G. W. Leibniz

Analysis Geometrica Propria

Translated by David Jekel and Matthew McMillan

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Gerhardt (ed.), *Leibnizens mathematische Schriften*, Bd. V (1858), S. 172–178 (Stück II).

Analysin Geometricam propriam eique connexum *calculum situs*, hic attentabimus nonnihil speciminis gratia, ne forte pereat cogitatio, quae aliis quod sciam in mentem non venit, et usus longe alios ab iis dabit, quos Algebra praestat. Sciendum enim est diversae considerationis esse magnitudinem et situm. Magnitudo datur etiam in illis quae situm non habent, ut numerus, proportio, tempus, velocitas, ubicunque scilicet partes existunt, quarum numero seu repetitione aestimatio fieri potest. Itaque eadem est doctrina magnitudinis et numerorum, Algebraque ipsa vel si mavis Logistica, tractans de magnitudine in universum, revera agit de numero incerto vel saltem innominato. Sed situs magnitudini vel partium multitudini formam quandam superaddit, ut in numeris figuratis. Hinc patet, Algebram continere Analysin proprie et per se ad Arithmeticam pertinentem, etsi ad Geometriam et situm transferatur, quatenus linearum et figurarum magnitudines tractantur. Sed tunc necesse est, Algebram multa supponere propria Geometriae vel situs, quae et ipsa resolvi debebant. Analysis igitur nostra resolutionem illam effectui dat nihil amplius assumens aut supponens nec tam magnitudini, quam ipsi per se situi accommodata. Nunc autem ad explicandam rem situs non nisi *congruentia* utemur, sepositis in alium locum *similitudine* et *motu*.

(1) *Congrua* sunt quae sibi substitui possunt in eodem loco, ut

A *properly Geometric Analysis*, and the *calculus of situation* connected with it, we will in some measure attempt here as a sample, lest perchance the idea be lost, which so far as I know has not come to mind for others, and which will provide far different applications from those furnished by Algebra. What should be understood is that magnitude and situs are different considerations. There is magnitude in things that don't have situs, such as number, proportion, time, velocity, and indeed wherever there exist parts of which an estimation can be made by number, i.e. by repetition. Therefore the doctrines of magnitude and of numbers are the same, and Algebra itself, or if you prefer Logistic, in treating of magnitude in general, actually treats of a variable or at least unspecified number. But situs adds a certain form over and above magnitude, or multitude of parts, as in figurate numbers. Hence, clearly, Algebra consists in Analysis properly and per se pertaining to Arithmetic, even if it is transferred to Geometry and situs, insofar as the magnitudes of lines and figures are treated. But then Algebra necessarily supposes much that is proper to Geometry or situs, which themselves also should be released [from it]. Our Analysis, therefore, effects this release assuming or supposing nothing more, and being not so much accommodated to magnitude as to situation itself per se. Moreover, for explaining the matter of situs we now use nothing but *congruence*, reserving *similarity* and *motion* to another place.

(1) Those are *Congruent* that can be substituted for one another in

ABC et CDA (fig. 65), quod sic designo $ABC \simeq CDA$. Nam $\sim$ mihi est signum similitudinis, et $=$ aequalitatis, unde congruentiae signum compono, quia quae simul et similia et aequalia sunt, ea congrua sunt.

the same place, as ABC and CDA (fig. 65), which I designate thus: $ABC \simeq CDA$. Namely, for me, $\sim$ is the sign of similarity, and $=$ of equality, from which I compose the sign of congruence, since those that are simultaneously similar and equal are congruent.

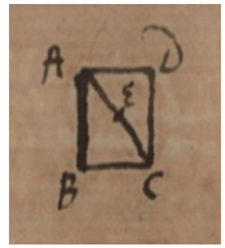

Figure 1

(2) *Eodem modo se habere* hic censentur vel *congruenter*, ea quae in congruis sibi respondent. Exempli causa $AB.ABC \simeq CD.CDA$, nempe AB est ad ABC, ut CD ad CDA; ita et $AB.AC \simeq CD.CA$ et $A.BC \simeq C.DA$, id est punctum A eodem modo se habet ad rectam BC, ut punctum C ad DA. Neque enim hic tantum de ratione aut proportione agitur, sed de relatione quacunque.

