eaque respondere poterunt primum ipsis $A.B$ deinde ipsis $X.Y.$, nempe $3\,L.\ 3\,M \between 6\,L.\ 6\,M$. Sit $A \infty 3\,L.$, $B \infty 3\,M$, $X \infty 6\,L$, $Y \infty 6\,M$ fiet $A.B \between X.Y$ Nihil autem prohibet esse $X \infty A$ unde fiet $A.B \between A.Y$. Potest etiam esse $X \infty C$ datae, unde $A.B \between C.Y$.

both pairs. It is demonstrated from the fact that $L.M$[19] could move preserving the situation between themselves, and they will be able to correspond at first to $A.B$ and later to $X.Y$, namely $3L.3M \between 6L.6M$. Let $A \infty 3L$, $B \infty 3M$, $X \infty 6L$, $Y \infty 6M$, then $A.B \between X.Y$. Furthermore, nothing prevents $X \infty A$, whence $A.B \between A.Y$. And $X \infty C$ can also be given, whence $A.B \between C.Y$.

(44) *Si* $A.B \between D.E$ *et* $B.C \between E.F$ *et* $A.C \between D.F$, *erit* $A.B.C \between D.E.F$. Vid. fig. 16. fig. 16.

(44) *If* $A.B \between D.E$ *and* $B.C \between E.F$ *and* $A.C \between D.F$, *then* $A.B.C \between D.E.F$. See Figure 18.

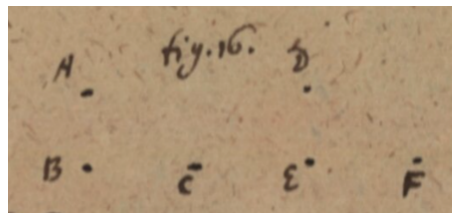


Figure 18

Nam nihil aliud significat $A.B \between D.E$ quam simul esse $A \between D$ et $B \between E$. situ $A.B$ et $D.E$ servato. Eodem modo ex $B.C \between E.F$ sequitur $B \between E$ et $C \between F$ situ $B.C$ et $E.F$ servato; et ex $A.C \between D.F$ sequitur $A \between D$ et $C \between F$ situ $A.C$ et $D.F$ servato. Habemus ergo simul $A.B.C \between D.E.F$ servato situ $A.B$ et $B.C$ et $A.C.$, itemque servato situ $D.E$ et $E.F$ et $D.F$ cum alias ex solis $A.B \between D.E$ et $B.C \between E.F.$ sequatur quidem simul $A \between D$ et $B \between E$ et $C \between F$. Sed servatis tantum sitibus $A.B$ et $B.C$ item $D.E$ et $E.F$ non vero exprimetur servari et situs $A.C$ et $D.F.$ nisi addatur $A.C \between D.F$.

For $A.B \between D.E$ just signifies $A \between D$ and $B \between E$ simultaneously, preserving the situation $A.B$ and $D.E$. In the same way, from $B.C \between E.F$ follows $B \between E$ and $C \between F$, preserving the situation $B.C$ and $E.F$; and from $A.C \between D.F$ follows $A \between D$ and $C \between F$, preserving the situation $A.C$ and $D.F$. Therefore, we have $A.B.C \between D.E.F$ simultaneously, preserving the situation of $A.B$ and $B.C$ and $A.C$, and likewise the situation of $D.E$ and $E.F$ and $D.F$; whereas from $A.B \between D.E$ and $B.C \between E.F$ alone, it would follow that $A \between D$ and $B \between E$ and $C \between F$ simultaneously. But if only the situations $A.B$ and $B.C$ and likewise $D.E$ and $E.F$ are preserved, it will not in fact be expressed that the situations $A.C$ and $D.F$ are also preserved unless $A.C \between D.F$ is added.

Hinc jam principium habemus ratiocinationem ad plura etiam puncta producendi.

From this we now have a principle for extending the reasoning to even more points.

[19] The manuscript has '$L.M$ $A.B$' with the $L.M$ seemingly added at the end of a line. It seems Leibniz forgot to cancel the '$A.B$' starting the next line after adding $L.M$.

(45) *Si sint* $A.B \between B.C \between A.C$ *erit* $A.B.C \between B.A.C$, vel alio ordine quocunque. fig. 17

(45) *If* $A.B \between B.C \between A.C$, *then* $A.B.C \between B.A.C$ (or any other order).

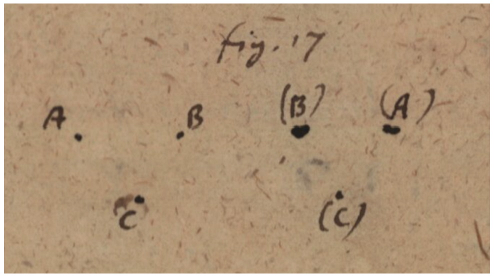


Figure 19

Nam si congruentibus $A.B$ et $(B).(A)$ ascribas congruentia C. et (C). congruenti modo quia $A.C \between (B).(C)$ et $B.C \between (A).(C)$ fient congruentia $A.B.C \between (B).(A).(C)$ sive $A.B.C \between B.A.C$ per praecedentem. Parentheses enim tantum confusionis ex repetitione vitandae causa ascripsi. Hinc patet quid sit congruenti modo ascribi, cum scilicet omnes combinationes ab una parte enuntiationis sunt congruentes omnibus ab altera parte. fig. 18.

For if we join some congruents C and (C) to the congruents $A.B$ and $(B).(A)$ in a congruent way, then since $A.C \between (B).(C)$ and $B.C \between (A).(C)$, by the preceding we will get congruents $A.B.C \between (B).(A).(C)$, i.e. $A.B.C \between B.A.C$. Parentheses are added just to avoid confusion from the repetition. From this it is clear what it is to be joined in a congruent way, namely when all combinations on one side of the statement are congruent to all on the other side.

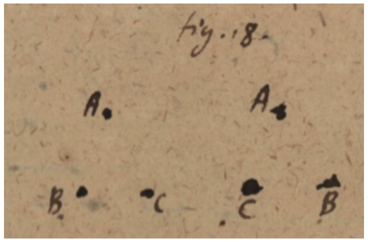


Figure 20

Unde patet si $A.B \between B.C \between A.C$ *fore*

$$A.B.C \between A.C.B \between B.C.A \between B.A.C \between C.A.B \between C.B.A.$$

It is clear from this that *if* $A.B \between B.C \between A.C$, *then:*

$$A.B.C \between A.C.B \between B.C.A \between B.A.C \between C.A.B \between C.B.A.$$

(46) *Si* $A.B.C \gamma A.C.B$ *sequitur* (*tantum*) $A.B \gamma A.C$ [sive triangulum esse isosceles], nam sequitur:

$$\overset{\frown}{A\frown.B\frown.C} \qquad A.B \gamma A.C \quad B.C \gamma C.B \quad A.C \gamma A.B \qquad \underset{\smile}{A\smile.C\smile.B}$$

ex quibus $B.C \gamma C.B$ per se patet, reliqua duo, $A.B \gamma A.C.$ et $A.C. \gamma A.B.$ eodem recidunt, hoc unum ergo inde duximus $A.B. \gamma A.C.$

(47) *Si* $A.B.C \gamma B.C.A$ *sequitur* $A.B \gamma B.C \gamma A.C$ [seu triangulum esse aequilaterum]. Nam fit

$$A.B \gamma B.C \qquad B.C \gamma C.A.$$

(48) *Si* $A.B.C. \gamma A.C.B$ *et* $B.C.A \gamma B.A.C$ *fiet* $A.B \gamma B.C \gamma A.C.$ Nam ob $A.B.C \gamma A.C.B.$ fit $A.B \gamma A.C.$ eodemque modo ob $B.C.A \gamma B.A.C.$ fit $B.C. \gamma B.A.$ sive $A.B. \gamma B.C.$ [itaque quandocunque in transposito punctorum ordine, unum ex tribus eundem in utroque ordine locum servat situsque posterior priori congruus est, inde tantum probari potest Triangulum esse isosceles, sed si nullum ex punctis servat locum, et nihilominus situs posterior priori congruit, Triangulum est aequilaterum[.

(49) *Si sit* $A.B \gamma B.C \gamma C.D \gamma D.A$ *et* $A.C \gamma B.D$ *erit*

$$A.B.C.D \gamma B.C.D.A \gamma C.D.A.B \gamma D.A.B.C$$
$$\gamma D.C.B.A \gamma A.D.C.B \gamma B.A.D.C \gamma C.B.A.D.$$

fig. 19

Hoc ex praecedentibus facile demonstratur, multaque alia hujusmodi, quae sufficiet demonstrari cum ipsis indigebimus. Nunc satis habebimus principium dedisse inveniendi haec solo calculo, sine inspectione figurae.

(50) Si tria puncta $A.B.C$ dicantur esse *sita in directum* tunc posito $A.B.C \gamma A.B.Y$ erit $C \infty Y$. Haec propositio est defini-

(46) *If* $A.B.C \gamma A.C.B$ *then* $A.B \gamma A.C$ *(only) follows* (i.e. the triangle is isosceles), for it follows that:

$$\overset{\frown}{A\frown.B\frown.C} \qquad A.B \gamma A.C \quad B.C \gamma C.B \quad A.C \gamma A.B \qquad \underset{\smile}{A\smile.C\smile.B}$$

of which $B.C \gamma C.B$ is clear *per se*, and the remaining two, $A.B \gamma A.C$ and $A.C \gamma A.B$, reduce to the same thing; from that we therefore deduce this one: $A.B \gamma A.C.$

(47) *If* $A.B.C \gamma B.C.A$, *then* $A.B \gamma B.C \gamma A.C$ *follows* (i.e. the triangle is equilateral). For then:

$$A.B \gamma B.C \qquad B.C \gamma C.A.$$

(48) *If* $A.B.C \gamma A.C.B$ *and* $B.C.A \gamma B.A.C$ *then* $A.B \gamma B.C \gamma A.C$ *will result.* For from $A.B.C \gamma A.C.B$ we get $A.B \gamma A.C$, and in the same way, from $B.C.A \gamma B.A.C.$ we get $B.C \gamma B.A$, i.e. $A.B \gamma B.C.$ (And so, whenever in a transposed order of the points, one of the three keeps the same place in both orders, and the latter situation is congruent to the former, from this one can prove only that the triangle is isosceles; but if none of the points keeps its place, and still the latter situation is congruent to the former, the triangle is equilateral.)

(49) *If we have* $A.B \gamma B.C \gamma C.D \gamma D.A$ *and* $A.C \gamma B.D$, *then*

$$A.B.C.D \gamma B.C.D.A \gamma C.D.A.B \gamma D.A.B.C$$
$$\gamma D.C.B.A \gamma A.D.C.B \gamma B.A.D.C \gamma C.B.A.D.$$

This is easy to show from the preceding, as are many others of the kind, which it will suffice to demonstrate when we need them. Now we have adequately given the principle of discovering these things by calculus alone, without inspection of a figure.

(50) If three points $A.B.C$ are said to be *situated in alignment*, then setting $A.B.C \gamma A.B.Y$ will yield $C \infty Y$. This proposition is the

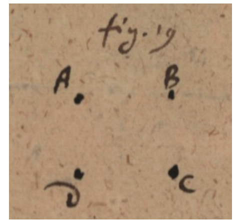


Figure 21

tio punctorum quae in directum sita dicuntur. fig. 20

definition of points which are said to be situated in alignment.

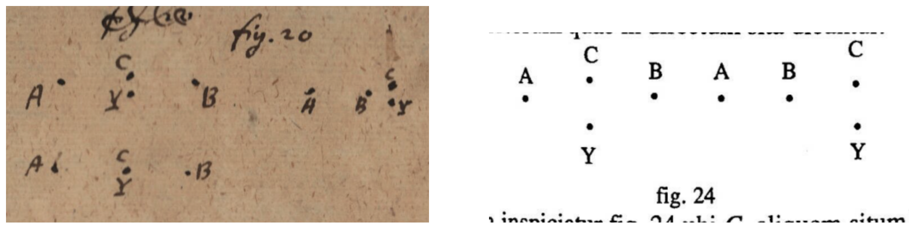


Figure 22

Nimirum inspiciatur fig. 20. ubi C. aliquem situm habet ad A. et B. Sumatur jam aliquod punctum Y eundem ad $A.B$. situm habens, id si diversum ab ipso C. assumi potest puncta, $A.B.C$. non sunt in directum sita, si vero necessario cum ipso C. coincidit, in directum sita dicentur.

Consider Figure 22, where C has some situation to A and B. Take now some point Y having the same situation to $A.B$; if it can be taken distinct from C, then points $A.B.C$ are not situated in alignment, but if it necessarily coincides with C, they are said to be situated in alignment.

(51) Datis punctis duobus semper assumi potest tertium quod cum illis sit in directum, sive si $A.B.Y$ ♉ $A.B.X$ erit $Y \infty X$. Nam datis punctis duobus $A.B$. semper assumi potest tertium Y. quod servato ad ipsa situ moveri potest ipsis immotis. Sed via quam motu suo describit potest esse semper minor ac minor, prout aliter atque aliter assumitur punctum Y. adeoque tandem sumi poterit tale, ut spatium motus evanescat, et tunc tria puncta erunt in directum.

(51) With two points being given, we can always assume a third one situated in alignment with them, i.e. if $A.B.Y$ ♉ $A.B.X$, then $Y \infty X$. For, with two points $A.B$ being given, we can always assume a third one Y that can move, preserving its situation to them, while they remain unmoved. But the path which it describes by its motion can be ever smaller and smaller, as the point Y is assumed now one way and now another, until at last we can take one such that the space of the motion vanishes, and then the three points will be in alignment.

Melius forte sic enuntiabimus: $A.B.\ 3\ Y \gamma A.B.\ 6\ Y$ erit $3\ Y \infty 6\ Y$ id est aliquod assignari posse Y quod servato situ ad $A.B.$ moveri seu locum mutare nequeat.

Aliter ista videor demonstrare posse hoc modo: Sit aliquod extensum, quod moveatur servato punctorum ejus situ inter se et duobus in eo sumtis immotis. Nam si quis id neget moveri posse, eo ipso fatetur puncta ejus servato ad puncta duo sumta situ moveri nequire, et adeo cum eo sita esse in directum per definitionem. Sed nulla ratio est cur puncta illa $A.B.$ immota durante eodem motu sumi possint haec sola, et non alia etiam sive nulla ratio est cur puncta extensi quod his duobus immotis movetur, servent situm ad haec duo solum immota, et non etiam ad alia, nam situs quem ipsa $A.B.$ inter se obtinent, nihil ad rem facit, itaque potuisset sumi aliquod Y. loco ipsius B. alium obtinens situm ad A quam ipsum B. obtinet. Verum quaecunque sumi possunt ut immota, ea manente eodem extensi motu sunt immota. Et quia sumtis duobus $A.B.$ immotis motus extensi est determinatus, seu determinatum est quaenam puncta ejus moveantur aut non moveantur, hinc duobus punctis sumtis immotis, determinata sunt alia plura quae servato ad ipsa situ moveri non possunt, seu quae cum ipsis jacent in directum.

(52) *Si sint* $E.A.B. \gamma F.A.B$ *et* $E.B.C. \gamma F.B.C.$ *erit* $E.A.C \gamma F.A.C.$ Vid. fig 21. fig. 21

Nam per $E.A.B \gamma F.B.A$ erit $E.A \gamma F.A$ et per $E.B.C \gamma F.B.C$ erit $E.C \gamma F.C.$ Jam si sit $E.A. \gamma F.A$ et $E.C \gamma F.C$ erit

$$\overparen{E^\frown.A^\frown.C} \gamma \overparen{F^\frown.A^\frown.C}$$

per prop. 44 (est enim $E.A \gamma F.A.$ et $E.C \gamma F.C.$ et $A.C \gamma A.C.$) Ergo si sit $E.A.B \gamma F.A.B$ et $E.B.C. \gamma F.B.C.$ erit $E.A.C \gamma F.A.C.$ Quod erat dem.

This is articulated significantly better as follows: $A.B.3Y \gamma A.B.6Y$ yields $3Y \infty 6Y$, that is, some Y can be assigned which cannot move or change its place while preserving its situation to $A.B$.

I seem to be able to show this differently, like this: suppose there is some extensum that moves while preserving the situation of its points among themselves and to two unmoved points taken in it. For if anyone denies that it can move, he thereby grants that its points cannot move while preserving their situation to the two assumed points, and so they are, by definition, situated in alignment with it. But there is no reason why only these points $A.B$ can be taken as unmoved during the same motion, and not others as well, i.e. there is no reason why the points of the extensum, which moves with these two unmoved, should preserve only the situation to these two unmoved points, and not also to others; for the situation that $A.B$ maintain between themselves makes no difference, and so one could have taken some Y in place of B that maintained another situation to A than B does. But whatever things can be taken as unmoved, those things are unmoved when the motion of the extensum remains the same. And because, with the two points being taken as unmoved, the motion of the extensum is determined, i.e. it is determined which of its points move and which do not move, hence with the two assumed points being unmoved, many others are determined which cannot move while preserving their situation to the same ones, i.e. which lie in alignment with them.

(52) If $E.A.B \gamma F.A.B$ and $E.B.C \gamma F.B.C$, then $E.A.C \gamma F.A.C$. See Figure 23.

For by $E.A.B \gamma F.A.B$ we have $E.A \gamma F.A$, and by $E.B.C \gamma F.B.C$ we have $E.C \gamma F.C$. Now if $E.A \gamma F.A$ and $E.C \gamma F.C$, then

$$\overparen{E^\frown.A^\frown.C} \gamma \overparen{F^\frown.A^\frown.C}$$

by Prop. 44 (indeed, $E.A \gamma F.A$ and $E.C \gamma F.C$ and $A.C \gamma A.C$). Therefore if $E.A.B \gamma F.A.B$ and $E.B.C \gamma F.B.C$, then $E.A.C \gamma F.A.C$. *Quod erat demonstrandum.*

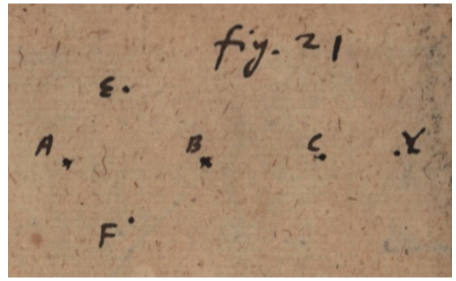


Figure 23

[*Erster Ansatz, gestrichen*]

(53) Inveniri possunt puncta quotcunque sita in directum cum duobus punctis datis. Sint enim duo puncta *A.B.* in eadem fig. 21. Sumantur alia duo *E* et *F* ita ut sit *E.A.B* ɣ *F.A.B*.

(54) Si puncta duo *E.F* eodem modo se habeant ad puncta quotcunque ut *A.B.C.* dicentur haec *puncta A.B.C. esse sita in directum*.

(55) Nimirum in fig. 21. si sit *E.A.B.C* ɣ *F.A.B.C* id est *E.A.B* ɣ *F.A.B.E.A.C.* ɣ *F.A.C.E.B.C.* ɣ *F.B.C.* id est porro *E.A* ɣ *F.A.E.B.* ɣ *F.B.E.C.* ɣ *F.C.* dicentur puncta *A.B.C.* esse in directum

[*Zweiter Ansatz*]

(53): Hinc etiam *erit E.A.B.C* ɣ *F.A.B.C posito E.A.B* ɣ *F.A.B* et *E.B.C* ɣ *F.B.C*. Nam etiam *E.A.C* ɣ *F.A.C* per praeced. habemus ergo: *E.A.B* ɣ *F.A.B* et *E.A.C.* ɣ *F.A.C.* et *E.B.C* ɣ *F.B.C* et *A.B.C* ɣ *A.B.C* id est habemus omnia quae ex hoc: *E.A.B.C* ɣ *F.A.B.C.* duci possunt; ergo habemus etiam *E.A.B.C.* ɣ *F.A.B.C.* [est egregius modus regredi-

(53) Hence also *setting E.A.B* ɣ *F.A.B and E.B.C* ɣ *F.B.C yields E.A.B.C* ɣ *F.A.B.C*. For indeed *E.A.C* ɣ *F.A.C* by the preceding; therefore we have: *E.A.B* ɣ *F.A.B* and *E.A.C* ɣ *F.A.C* and *E.B.C* ɣ *F.B.C* and *A.B.C* ɣ *A.B.C*, that is, we have everything which may be derived from this: *E.A.B.C* ɣ *F.A.B.C*; therefore we now also have *E.A.B.C* ɣ *F.A.B.C*. (Evidently this is an excellent way of regressing[20]

[20] Cf. the footnote to (6).

endi, nimirum ex consequentibus omnibus totam naturam antecedentis exhaurientibus ad antecedens.]