(2) Those are reckoned to relate [se habere] in the same manner or *congruently* that correspond to each other in congruents. For example $AB.ABC \simeq CD.CDA$, for of course AB is to ABC as CD to CDA; and so $AB.AC \simeq CD.CA$ and $A.BC \simeq C.DA$, that is, the point A relates in the same manner to the line BC as the point C to DA. Indeed here it is done not so much by ratio or proportion, but by any relation whatever.

(3) *Axioma*: Si determinantia sint congrua, talia erunt etiam determinata, posito scilicet eodem determinandi modo. Exempli gratia si $A.B.C \simeq D.E.F$, etiam circumferentia circuli per $A.B.C$ congruet circumferentiae circuli per $D.E.F$, quia datis tribus punctis circumferentia circuli est determinata. Et in universum in calculo congruentiarum substitui possunt determinata pro determinantibus sufficientibus, prorsus ut in vulgari calculo aequalia aequalibus substituuntur. Vid. infra § 26. § 30. § 32.

(3) *Axiom*: If the determiners are congruent, so too will be those determined, supposing of course the same mode of determining. For example, if $A.B.C \simeq D.E.F$, then also the circumference of the circle through $A.B.C$ is congruent to the circumference of the circle through $D.E.F$, since the circumference of a circle is determined by three given points. And in general, in the calculation of congruents, determined things may be substituted for those sufficient for determining, just as equals are substituted for equals in common calculation. See below §26, §30, §32.

(4) Ut autem calculus melius intelligatur, notandum est, cum dicitur $A.B.C \simeq D.E.F$, idem esse ac si simul diceretur $A.B \simeq D.E.$ et $B.C \simeq E.F$ et $A.C \simeq D.F$, ita ut haec ex illo fieri

(4) It should be observed, so that the calculus is better understood, that when we say $A.B.C \simeq D.E.F$, it is the same as if we said simultaneously $A.B \simeq D.E.$ and $B.C \simeq E.F$ and $A.C \simeq D.F$, so that these can be made

possint *divellendo*, et illud ex his *conjungendo*. Vid. infra § 26. 27. § 29. § 30. § 31. § 32.

(5) *Terminus communis* est locus qui inest duobus locis, ita ut pars eorum non sit. Sic punctum E est locus, qui inest rectis AE, CE, pars autem est neutrius. Itaque terminus earum communis esse dicetur.

(6) *Sectio* est duarum partium totum constituentium nec partem communem habentium terminus communis totus. Ita AC est terminus communis totus triangulorum ABC, CDA constituentium totum $ABCD$ nec habentium partem communem.

(7) Hoc ita calculo exprimemus, per quem Geometria ad Logicam refertur. Punctum omne in proposito loco existens communi nota vel litera designetur (exempli gratia) X, et ipse locus designetur per $\bar{X}$, lineam super litera ducendo. Si quaevis loci puncta sint Y et Z, loca erunt $\bar{Y}$ vel $\bar{Z}$. Sit ergo totum $\bar{X}$, partes constituentes sint $\bar{Y}$ et $\bar{Z}$, et sectio sit $\bar{V}$, formari poterunt hae propositiones: Omne Y est X, omne Z est X, quia $\bar{Y}$ et $\bar{Z}$ *insunt* ipsi $\bar{X}$. Sed et quod non est Y nec Z, id non est X, posito $\bar{Y}$ et $\bar{Z}$ esse *partes constituentes* seu exhaurientes totum $\bar{X}$. Porro omne V est Y, et omne V est Z, quia $\bar{V}$ est ipsis $\bar{Y}$ et $\bar{Z}$ *commune*, seu utrique inest. Denique quod est Y et Z simul, id etiam est V, quia $\bar{V}$ est *sectio* seu terminus communis totus, scilicet qui continet quicquid utrique commune est, partem enim (seu aliquid praeter terminum) non habent communem. Hinc omnes Logicae subalternationes, conversiones, oppositiones et consequentiae hic locum interdum cum fructu habent, cum alias a realibus proscriptae fuerint visae, hominum vitio, non propria culpa.

(8) *Coincident* loca $\bar{X}$ et $\bar{Y}$, si omne X sit Y, et omne Y sit X. Hoc ita designo: $\bar{X} \cong Y$.

from that by *disjoining*, and that from these by *conjoining*. See below §26, 27, §29, §30, §31, §32.