(54) *Si sit* $E.A$ ☋ $F.A$ $E.B$ ☋ $F.B$ $E.C$ ☋ $F.C$ *erit*

$$E.\widehat{A^\frown.B^\frown.C} \; ☋ \; F.\widehat{A^\frown.B^\frown.C},$$

nam quae supersunt combinationes, utrinque comparandae $A.B$ et $B.C$ et $A.C$ utrobique coincidunt.

(55) $A.B.X$ ☋ $A.B.Y$ seu datis duobus punctis, $A.B.$ inveniri possunt duo alia $X.Y.$ ita ut X et Y eodem modo se habeant ad $A.B$. Vid. fig. 22. fig. 22

to the antecedent from all the consequences, exhausting the complete nature of the antecedent.)

(54) *If* $E.A$ ☋ $F.A$, $E.B$ ☋ $F.B$, $E.C$ ☋ $F.C$, *then*

$$E.\widehat{A^\frown.B^\frown.C} \; ☋ \; F.\widehat{A^\frown.B^\frown.C},$$

since the combinations remaining on the two sides in need of comparison, $A.B$ and $B.C$ and $A.C$, coincide on the two sides.

(55) $A.B.X$ ☋ $A.B.Y$, i.e. two points $A.B$ being given, two others $X.Y$ may be found such that X and Y have the same relation to $A.B$. See Figure 24.

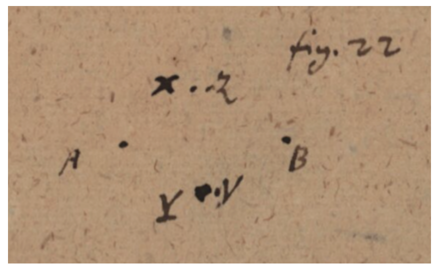


Figure 24

Potest enim reperiri $A.X$ ☋ $A.Y$ et $B.Z$ ☋ $B.V$ per prop. 43. Ponatur $Z \infty X$ (hoc enim fieri potest per prop. 43. seu Z potest esse data seu jam assumta X. quia $A.B$ ☋ $A.V$) itemque ponatur $A.X$ ☋ $B.X$. (nam et in $A.X$ ☋ $A.Y$ potest X. esse data quia datur $A.C.$ ☋ $A.Y$ per prop. 43) erit $V.B$ (☋ $B.Z$ ☋ $B.X$ ☋ $A.X$) ☋ $A.Y$. Ergo $V.B.X$ ☋ $Y.A.X$ ☋ $X.B.V$ in quo omnia hactenus determinata continentur. Ergo potest poni $V \infty Y$, nihil enim in hactenus determinatis obstat. Fiet $Y.B.X$ ☋ $Y.A.X$. Ergo $Y.B.$ ☋ $Y.A.B.X.$ ☋ $A.X$. Rursus $Y.B.X$ ☋ $X.B.Y$. Ergo $Y.B.$ ☋ $X.B$. Ergo fit: $Y.B.$ ☋ $X.B.$ ☋ $Y.A.$ ☋ $A.X$. Ergo $A.B.X$ ☋ $A.B.Y$.

Indeed $A.X$ ☋ $A.Y$ and $B.Z$ ☋ $B.V$ can be obtained by Prop. 43. Setting $Z \infty X$ (this can be done by Prop. 43, i.e. Z can be given, i.e. with X being now assumed, since $A.B$ ☋ $A.V$) and likewise setting $A.X$ ☋ $B.X$ (for also in $A.X$ ☋ $B.Y$, X can be given since $A.C$ ☋ $A.Y$ is given by Prop. 43), yields

$$V.B\ (☋\ B.Z\ ☋\ B.X\ ☋\ A.X)\ ☋\ A.Y.$$

Therefore $V.B.X$ ☋ $Y.A.X$ ☋ $X.B.V$, in which everything determined up to now is contained. Therefore we can set $V \infty Y$; for nothing prevents it in what has been determined up to now. Then $Y.B.X$ ☋ $Y.A.X$. Therefore $Y.B$ ☋ $Y.A$, $B.X$ ☋ $A.X$. Again, $Y.B.X$ ☋ $X.B.Y$. Therefore $Y.B$ ☋ $X.B$. Therefore $Y.B$ ☋ $X.B$ ☋ $Y.A$ ☋ $A.X$ results. Therefore $A.B.X$ ☋ $A.B.Y$.

.... fig. 23.

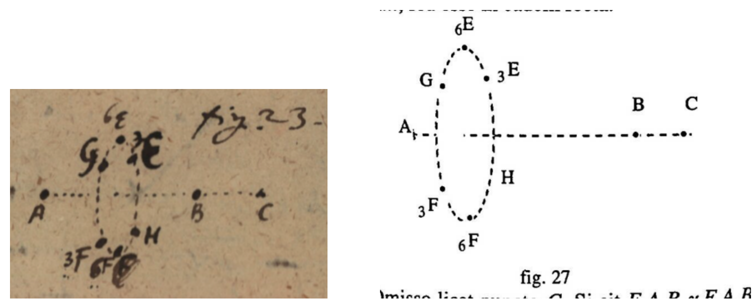


Figure 25

(56) Si tria puncta $E.F.G$ sumta distributive eodem modo se habeant ad tria puncta $A.B.C$ sumta collective, erunt tria priora in eodem arcu circuli, tria posteriora in eadem recta seu jacebunt in directum. Hanc propositionem annotare placuit, ratio patebit ex sequentibus.

(57) Si sit $E.A.B.C$ ɣ $F.A.B.C$ ɣ $G.A.B.C$ et sit E non ∞ F, E non ∞ G, F non ∞ G. dicentur *puncta quotcunque* $A.B.C$ *sita esse in directum,* seu esse in eadem recta.

(58) Omisso licet puncto C. Si sit $E.A.B$ ɣ $F.A.B$ ɣ $G.A.B$ erunt puncta $E.F.G$ in *eodem plano.*

(59) Iisdem positis erunt puncta $E.F.G.$ in eodem *arcu circuli.*

(60) Inter duo quaevis congrua assumi possunt infinita alia congrua, nam unum in locum alterius servata forma sua transire non posset, nisi per congrua.

(61) Hinc a quolibet puncto ad quodlibet punctum duci potest linea. Nam punctum puncto congruum est.

(56) If three points $E.F.G$, taken distributively, have the same relation to three points $A.B.C$, taken collectively, the former three will be in the same arc of a circle, the latter three in the same line, i.e. will lie in alignment. It seems good to make note of this proposition; the reason will become clear from what follows.

(57) If $E.A.B.C$ ɣ $F.A.B.C$ ɣ $G.A.B.C$, and if E *not* ∞ F, E *not* ∞ G, F *not* ∞ G, then we say *(however many) points* $A.B.C$ *are situated in alignment*, i.e. are on the same line.

(58) The point C may be omitted. If $E.A.B$ ɣ $F.A.B$ ɣ $G.A.B$, then the points $E.F.G$ are in *the same plane*.

(59) From the same things being supposed, the points $E.F.G$ will be in the same *arc of a circle*.

(60) Between any two congruent things, infinitely many other congruents can be taken, for it is only through congruents that the one can pass into the place of the other while preserving its form.

(61) Hence a curve can be drawn from any point to any other point. For a point is congruent to a point.

(62) Hinc et a quolibet puncto per quodlibet punctum duci potest linea.

(62) Hence a curve can be drawn from any point through any point.

(63) Linea duci potest quae transeat per puncta quotcunque data.

(63) A curve can be drawn which passes through any given point.

(64) Eodem modo ostendetur per lineas quotcunque datas transire posse superficiem. Nam si congruae sunt, patet lineam generantem successive in omnibus esse posse. Si congruae non sunt, patet lineam generantem durante motu, ita augeri minui et transformari posse, ut dum illuc venit, congrua fiat.

(64) In the same way it is shown that a surface could pass through any given curves. For if they are congruent, clearly the generating curve could be in all of them in succession. If they are not congruent, clearly the generating curve could be augmented, diminished, and transformed during its motion, such that when it arrives there it will be congruent.

(65) Unumquod[que] in spatio positum potest, servata forma sua; seu cuilibet in spatio existenti infinita alia congrua assignari possunt.

(65) Each thing can be put in space, preserving its form; i.e. to anything existing in space, an infinity of other congruent things can be assigned.

(66) Unumquodque servata forma sua moveri potest infinitis modis.

(66) Each thing can move, preserving its form, in an infinity of ways.

(67) Unumquodque ita moveri potest servata forma sua, ut incidat in punctum datum.

(67) Each thing can so move, preserving its form, that it falls on a given point.

Generalius: unumquodque servata forma sua ita moveri potest, ut incidat in aliud cui congruum in ipso designari potest. Nam congruum unum transferri potest in locum alterius, nec quicquam prohibet id in quo congruum illud est quod transferri debet, simul cum ipso transferri, quia ratio separationis nulla est: et quod uni congruorum aptari potest, poterit et alteri congruorum similiter aptari.

More generally: each thing can so move, preserving its form, that it falls on another thing to which something congruent can be specified in it. For one congruent thing can be translated into the place of another; and nothing prohibits that in which the congruent thing is, which needs to be translated, from being translated simultaneously with it, since there is no reason for a separation: what can be fitted to one of the congruents can also be fitted in a similar way to the other of the congruents.

(68) $A \gamma B$ id est assumto puncto quodlibet alid congruum est.

(68) $A \gamma B$,[21] that is, to an assumed point, any other point is congruent.

(69) $A.B \gamma B.A$ ut supra.

(69) $A.B \gamma B.A$, as above.

(70) $A.B \gamma X.Y$. Eodem modo $A.B.C \gamma X.Y.Z$ et $A.B.C.D \gamma X.Y.Z.\Omega$. et ita porro. Hoc enim nihil aliud est, quam quotcunque puncta posse moveri servato situ inter se. Situm autem eorum inter se servari intelligi potest, si ponatur esse extrema Lineae cujusdam rigidae qualiscunque.

(70) $A.B \gamma X.Y$. In the same way, $A.B.C \gamma X.Y.Z$, and $A.B.C.D \gamma X.Y.Z.\Omega$, and so on. Indeed, this is just the fact that however many points can move while preserving the situation among themselves. Now, we can understand that their situation among themselves is preserved if they are set as the extremes of a rigid curve of any kind.

(71) $A.B \gamma C.X.A.B.C \gamma D.X.Y$. etc.

(71) $A.B \gamma C.X$. $A.B.C \gamma D.X.Y$. Etc.

Hoc enim nihil aliud est quam quotcunque puncta, ut $A.B.C$. posse moveri servato situ inter se; ita ut unum ex ipsis A incidat in punctum aliquod datum D. reliquis duobus $B.C$ in alia quaecunque $X.Y$ incidentibus.

Indeed, this is just the fact that however many points, such as $A.B.C$, can move while preserving the situation among themselves, such that one of them, A, falls on some given point D, with the two remaining ones falling on any particular others, $X.Y$.

(72) Si $A.B.C$ non γ $A.B.Y$ nisi $C \infty Y$. tunc puncta $A.B.C$ dicentur *sita in directum* (vid. fig. 20) seu C. *erit situm in directum cum ipsis* $A.B$. si unicum sit quod eum situm ad $A.B$. habeat. fig. 20

(72) If $A.B.C$ is not γ $A.B.Y$ unless $C \infty Y$, then the points $A.B.C$ are said to be *situated in alignment* (see Figure 26), i.e. C *is situated in alignment with* $A.B$ if what has that situation to $A.B$ is unique.

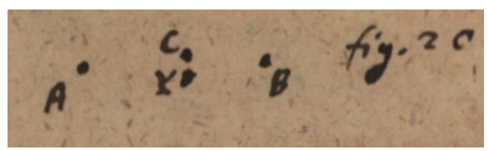


Figure 26

An autem talis punctorum situs reperiatur postea inquirendum erit.

Now whether such a situation of points can be obtained is something to be investigated later.

[*Erster Ansatz, gestrichen*]

[21]The manuscript shows Y and B written in the same place; probably the Y was amended into the B.

(73) $A.B.X \between A.B.Y$. Nam sit $A.X. \between A.Y.$ et $B.X \between B.V.$ per 71. (Nam $A.B. \between C.Y.$ dicta prop. 71. Pro puncto B dato, ponatur punctum X. aliquod. Et ponatur C datum ∞A. fiet $A.X \between A.Y$. Eodem modo fiet $B.X \between B.V$.) Ostendendum fieri posse $V \infty Y$. Ponatur $A.X. \between B.X$. (Quia X. arbitrarium est) fiet $B.V \between A.Y$. (Nam $A.Y. \between A.X. \between B.X. \between B.V$.) Itaque $V.B.X \between Y.A.X \between X.B.V$ per 44. In quibus continentur omnia quae assumsimus, quibus non repugnat sumi $V \infty Y$ quo posito fiet $Y.B.X \between Y.A.X \between X.B.Y$. Unde fiet: $A.Y \between B.Y \between A.X \between B.X$. Ac proinde per 44. fiet: $A.B.X \between A.B.Y$.

[*Zweiter Ansatz*]

Linea autem cujus omnia puncta sita sunt in directum, dicetur *recta*. Sit enim $A.B.zY \between A.B.zX$ atque ideo $zY \infty zX$, erit $\bar{z}Y$ ($\infty \bar{z}X$) *Linea recta*; id est si punctum Y ita moveatur, ut situm semper ad puncta $A.B$. servet qui ipsi uni competere possit sive determinatum, minimeque varium ac vagum, describi ab eo lineam rectam.

A curve, all of whose points are situated in alignment, is called *straight*. Indeed, let $A.B.zY \between A.B.zX$ and thus $zY \infty zX$, then $\bar{z}Y$ ($\infty \bar{z}X$) will be *a straight line*; that is, if the point Y so moves that it always preserves the situation to the points $A.B$ which can belong only to itself, i.e. which is determined, and varies or wanders minimally, then it describes a straight line.

(73) Si $A.B.C \between A.B.D$ erit $\between A.B.zY$, vid. fig. nam erit $C \infty 3\,Y$. et $D \infty 6\,Y$ nempe C et D erunt loca ipsius Y moti ita ut servet situm eundem ad $A.B$. inter quae alia necessario erunt indefinita nempe designanda per zY. Linea autem $\bar{z}Y$ vocetur *Circularis*. fig. 24

(73) If $A.B.C \between A.B.D$, then $\between A.B.zY$, see Figure 27, for then $C \infty 3Y$ and $D \infty 6Y$, namely C and D are places of the motion of Y such that it preserves the same situation to $A.B$, between which there will necessarily be other indefinite ones, indicated of course through zY. Now the curve $\bar{z}Y$ is called *circular*.

Notandum autem hanc Lineae circularis descriptionem ea priorem esse quam dedit Euclides. Euclidea enim indiget recta et plano. A nostra procedit qualiscunque assumatur rigida linea, modo in ea duo sumantur puncta quibus immotis ipsa linea vel saltem aliquod ejus punctum moveatur; hoc enim punctum ad puncta duo assumtatu eundem servabit situm, cum omnia sint in linea rigida. Id ergo punctum describit lineam Circularem per hanc definitionem nostram. Si quis vero neget in

It should be noted, moreover, that this description of a circular curve is prior to the one Euclid gave. For Euclid requires the straight line and the plane. From ours it works whatever kind of rigid curve is assumed, just taking two points in it which are unmoved while the curve itself, or at least some point in it, does move, for this point keeps the same situation to the two assumed points as they are all in the rigid curve. Therefore this point describes a circular curve by this definition of ours. If someone denies that such a point could be found on the rigid

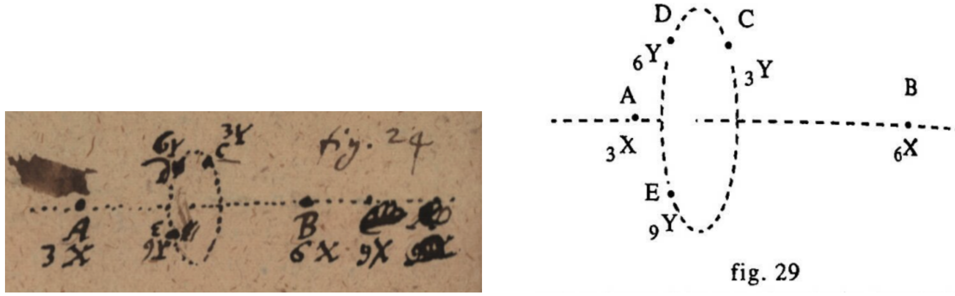


Figure 27

Linea rigida tale punctum inveniri posse, quod datis duobus immotis moveatur; necesse erit per definitionem praecedentem prop. 72. omnia Lineae rigidae puncta in directum esse sita, sive necesse erit dari Lineam rectam. Alterutrum ergo hoc modo admittere necesse est lineam rectam possibilem esse, vel circularem. Alterutra autem admissa alteram postea inde ducemus. Hic obiter notandum, quia ut suo loco patebit per tria quaelibet data puncta transire potest arcus circularis, hinc tribus datis punctis inveniri unum posse, quod ad tria illa eodem se habeat modo, nempe $X.C \gamma X.D \gamma X.E$, idque saepius fieri posse seu diversa reperiri posse X. pro iisdem $C.D.E$ omniaque X in unam rectam cadere quae transeat per circuli centrum, sitque ad planum ejus ad angulos rectos.

(74) Sit Linea quaelibet $\overline{z}Y$, vid. fig. 24, in ea poterunt sumi quotcunque puncta 3 Y. 6 Y. 9 Y. 12 Y. etc. ita ut sit 3 Y. 6 $Y \gamma$ 6 Y. 9 $Y \gamma$ 9 Y. 12 Y etc. Nam generaliter si qua sit linea parva, cujus unum extremum sit in alia linea, poterit prior ita moveri extremo ejus duabus lineis communi immoto, ut altero quoque extremo posteriori lineae occurrat, itaque hoc motu partem unam abscindet, jamque novo puncto invento immoto manente rursus aliam, et ita porro. Sed jam observo, ne id quidem necesse esse, et sufficere Unam lineam eidem

curve which moves while the two given ones remain unmoved, by the preceding definition (Prop. 72) it will be necessary that all the points of the rigid curve are situated in alignment, i.e., it will be necessary that a straight line is given. It must, therefore, be accepted in this way that either a straight line is possible, or a circle is. Once either one has been accepted, we then deduce the other one from it. It should be noted here by the way, as will be clear in its own place, that an arc of a circle can pass through any three given points, hence from three points being given, we can find one which relates in the same way to those three, namely $X.C \gamma X.D \gamma X.E$, and this can be done many times, i.e. various X can be found for the same $C.D.E$, and all X fall on one straight line which passes through the centre of the circle and is at right angles to its plane.

(74) Let there be some curve $\overline{z}Y$, see Figure 28, in which however many points $3Y.6Y.9Y.12Y$ etc. can be taken, such that

$$3Y.6Y \gamma 6Y.9Y \gamma 9Y.12Y \text{ etc.}$$

For in general, if there is any sufficiently small curve with one extreme in another curve, the former can so move, with its extreme point common to both curves being unmoved, such that it also meets the latter curve with its other extreme, and so one part will be cut off by this motion, and then again another one with a new point being found and remaining unmoved, and so on. But now I observe that this is lineae suis extremis applicari saepius quomodocunque ita ut plures ejusdem lineae partes assignentur quarum extrema aliorum extremis sint congrua. fig. 25

not actually necessary, and it suffices that one curve attaches to the same curve by its extremes in many ways, such that several parts of the same curve are assigned whose extremes are congruent to the extremes of the others.