(5) A *common boundary* is a locus which is-in two loci, such that it is not a part of them. In this way a point E is a locus which is-in the lines AE, CE, but is a part of neither. Therefore it is called their common boundary.

(6) A *section* is the entire common boundary of two parts constituting a whole and not having a common part. Thus AC is the entire common boundary of triangles ABC, CDA constituting the whole $ABCD$ and not having a common part.

(7) We will express this by the calculus through which geometry is traced back to logic thus: Every point existing in a proposed locus is designated by a common mark or letter (for example) X, and the locus itself is designated by $\bar{X}$, drawing a line over the letter. If some points of the locus are Y and Z, [their] loca will be $\bar{Y}$ or $\bar{Z}$. Let, therefore, the whole be $\bar{X}$, the constituting parts be $\bar{Y}$ and $\bar{Z}$, and $\bar{V}$ be the section, then these propositions can be formed: Every Y is X, every Z is X, since $\bar{Y}$ and $\bar{Z}$ *are-in* $\bar{X}$. But also, whatever is neither Y nor Z is not X, having set $\bar{Y}$ and $\bar{Z}$ to be *parts constituting* or i.e. exhausting the whole $\bar{X}$. Further, every V is Y, and every V is Z, since $\bar{V}$ is *in common* to $\bar{Y}$ and $\bar{Z}$, or i.e. is-in both. Finally, whatever is simultaneously Y and Z is also V, since $\bar{V}$ is the *section* or i.e. the entire common boundary, namely that contains whatever is common to both, and indeed they don't have a common part (or i.e. something besides a boundary). Hence all subalterns, conversions, oppositions, and inferences of logic have a place here, sometimes fruitfully, while other times they seem to be precluded from real [use] by human failing, not their own deficiency.

(8) The loca $\bar{X}$ and $\bar{Y}$ are *coincident* if every X is Y, and every Y is X. This I denote thus: $\bar{X} \cong \bar{Y}$.

(9) *Punctum* est locus, in quo nullus alius locus assumi potest, itaque si in puncto ⊙ assumatur locus ☽, coincidet ☽ ipsi ⊙, et vicissim si ☽ insit in ⊙ et ex hoc solo concludatur coincidere ⊙ et ☽, erit ⊙ punctum.

(9) A *point* is the locus in which no other locus can be taken; therefore if the locus ☽ is taken in a point ⊙, then ☽ is coincident with ⊙, and if in turn ☽ is-in ⊙ and from this alone it is deduced that ⊙ and ☽ coincide, then ⊙ is a point.

Spatium absolutum est contrarium puncto; nam in spatio omnis alius locus assumi potest, ut in puncto nullus, ut ita punctum sit simplicissimum in situ, et velut minimum, spatium vero sit diffusissimum et velut maximum.

Absolute space is the opposite of a point, for in space every other locus can be taken, as none [can be] in a point, so that the point is the simplest with respect to situation, and as it were the minimum, while indeed space is the most diffuse, and as it were the maximum.

(10) *Corpus* (mathematicum scilicet) seu *solidum* est locus, in quo plus est quam terminus. Atque hoc scilicet volumus, cum solido tribuimus profunditatem. Contra quicquid in superficie aut linea est, terminus intelligi potest, et commune esse alicui cum alio partem communem cum ipso non habente. Analogia hic etiam est inter punctum et solidum, quod quicquid puncto inest, punctum est; contra cui solidum inest, id solidum est. Item, punctum non potest cuiquam inesse ut pars; at solidum nulli aliter inesse potest quam ut pars.

(10) A *body* (mathematical, of course) or i.e. *solid* is a locus in which there is more than boundary. And this is what we mean when we attribute depth to a solid. Conversely, whatever is in a surface or curve can be understood to be a boundary, and is in common to something and another thing not having a common part with the first. There is an analogy here too, between point and solid, inasmuch as whatever is-in a point, is a point; and conversely whatever a solid is-in, is a solid. Likewise, a point cannot be-in something as a part; but a solid cannot be-in anything otherwise than as a part.