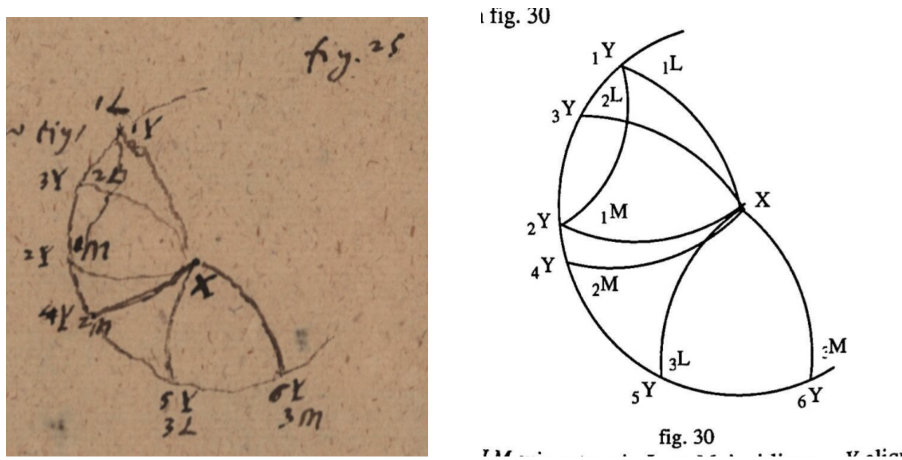


Figure 28

Ut in fig. 25. linea rigida LM suis extremis L. et M. ipsi lineae $\bar{z}Y$ aliquoties in 1 Y. 2 Y. et 3 Y. 4 Y. et 5 Y. 6 Y quae coincident ipsis 1 L. 1 M. et 2 L. 2 M. et 3 L. 3 M. Nam si semel $L.M$ ipsi $\bar{z}Y$ applicari possit infinitis modis applicari potest, si posteriores applicationibus quantumvis parum distent. Jam tam ex L et M educantur duae lineae congruae eodem modo se habentes ea quae ex L educitur ad L, quo illa quae ex M educitur ad M quae sibi occurrant in X. Sitque 1 LX 1 M ȣ 2 LX 2 M, et ita porro, id est quae ex 1 L et 1 M educuntur eousque producantur ut non ante sibi occurrant quam ubi ex 2 L 2 M eodem modo eductae sibi occurrunt. Unde patet punctum X. eodem modo se habere ad omnia Y.

As in Figure 28, the rigid curve LM [is attached[22]] by its extremes L and M to the curve zY a number of times in $1Y.2Y$, or $3Y.4Y$, or $5Y.6Y$, which coincide with $1L.1M$, and $2L.2M$, and $3L.3M$. For if $L.M$ can be attached to $\bar{z}Y$ once, it can be attached in an infinite number of ways, provided the later ones were to be separated by however little from the [earlier] attachments.[23] Now let two congruent curves be drawn from L and M, with the one drawn from L relating in the same way to L as the one drawn from M relates to M, and which meet each other in X. And let $1LX1M$ ȣ $2LX2M$, and so on, that is, those drawn from $1L$ and $1M$ are extended so that they do not meet each other before the place where those (drawn in the same way) from $2L2M$ meet each other. It is clear from this that the point X relates in the

[22]Leibniz cancelled *applicatur* here, but he did not supply another verb.

[23]Latin: *si posteriores applicationibus quantumvis parum distent*; it would seem Leibniz meant *si posteriores prioribus applicationibus* or similar.

assignata, et si quidem linea talis est, ut ejusmodi punctum habeat, quod ad omnia ejus puncta eodem sit modo, id hoc modo inveniri. Si autem circularis sit linea ut hoc loco sufficit punctum aliquod ad tria lineae circularis puncta se eodem modo habens inveniri, id enim eodem modo se habebit ad alia omnia. Cujus rei ratio est, quia ex tribus punctis datis $C.D.E.$ vide fig. 24 posito esse $C.D.$ ɣ $D.E.$ methodo paulo ante dicta ad fig. 25. punctum aliquod certum determinari, ac proinde aliis tribus punctis quibuslibet in circulo assumtis prioribus tribus congruentibus, eodem modo lineas concurrentes congruas ducendo, necesse esse deveniri semper ad idem X. Hinc cum ex tribus datis punctis $D.C.E.$ modo diversa inveniri possint puncta X. prout lineae congruentes aliter atque aliter ducuntur, seu citius tardiusque convergunt, patet etiam alia atque inveniri posse puncta X. eaque omnia in unam lineam cadere.

same way to all those assigned Y, and indeed if the curve is such that it has a point of this kind, which relates in the same way to all of its points, it is found in this way. If, moreover, the curve is circular, as here, it suffices to find some point relating in the same way to three points of the circular curve, for then it will relate in the same way to all the others. The reason for this is that, from three given points $C.D.E$ (see Figure 27) with $C.D$ ɣ $D.E$ being set, some specific point is determined by the method written a little earlier for Figure 28, and accordingly, taking any other three points on the circle congruent to the first three, drawing congruent curves in the same way that meet each other, it is necessary that one will always arrive at the same X. Hence, since from three given points $D.C.E$ distinct points X can be found only according as the congruent curves are drawn in various distinct ways, i.e. converge faster or slower, clearly other points X could also be found, and they will all fall on one line.

(75) Sed eadem sine circulo simplicius consequi possumus.

(75) However, the same thing can be inferred more simply without a circle.

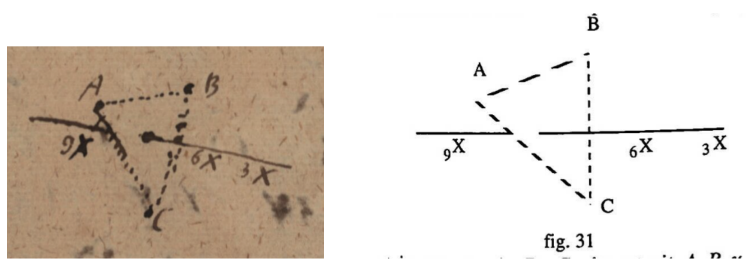


Figure 29

Sint tria puncta $A.B.C.$ ita ut sit $A.B$ ɣ $B.C$ ɣ $A.C$ invenianturque puncta X. ita ut sit $A.X$ ɣ $B.X$ ɣ $C.X$ idque quoties libet, sive quod idem est moveatur punctum X. ita ut quivis ejus locus ut zX eodem modo se habeat ad $A.B.C.$ id est ut sit $A.\,3\,X.$ ɣ $B.\,3\,X.$ ɣ $C.\,3\,X.$ tunc puncta zX erunt in directum posita, sive $\bar{z}X$ erit *linea recta*. Atque ita apparet quid velit

Let there be three points, $A.B.C$, such that $A.B$ ɣ $B.C$ ɣ $A.C$, and suppose points X are found such that $A.X$ ɣ $B.X$ ɣ $C.X$, and this [happens] as many times as you like, or what is the same thing: let the point X move such that any of its places, say zX, relates in the same way to $A.B.C$, that is, such that $A.3X$ ɣ $B.3X$ ɣ $C.3X$; then the points zX will be set in alignment, i.e. $\bar{z}X$ will be *a straight line*. And thus

Euclides, cum ait Lineam rectam ex aequo sua interjacere puncta, id est non subsultare in ullam partem, seu non aliter ad punctum A quam B vel C durante motu se habere. Hinc autem modus quoque habetur puncta X. rectae $\bar{z}X$ inveniendi. Nimirum si ex A linea educatur quaecunque eodem modo se habens ad B et C, itemque alia per B priori congruens et congruenter posita, id est, ut punctum hujus puncto illius respondens eodem modo se habent ad $B.A.C$ ut punctum illius ad $A.B.C$. eaque lineae producantur, donec sibi occurrant, occurrent sibi necessario in puncto X quod se habet eodem modo ad $A.B.C$. Et si per punctum C. etiam talis linea ducta fuisset congrua congruenterque prioribus ea ipsis occurrisset in puncto eodem X. Hinc autem quotvis ejusmodi puncta inveniri possunt, adeoque et Linea recta describi poterit per puncta.

(76) Resumamus aliqua: a puncto quolibet ad quodlibet ducta intelligi potest linea, eaque *rigida*.

(77) $A.B$. significat situm ipsorum A et B inter se, id est tractum aliquem rigidum per A et B. quem tractum nobis sufficit esse lineam. Ita $A.B.C$. significat tractum alium rigidum per $A.B.C$.

(78) Quicquid in spatio ponitur id moveri potest, sive punctum sit sive linea, sive alius tractus sive cuilibet in extenso assignari potest aliud congruum. Hinc $A \between X$. $A.B \between X.Y$ $A.B.C \between X.Y.Z$ vel $A.B.C \between \omega\overline{X.Y.Z}$.

(79) Datis duobus diversis in extenso poni potest unum quiescere, alterum moveri.

(81) Si aliquod eorum quae sunt in tractu rigido moventur, ipse tractus rigidus movetur.

it appears what Euclid intends when he says that a straight line lies equitably between its points,[24] that is, it doesn't swerve to either side, nor relate during its motion to point A in any other way than it does to B or C. Hence, moreover, one also has a method of finding the points X of the line $\bar{z}X$. Namely, if any curve whatever is drawn out from A relating in the same way to B and C, and likewise another through B that is congruent to the first one and congruently set, that is, so that a point of this corresponding to a point of that would relate in the same way to $B.A.C$ as the point of that does to $A.B.C$, and these curves are extended until they meet each other, then they will necessarily meet in a point X which relates in that same way to $A.B.C$. And if such a curve were also drawn through point C, congruent and congruently to the first ones, it would meet them in the same point X. Hence, however many points of this kind could be obtained, and so indeed a straight line can be described through the points.

(76) Let us revisit a few things: from any point to any point, one can understand a curve to be drawn, and that *rigid*.

(77) $A.B$ signifies the situation of A and B between themselves, that is, some rigid tract through A and B, which tract, it will suffice for us, is a curve. Thus $A.B.C$ signifies another rigid tract through $A.B.C$.

(78) No matter what may be put in space, it can move, whether a point, or a curve, or another tract, or anything to which another congruent thing can be assigned in the extensum. Hence $A \between X$, $A.B \between X.Y$, $A.B.C \between X.Y.Z$, or $A.B.C \between \omega\overline{X.Y.Z}$.

(79) Given two distinct things in an extensum, one can be set still, the other to move.

(81) If some of the things which are in a rigid tract move, the rigid tract itself moves.

[24] Latin: *ex aequo sua interjacere puncta*.

(82) Omnis tractus ita moveri potest ut punctum ejus datum incidat in aliud datum. $A.B.C \between D.X.Y$.

(83) Omnis tractus moveri potest uno ejus puncto manente immoto. $A.B.C \between A.X.Y$.

(84) *Recta* est tractus qui moveri non potest, duobus punctis in eo quiescentibus; sive si quidam tractus moveatur duobus punctis manentibus immotis si alia praeterea in eo puncta ponantur manere quiescentia; omnia ea puncta dicentur in directum sita, sive cadere in tractum qui dicitur recta.

(82) Every tract can so move that a given one of its points falls on another given one. $A.B.C \between D.X.Y$.

(83) Every tract can move with one of its points remaining unmoved. $A.B.C \between A.X.Y$.

(84) The *straight line* is a tract which cannot move while two points in it remain still, i.e. if a certain tract moves with two of its points remaining unmoved, and if moreover some other points in it are set to remain still, all these points are said to be situated in alignment, or to fall in a tract which is called a straight line.

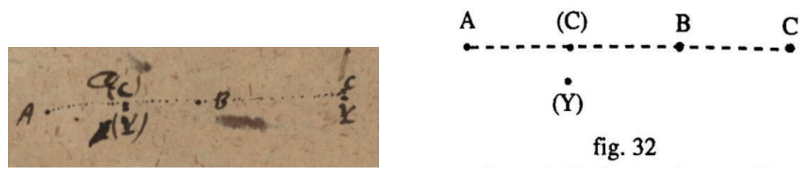


Figure 30

Seu si $A.B.C. \between A.B.Y.$ necesse est esse $C \infty Y$. hoc est si punctum aliquod reperiatur C. situm in directum cum punctis $A.B$. non potest tractus $A.B.C$ (vel $A.C.B$) ita moveri manentibus $A.B$. immotis, ut C. transferatur in Y. atque ita congruat tractus $A.B.Y$. priori $A.B.C$. Sive quod idem est, non potest praeter punctum C. aliud adhuc reperiri Y, quod ad puncta fixa $A.B$. eundem quem ipsum C. situm habeat, sed necesse est si tale quod Y. assumatur ipsum ipsi C coincidere seu esse $Y \infty C$. Unde dici potest punctum C. sui ad puncta $A.B$. situs esse exemplum unicum. Et punctum quod ita moveatur ut ad duo puncta fixa situm servet in sua specie unicum, movebitur in recta. Nempe si sit: $A.B.Y \between A.B.X$ sitque ideo $Y \infty X$ erit $\overline{\varpi}X$ ($\infty\ \overline{\varpi}Y$) linea recta. An autem dentur hujusmodi puncta in directum sita, et an tractum componant, et utrum tractus ille linea sit, non sumendum, sed demonstrandum est. Via

I.e. if $A.B.C \between A.B.Y$, then necessarily $C \infty Y$, that is, if some point C is found situated in alignment with the points $A.B$, then the tract $A.B.C$ (or $A.C.B$) cannot so move, while $A.B$ remain unmoved, that C is translated to Y, and thus the tract $A.B.Y$ is congruent to the first one $A.B.C$. Or what is the same, another point Y cannot be found besides the one C, which has the same situation to the fixed points $A.B$ that C has, but rather it is necessary that if such a point Y is taken, it coincides with C, i.e. $Y \infty C$. From this we can say that the point C is the unique instance with its situation to the points $A.B$. And a point which so moves that it preserves a situation unique of its kind [*in sua species*] to two fixed points, moves in a straight line. Namely, if $A.B.Y \between A.B.X$ and for that reason $Y \infty X$, then $\overline{\varpi}X$ ($\infty\ \overline{\varpi}Y$) will be a straight line. Now whether two such points situated in alignment are given, and whether they form a tract, and whether this tract is a curve, should not be assumed but rather demonstrated. But the path autem puncti ita moti, utique *linea recta* erit, quae quidem si per omnia puncta hujusmodi transit, utique locus omnium punctorum duobus punctis in directum junctorum, non alius tractus quam linea erit.

(85) Si duobus in tractu $A.C.B.$ punctis $A.B$ manentibus immotis, moveatur ipse tractus, linea quam punctum ejus motum C describet dicetur *circularis*.

of a point so moved will certainly be a *straight line* which, if indeed it passes through all points of this kind,[25] namely the locus of all points joined in alignment with the two points, will not be any other tract than a curve.

(85) If two points $A.B$ remain unmoved in a tract $A.C.B$, and the tract itself moves, the curve which the point C describes by its motion is called *circular*.

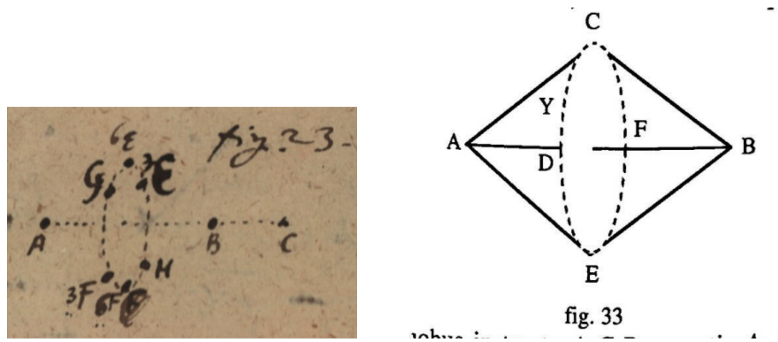


Figure 31

An autem possit tractus aliquis moveri duobus punctis manentibus immotis, etiam non sumendum, sed demonstratione definiendum est.

$A.C.B$ ♉ $A.Y.B$ dicetur ϖY linea *circularis*. Et si sint quotcunque puncta $C.D.E.F.$ sitque $A.B.C$ ♉ $A.B.D$ ♉ $A.B.E$ ♉ $A.B.F$ dicentur esse in una eademque circulari. Haec definitio lineae circularis non praesupponit dari rectam et planum; quod facit Euclidis definitio.

(86.) Locus omnium punctorum quae eodem modo se habent

But whether some tract could move, with two points remaining unmoved, also should not be assumed but needs to be settled [*definiendum*] by a demonstration.

$A.C.B$ ♉ $A.Y.B$: then ϖY is called a *circular* curve, and if there are however many points $C.D.E.F$ and $A.B.C$ ♉ $A.B.D$ ♉ $A.B.E$ ♉ $A.B.F$, they are said to be in one and the same circle. This definition of the circular curve does not presuppose a straight line and plane to be given, whereas Euclid's definition does.

(86) The locus of all points which relate in the same way to A as to B

[25] At first Leibniz concluded the sentence with: 'certainly this line will be nothing but a curve' (*utique ipsa recta non nisi linea erit*). He cancelled that in favour of the presented version. This evidences what is reasonably apparent, namely that for Leibniz it is not patently clear that a tract's being a *curve* (i.e. a *linea*) is not logically prior to its being *straight* (i.e. *in directum*).

ad A quemadmodum ad B, appellabitur *planum*. Sive si sit $A.Y \mathbin{\gamma} B.Y$ erit $\overline{Y}$. *planum*.

(87) Hinc si sit $A.C.B. \mathbin{\gamma} A.Y.B.$ sitque $A.C. \mathbin{\gamma} C.B.$ (adeoque et $A.Y. \mathbin{\gamma} B.Y.$) erit Linea $\overline{\omega Y}$ circularis in plano. An autem omnis circularis sit in plano postea definiendum est.

(88) Si sint $A.B \mathbin{\gamma} B.C \mathbin{\gamma} A.C$ sitque $A.Y. \mathbin{\gamma} B.Y. \mathbin{\gamma} C.Y.$ erit $\overline{\omega Y}$ Linea Recta.

(89) Si sit $A.Y \mathbin{\gamma} A.\ (Y)$. erit $\overline{Y}$ superficies *sphaerica*.

(90) Si sit $Y. \mathbin{\gamma} (Y)$ erit Locus omnium Y. seu Y. extensum absolutum, sive *Spatium*. Nam locus omnium punctorum inter se congruentium est locus omnium punctorum in universum. Omnia enim puncta congrua sunt.

(91) Idem est si sit $Y. \mathbin{\gamma} A.$ nam (ex characterum significatione) si $Y \mathbin{\gamma} A$ erit $(Y) \mathbin{\gamma} A$.

Darüber: [γ. significat congruitatem. ∞ coincidentiam. Cum dico $A.B \mathbin{\gamma} A.Y.$ possem quidem dicere distantiam AB. aequari distantiae AY. Sed quia postea cum tria vel plura adhibentur ut $A.B.C \mathbin{\gamma} A.B.Y.$ non hoc tantum volumus triangulum ABC. triangulo ABY aequari, sed praeterea simile esse, id est congruere, ideo signo γ. potius utor.] Ergo $Y \mathbin{\gamma} (Y)$. Nimirum locus omnium punctorum Y dato puncto A congruorum, utique est etiam spatium ipsum interminatum, omnia enim puncta cuilibet dato congruunt.

(92) Proximum est: $A.Y \mathbin{\gamma} A.\ (Y)$. Locus omnium Y. seu $\overline{Y}$ dicatur *Sphaerica*. Quae est locus omnium punctorum ejusdem ad datum punctum situs existentium. Datum autem

will be called a *plane*. I.e. if $A.Y \mathbin{\gamma} B.Y$, then $\overline{Y}$ is a *plane*.

(87) Hence if $A.C.B \mathbin{\gamma} A.Y.B$, and if $A.C \mathbin{\gamma} C.B$ (and thus also $A.Y \mathbin{\gamma} B.Y$), then the curve $\overline{\omega Y}$ will be circular in a plane. Whether every circle is in a plane needs to be settled later.

(88) If $A.B \mathbin{\gamma} B.C \mathbin{\gamma} A.C$ and $A.Y \mathbin{\gamma} B.Y \mathbin{\gamma} C.Y$, then $\overline{\omega Y}$ will be a straight line.

(89) If $A.Y \mathbin{\gamma} A.(Y)$, then Y will be a *spherical* surface.

(90) If $Y \mathbin{\gamma} (Y)$, then the locus of all Y, or $\overline{Y}$, is absolute extension [*extensum absolutum*], or *space*. For the locus of all points congruent to each other is the locus of all points in general. Indeed, all points are congruent.

(91) The same if $Y \mathbin{\gamma} A$, for (by the signification of the characters) if $Y \mathbin{\gamma} A$, then $(Y) \mathbin{\gamma} A$.

[[26] γ signifies congruence. ∞ coincidence. When I say $A.B \mathbin{\gamma} A.Y$, I could indeed say that the distance AB is equal to the distance AY. But since later on when three or more are brought, as in $A.B.C \mathbin{\gamma} A.BY$, by this we do not mean only that the triangle ABC is equal to the triangle ABY, but moreover that it is similar, that is, congruent, and for that reason it is better to use the sign γ.] Therefore $Y \mathbin{\gamma} (Y)$. Namely the locus of all points Y congruent to a given point A is certainly also unbounded space itself, for all points are congruent to any given one.

(92) Next: $A.Y \mathbin{\gamma} A.(Y)$. The locus of all Y, i.e. $\overline{Y}$, is called a *sphere*. This is the locus of all points being of the same situation to a given point. And the given point is called the *center*.