(11) *Planum* est sectio solidi utrinque eodem modo se habens ad ea, quae solidi terminos non attingunt, seu utrinque eodem modo se habens ad ea, quae fiunt in una parte ut in alia. Si pomum plano secas, duorum segmentorum extrema, quibus cohaerebant, distingui invicem non possunt.

(11) A *plane* is a section of a solid having the same relation on both sides to what does not touch the boundaries of the solid, or i.e. having the same relation on both sides to what ends up in one part as in the other. If you slice an apple by a plane, the faces [extrema] of the two pieces [segmentorum], where they were conjoined, cannot be distinguished from each other.

(12) Itaque si solidum interminatum sit, absolute verum est, planum secans utrinque eodem se modo habere. Sin terminatum sit solidum, sufficit terminos in rationes non venire. Et utrinque eodem modo facta etiam ad sectionem se eodem modo habebunt.

(12) Therefore if it is an unbounded solid, it is absolutely true that the dividing plane has the same relation to both sides. If, on the contrary, the solid is bounded, it's enough that the boundaries do not enter the reckoning. And something done the same way on both sides will indeed have the same relation to the section itself.

(13) *Recta* est sectio plani utrinque eodem modo se habens ad

(13) A *line* is a section of a plane having the same relation on both

ea, quae terminos plani non attingunt.

sides to what does not touch the boundaries of the plane.[1]

(14) Sit planum interminatum AA (fig. 66), ejusque sectio BB utrinque se habens eodem modo, erit BB recta interminata.

(14) If a plane AA is unbounded (fig. 66), and its section BB has the same relation on both sides, then BB will be an unbounded line.

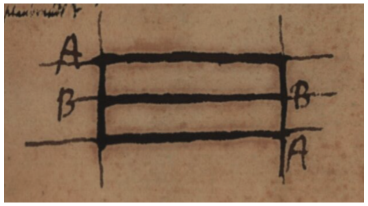

Figure 2

(15) Sed et si planum sit terminatum CC (fig. 67), quaecunque ejus figura sit, si tamen tegamus terminos ut non appareant, vel rationem eorum nullam habeamus, reperiemus rectam secantem DD utrinque se eodem modo habere, eaque erit terminata.

(15) But if a plane CC is bounded (fig. 67), no matter what its shape, if we nonetheless cover the boundaries so that they don't appear or we in no way take them into account, then we will discover a dividing line DD to have the same relation on both sides, and it will be bounded.

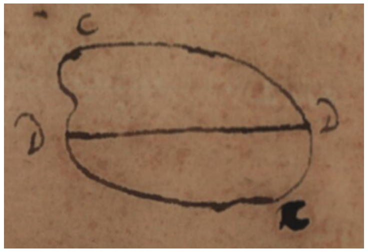

Figure 3

(16) *Curva* vero diversimode se habet utrinque, cum ab uno latere sit concava, ab alio convexa.

(16) A curve, however, has a different relation on each side, since it is concave from one side, and convex from the other.

[1]Here Leibniz makes the following remark: What if someone doubted whether the plane could be so divided? Is it preferable, then, to form a line by section of two planes?

Quae sequentur, nunc quidem omnia in plano intelligantur.

Now, everything that follows should be understood in the plane.[2]

(17) Si sit recta (fig. 68), in qua puncta A et B, et extra eam punctum C ab uno latere, tunc oportet dari posse aliud D ab altero latere, quod eodem modo se habeat ad A et B, quo C se ad ea habet. Nam alioqui cum puncta haec sint in recta ex hypothesi, unum latus non ita ut alterum ad rectam se haberet, contra rectae definitionem. — Dato igitur $C.A.B$ inveniri potest D, ut sit $C.A.B \simeq D.A.B$.

(17) If there is a line (fig. 68) in which there are points A and B, and outside of it a point C on one side, then necessarily there is another D on the other side, which has the same relation to A and B that C has to them. For otherwise, since these points are in the line by hypothesis, one side would not relate to the line in such a way as the other, contrary to the definition of a line. Therefore, given $C.A.B$, one can find D such that $C.A.B \simeq D.A.B$.