[26] Leibniz added this portion, in his own brackets, in a page header. It is not clear when he added it.

punctum dicitur *Centrum.*

(93) Idem est si sit $A.B \between A.Y$. Nam ideo erit et $A.B. \between A.\,(Y)$. ac proinde $A.Y \between A.Y$. Ubi nota ipsum B. esse ex numerorum Y. seu esse bY. Si enim Y. omnia puncta comprehendit quae eum habent situm ad A, quem B habet, utique ipsum B. comprehendet, quod eum utique situm ad A habet quem habet. Sphaerica est locus omnium punctorum dati ad punctum datum situs (id est dati puncti situi congrui situs) existentium.

(93) It is the same if $A.B \between A.Y$. For then also $A.B \between A.(Y)$ and accordingly $A.Y \between A.Y$, where we note that B is from the number of Y, or is bY. If indeed Y covers all points having the situation to A that B has, then certainly it covers B, which certainly has the situation to A that it has. The sphere is the locus of all points being of a given situation to a given point (that is, of a situation congruent to the situation of a given point).

(94) Si sit $A.Y \between B.Y$. locus omnium Y. seu $\overline{Y}$. dicatur *planum* sive locus punctorum ut Y, quorum unumquodque ad unum ex duobus datis punctis A. eodem modo situm est, quemadmodum ad alterum B, est *planum.* Notandum est hanc Loci expressionem in aliam converti non posse in qua simul sint Y. et (Y).

(94) If $A.Y \between B.Y$, the locus of all Y, or $\overline{Y}$, is called a *plane*, i.e. the locus of points Y, each of which is situated in the same way to one point A of two given points as it is to the other B, is a *plane*. It should be noted that this expression of a locus could not be converted to the other in which there are Y and (Y) together.

(95) Si sit $A.B. \between C.Y$. erit $\overline{Y}$ *sphaerica.* Nam erit: $A.B. \between C.Y. \between C.dY$ sit $dY. \infty D$. fiet: $C.D. \between C.Y$. adeoque locus erit sphaerica per 92.

(95) If $A.B \between C.Y$, then $\overline{Y}$ will be a *sphere*. For then:

$$A.B \between C.Y \between C.dY.$$

Let $dY \infty D$, then $C.D \between C.Y$, and thus the locus will be a sphere by Prop. 92.

(95) Y. et (Y) significant quodcunque punctum loci alicuius, et quodcunque aliud praeter prius. zY. significat quodlibet punctum loci, seu omnia loci puncta distributive. Idem etiam significat Y absolute positum. dY. significat unum aliquod peculiare punctum loci. $\overline{Y}$ significat omnia puncta loci collective, seu totum locum. Si locus sit Linea hoc ita significo $\overline{\omega Y}$. Si sit superficies ita $\overline{\omega\not\psi Y}$. Si solidum ita: $\overline{\omega\not\psi\varphi Y}$.

(95) Y and (Y) signify any point of some locus and any other besides the first. zY signifies whatever point of a locus, i.e. all points of a locus distributively. The same is signified by Y put absolutely. Then dY signifies some one particular point of the locus. $\overline{Y}$ signifies all points of the locus collectively, or the whole locus. If the locus is a curve, I signify this by $\overline{\omega Y}$. If it is a surface, by $\overline{\omega\not\psi Y}$. If a solid, by: $\overline{\omega\not\psi\varphi Y}$.

(96) Si sit $A.B.C \between A.B.Y$, (sive si sit $A.B.Y \between A.B.\,(Y)$) tunc locus omnium Y. seu $\overline{Y}$ dicetur *Circularis*. Id est si plurium punctorum idem sit situs (vel datus) ad duo data puncta, *Locus* erit *Circularis.*

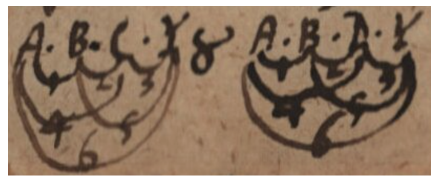
(96) If $A.B.C \between A.B.Y$ (i.e. if $A.B.Y \between A.B.(Y)$), then the locus of all Y, or $\overline{Y}$, is called a *circle*. That is, if the situation to two given points is the same for many points (or it is given), the locus will be a *circle*.

(97) Si sit $A.Y \between B.Y \between C.Y$ tunc Locus omnium Y. seu $\overline{Y}$ dicetur *recta*.

(97) If $A.Y \between B.Y \between C.Y$, then the locus of all Y, or $\overline{Y}$, is called a *straight line*.

(98) Si sit $A.B.C.Y \between A.B.D.Y$ erit $\overline{Y}$ *Planum* seu si $C.D$. duo puncta eodem modo sint ad tria $A.B.Y$. erunt haec tria in eodem plano et si duobus ex his datis $A.B$ quaeratur tertium Y., locus omnium Y. erit planum. Ubi patet ipsa $A.B$. sub Y. comprehendi.

(98) If $A.B.C.Y \between A.B.D.Y$, then $\overline{Y}$ will be a *plane*, i.e. if $C.D$ are two points relating in the same way to the three $A.B.Y$, these three will be in the same plane, and if from two of these given ones, $A.B$, a third Y is sought, the locus of all Y will be a plane. It is clear from this that $A.B$ are included under Y.



Demonstrandum est hunc locum cum altero qui est prop. 94 coincidere. Hoc ita fiet: (1) (2) $C.Y \between D.Y$ locus est ad planum per prop. 92. Sint 3 $Y. \infty A$ et 7 $Y \infty B$ erit $C.A \between D.A$ (3) et $C.B \between D.B$. Ergo fiet:

$$A.B.C.Y \between A.B.D.Y.$$

1 2 3 1 2 3
4 5 4 5
6 6

Nam 1 patet per se et 2 per (3) et 3 per (1) et 4 per (2) et 5 per se et 6 per se.

It needs to be demonstrated that this locus coincides with the other one which is in Prop. 94.[27] This will be done as follows: $C.Y \overset{(1)}{\between} D.Y$ is the locus for a plane by Prop. 94. Let $3Y \infty A$ and $7Y \infty B$, then $C.A \overset{(2)}{\between} D.A$ and $C.B \overset{(3)}{\between} D.B$. Therefore: Namely 1 is clear *per se*, and 2 through (3), and 3 through (1), and 4 through (2), and 5 *per se*, and 6 *per se*.

(99) Si sit $A.Y \between B.Y \between C.Y$ locus $\overline{Y}$ erit *punctum,* sive Y satisfaciens non erit nisi unicum sive erit $Y \infty (Y)$. Haec proposito demonstranda est.

(99) If $A.Y \between B.Y \between C.Y$, the locus $\overline{Y}$ will be a *point*, or the Y satisfying it will be not but unique, i.e. $Y \infty (Y)$. This proposition should be demonstrated.

(100) Habemus ergo loca ad punctum, ad Rectam, ad Circularem, ad Planum, ad Sphaericam, solis congruentis mira

(100) We therefore have loci for a point, a line, a circle, a plane, and a sphere, expressed with wonderful simplicity only with congruences;

[27] Manuscript: 92.

simplicitate expressa sed haec partim vera, partim possibilia esse; et cum aliis definitionibus coincidere nostras demonstrandum est.

but some of these are real, some possible, and it needs to be demonstrated that our definitions coincide with others.

(101) Si tractus sive extensum quodcunque moveatur uno puncto existente immoto, aliud quodcunque ejus punctum movebitur in Sphaerica. Pono autem ipsum extensum esse rigidum, seu partes situm eundem servare. Habebimus ergo modum inveniendi sphaericae puncta quotcunque. Potest etiam $A.B.$ ɣ $A.Y.$ esse data; si tractus transeat per duo puncta $A.B$. Tractum autem aliquem (sive linea sit sive superficies sive solidum) per duo data puncta ducere, et tractum movere uno puncto immoto, utique in potestate est.

(101) If any tract or extensum whatever moves with one point being unmoved, any other point of it moves on a sphere. Now let the extensum be rigid, i.e. let its parts preserve the same situation. We will then have a way of obtaining however many points of a sphere. For $A.B$ ɣ $A.Y$ can also be given if the tract passes through two points $A.B$. Certainly it is in our power to draw some tract (whether it be a curve, a surface, or a solid) through two given points, and to move the tract with one point being fixed.

(101) Si per duo data puncta transeat duo tractus congrui, *congruenter,* id est ita ut puncta respondentia in duobus tractibus situs habeant ad duo puncta data unumquodque ad suum, congruos, moveanturque aut etiam si opus sit crescant etiam congruenter, donec sibi occurrant loca in quibus puncta eorum respondentia sibi occurrent, erunt puncta plani illius, quod ad duo puncta data eodem modo se in quolibet puncto suo habere, definivimus. Posse autem congruenter moveri, posse congrue ac congruenter produci, donec occurrant, postulo.

(101)[28] If two congruent tracts pass *congruently* through two given points, that is, such that corresponding points in the two tracts have congruent situations to the two given points, each one to its own, and they move, or even lengthen if necessary, also congruently, until they meet each other, then the places at which their corresponding points meet each other will be points of the plane which we have defined to relate in the same way at any of its points to the two given points. I claim, moreover, that they can move congruently, or be extended congruously[29] and congruently, until they meet.

(102) Si jam sphaericam sphaerica, aut plano secemus, habebimus cicularem, si planum planom, habebimus rectam. Si rectam recta, punctum. Ostendendum autem est has sectiones fieri posse, et punctorum Sphaericae et Sphaericae, vel plano et plano, vel plano et Sphaerica communium, esse tractum. Si Sphaerica planum vel Sphaericam tangat locus, etiam est punctum, cum scilicet circularis fit minima seu evanescit.

(102) If we now section a sphere by a sphere, or by a plane, we will have a circle; if a plane by a plane, we will have a line; if a line by a line, a point. But it needs to be shown that these sections can be made, and that [the locus] of points in common to a sphere and a sphere, or a plane and a plane, or a plane and a sphere, is a tract. If a sphere is tangent to a plane or a sphere, the locus is a point, namely when the circle is made minimal, i.e. vanishes.

[28]The manuscript has two paragraphs '(101)'.
[29]Latin: *congruè ac congruenter*. It is not clear whether the two adverbs are meant to carry different meanings.

(103) Caeterum omnes definitiones rectae communes nondum satis perfectae sunt, nam semper adhuc demonstratione opus est quod talis recta sit possibilis. Quod tamen videtur ponendum inter facillima. Tali itaque opus definitione, ut statim appareat rectam esse possibilem. Si definias minimum, etiam hoc dubitari potest an detur minima a puncto ad punctum. Si definias eam cujus puncta non magis diduci possunt, praesupponis distantiam, seu viam minimam, diduci enim est majorem distantiam acquirere. Equidem aliud est exhibere rectam materialem, seu ducere eam, aliud est cogitatione complecti.

(104) Quando duo Puncta simul percipiuntur eo ipso percipitur ipsorum situs eorum ad se invicem. Sunt autem duo quilibet situs inter duo puncta similes, adeoque sola comperceptione seu magnitudine distinguuntur. Est ergo situs discrimen magnitudo cujusdam extensi cum duo puncta simul percipiuntur, eo ipso percipitur extensum quoddam.

(105) *Recta* est extensum quod duobus punctis simul perceptis eo ipso percipitur.

(106) Puncto uno percepto, rursumque alio percepto separatim, nulla notari potest varietas.

(107) Si simul percipiantur *A.B*. rursusque simul percipiantur *C.D*. discrimen situs est. Id tamen quod percipitur perceptis *A.B*. et quod percipitur perceptis *C.D*. similia sunt: patet enim nihil in singulis notari posse, quod non in utrisque notetur, itaque id quod percipitur perceptis *A.B*. et quod percipitur perceptis *C.D*., si ex se invicem discerni possunt sola magnitude discernentur, itaque simul perceptis *A.B*. simul

(103)[30] As for the rest, all the common definitions of a straight line are not yet sufficiently perfect, since a demonstration is still always needed that such a line is possible. But it seems that this should be regarded among the easiest things. And so we need such a definition that it is immediately apparent that the straight line is possible. If you define it as the minimal one, this too can be doubted, whether there is a minimal one from a point to a point. If you define it as that whose points cannot be drawn farther apart, you presuppose distance, or a minimal path, for to be drawn apart is to acquire a greater distance. Of course, it is one thing to exhibit a material straight line, i.e. to draw it, and another thing to comprehend it in thought.

(104) When two points are perceived simultaneously, *eo ipso* their situation to each other is perceived.[31] But any two situations between two points are similar, and thus are distinguished only by coperception, i.e. by magnitude. The difference of situation is therefore the magnitude of a certain extensum, when two points are perceived simultaneously and *eo ipso* a certain extensum is perceived.

(105) The *straight line* is the extensum that, when two points are simultaneously perceived, is perceived *eo ipso*.

(106) When one point is perceived, and then again another is perceived separately, no diversity can be noted.

(107) If *A.B* are simultaneously perceived, and then again *C.D* simultaneously perceived, a difference is one of situation. Nevertheless, that which is perceived when *A.B* are perceived and that which is perceived when *C.D* are perceived, are similar: for clearly nothing can be noted in each individual which cannot be noted in both of them, and so that which is perceived when *A.B* are perceived and that which is perceived when *C.D* are perceived, if they can be distinguished from

[30] Paragraphs 103 through the end of the essay were circled and a note provided: 'let the included portion be omitted or moved back somewhere else'.

[31] This sentence underwent a few revisions that may be of interest. It was initially '…*eo ipso* a certain curve is perceived', then '…*eo ipso* their distance, i.e. situs, is perceived', then '…*eo ipso* their situs, i.e. distance, is perceived', then the version presented.

percipitur aliquid magnitudinem habens. Cum duo simul in spatio esse percipiuntur, eo ipso percipitur via ab uno ad aliud. Et cum sunt congrua eo ipso concipitur via unius in alterius locum. Sunt autem duo puncta congrua. Itaque quod percipitur, duobus punctis simul perceptis est Linea seu via puncti. Patet etiam viam hanc talem esse, ut sive ab A ad B, rendas, sive a B ad A. via sit eadem. Idemque sit si simul A tendat versus B. et B versus A. et locus in quo erit A. congruus erit loco in quo erit B. congruenterque positus, id est ut locus puncti venientis ab A erit ad A. ita locus puncti venientis ad B. erit ad B., imo etiam ut locus puncti venientis a B erit ad A. ita locus respondens puncti venientis ab A erit ad B. Punctum erit in quo sibi occurrant quod eodem modo erit a A quo ad B. Cum concipimus duo puncta ut simul existentia, inquirimusque rationem cur simul existentia dicamus, cogitabimus esse simul percepta, vel certe posse simul percipi. Quando aliquid percipimus velut existens, eo ipso percipimus esse in spatio, id est posse alia existere indefinitis quae ab ipso nullo modo possint discerni. Sive quod idem est posse moveri, sive posse tam in uno loco quam in alio esse, et quia non potest simul esse in pluribus locis, nec moveri in instanti, ideo locum illum percipimus ut continuum. Sed quia indefinitum adhuc est, quorsum moveatur, potest enim moveri multis modis qui inter se discerni non possunt; hinc determinatur animus ad certum aliquem motum, si aliud praeterea ponamus, congruum priori, eo ipso enim cogitatur unum posse pervenire in alterius locum. Idque cum pluribus modis fieri possit, determinatur tamen unicus, ad quem considerandum nullo alio praeterea assumto opus est, quam his duobus positis. Id est ex positis duobus congruis in spatio ponitur via unius ad alterum, ipsa connectens. Simplicissima autem positio est puncti. Nam etsi ponas aliud tamen cum ex motibus diversis unius ad alterum unus determinetur, quo puncta respondentia ad se determinate moventur, patet animum tan-

each other, will be distinguished only by magnitude, and so $A.B$ being simultaneously perceived, something having magnitude is simultaneously perceived. When two things are simultaneously perceived to be in space, *eo ipso* a path from one to the other is perceived. And when they are congruent, *eo ipso* a path of one into the place[32] of the other is conceived. But two points are congruent. And so what is perceived, two points being simultaneously perceived, is a curve, i.e. the path of a point. It is also clear that this path is such that, whether you give the line from A to B, or from B to A, the path is the same. And it will be the same if A tends toward B and simultaneously B toward A, and the locus in which A will be is congruent to the locus in which B will be, and congruently set. That is, as the locus of a point coming from A will be to A, so the locus of a point coming from[33] B will be to B, and also, as the locus of a point coming from B will be to A, so the corresponding locus of a point coming from A will be to B. When we conceive of two points as simultaneously existing, and we examine the reason why we say they simultaneously exist, we will think that they are simultaneously perceived, or at least are able to be simultaneously perceived. When we perceive something as though existing, *eo ipso* we perceive it to be in space; that is, indefinitely many others can exist which could in no way be distinguished from it. Or what is the same, it can move, i.e. it can be in one place as well as in another, and since it cannot be simultaneously in several places, nor move in an instant, we therefore perceive the locus as a continuum. But since it is still indefinite to where it moves, for indeed it can move in many ways which cannot be distinguished from each other, hence, if we suppose another thing besides, congruent to the first, and indeed *eo ipso* the one is thought to be able to pass through into the place of the other, then the mind is determined to some specified motion. And although it could be done in several ways, nonetheless a unique one is determined, for considering which there is no need to assume anything else besides these two being supposed. That is, from supposing two congruent ones to be set in space, a path from one to the other is supposed, connecting them. But the simplest position is that of a

[32] We use either 'place' or 'locus' where the Latin is just 'locus' in this vicinity.

[33] *ad* ('to') in the manuscript. Leibniz corrected at least three other instances of *ad* to *ab* in this paragraph, and likely this one was missed.

dem incidere in considerationem duorum punctorum, ea enim congrua esse constat per se.

[*Verworfener Abschnitt*] Itaque primum cogito: punctum *A* per se existere potest. Punctum existit in Spatio. Punctum coexistit aliis punctis. Punctum moveri potest. Punctum coexistere potest aliis punctis infinitis. *Punctum* est in extenso simplicissimum. Puncta duo per se invicem discerni non possunt. Ergo punctum aliud per se etiam existere potest. Seu quodlibet punctum per se existere potest. Punctum aliquod existit. Quicquid existit existere potest. Ergo punctum aliquod existere. *Extensum* existit.

[*Schlussabschnitt*] *Extensum* est continuum. In extenso possunt fieri partes. In extenso possunt fieri partes quae existunt simul. In extenso possunt fieri partes infinitis modis. Extensi pars extensa est. In uno extenso existere possunt multa. In uno extenso existere possunt infinita. In uno extenso existere possunt infinita similia. In uno extenso existere possunt infinita congrua. Si quid in extenso existit, eique congruum est, ei coincidit. Si quid in extenso existit eique congruum non est, possunt infinita existere in eodem extenso, quae priori non coincidunt, sed tamen congrua sunt. Duo mobilia quae un extenso sumuntur, sibi congruere possunt, sive diversis temporibus ita locari possunt, ut prior status a posteriore discerni non possit. Locus ipse extensi extensus est. Locus extensi congruit extenso. Locus est immobilis.

point. For even if you suppose otherwise, nonetheless, since one of the various motions of one to the other would be determined according to which the corresponding points move determinately towards each other, clearly the mind eventually settles upon the consideration of two points, for they are established to be congruent *per se*.

[[34]And so I think first: a point *A* can exist *per se*. A point exists in space. A point coexists with other points. A point can move. A point can coexist with infinitely many other points. A *point* is the simplest in extension.[35] Two points cannot be distinguished *per se* in turns. Therefore another point can also exist *per se*. I.e. any point whatever can exist *per se*. Some point exists. Whatever exists can exist. Therefore some point [can] exist. An *extensum* exists.]

An *extensum* is a continuum. Parts can be made in an extensum. Parts can be made in an extensum which simultaneously exist. Parts can be made in an extensum in infinitely many ways. A part of an extensum is extended. Many [parts] can exist in one extensum. Infinitely many [parts] can exist in one extensum. Infinitely many similars can exist in one extensum. Infinitely many congruents can exist in one extensum. If something exists in an extensum, and it is congruent to it, it coincides with it. If something exists in an extensum and it is not congruent to it, there can exist infinitely many in the same extensum which do not coincide with the first, but nonetheless they are congruent. Two moveable things that are taken in an extensum can be congruent to each other, i.e. can be located at different times such that the earlier state could not be distinguished from the later one. The locus itself of an extensum is extended. The locus of an extensum is congruent to the extensum. The locus is immovable.

[34]From here to the end, the script style is different, suggesting that Leibniz wrote this last part in a much later sitting. He also cancelled in bulk the part in brackets, which we include because of its potential philosophical interest.

[35]Latin *in extenso simplicissimum*; this could be 'the simplest in terms of extension' or 'the simplest thing in an extensum'.