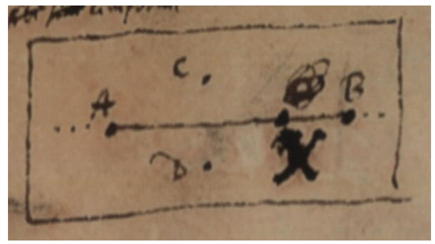

Figure 4

(18) Itaque si detur punctum X suae ad duo puncta $A.B$ relationis unicum, id non poterit esse ab alterutro latere rectae per $A.B$; alioqui contra hypothesin daretur aliud ei geminum, per praecedentem. — Hinc necesse est ut cadat in ipsam rectam, ubi gemina alibi in unum coeunt, cum recta sit utriusque lateris terminus communis.

(18) Therefore if there is a point X unique with its relation to two points $A.B$, it couldn't be on either side of the line through $A.B$, otherwise, by the preceding, it has some twin, contrary to hypothesis. Hence it necessarily falls on the same line on which what are twins elsewhere merge into one, since the line is also the common boundary of either side.[3]

(19) Recta igitur (terminata scilicet) est locus omnium punctorum suae ad duo in ipsa puncta relationis unicorum. Sit

(19) A straight line (bounded, of course) is consequently the locus of all points unique with their relation to two points in itself. If

[2]It appears necessary for some property of the plane to enter the argument, such as that two planar figures with the same boundary are congruent, or i.e. that the interior is uniform. Marginal note by Leibniz.

[3]Leibniz remarks: It still needs to be demonstrated, conversely, that every point in a line is unique of its relation to two points in it.

So in turn, every point in the line through $A.B$ is unique of its relation to those two points. Let such a point be X; if it is not unique, then there exists Ω which is to $A.B$ as X is to $A.B$; since X is in the line through $A.B$ by hypothesis, then so too is Ω; there is therefore some order in the straight line among the four points $A.B.X.\Omega$, therefore $A.B.X$ and $A.B.\Omega$ are not $\simeq$. Supposing that the locus is a *curve*, there is not an order of all points in the (linear?) curve. Therefore there are two points in it having the same relation to two points in the same [curve], and so the locus determined with respect to the two is a curve.

$X.A.B \simeq Z.A.B$, atque ideo X coincidat ipsi Z, erit $\bar{X}$ recta (interminata) per $A.B$.

$X.A.B \simeq Z.A.B$ and for that reason X coincides with Z, then $\bar{X}$ is the line (unbounded) through $A.B$.[4]

(20) Unde jam colligitur, duas rectas non posse sibi occurrere nisi in uno puncto, seu duas rectas, quae habeant duo puncta A et B communia, productas coincidere inter se, cum utraque sit locus omnium (atque adeo eorundem) punctorum suae ad puncta A et B relationis unicorum. Atque ita datis duobus punctis determinata est recta, in quam cadunt.

(20) From this it is now obtained that two lines could not intersect each other except in one point, or that two lines, which have two points A and B in common, coincide with each other when extended, since either one is the locus of all points (and for that reason of the same [points]) unique with their relation to the points A and B. And so, given two points, a line is determined on which they fall.

(21) Hinc porro duae rectae, quae scilicet productae non coincidunt, non possunt habere segmentum commune. Nam si segmentum commune AB (fig. 69) habeant, duo etiam puncta minimum habebunt communia $A.B$, ergo productae coincident.

(21) Hence, again, two lines which, extended of course, do not coincide, cannot have a segment in common. For if they have a segment AB in common (fig. 69), then they also have at least the two common points $A.B$; therefore, [being] extended, they coincide.

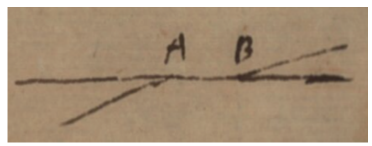

Figure 5

(22) Similiter duae rectae non possunt claudere spatium, alioqui bis sibi occurrerent, adeoque duo puncta communia haberent (fig. 70).

(22) Similarly, two lines cannot enclose space, otherwise they will intersect each other twice, and thus they will have two common points (fig. 70).

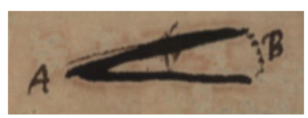

Figure 6

(23) Ita ex nostra rectae definitione demonstrantur Axiomata,

(23) Thus the Axioms that Euclid assumes without demonstration

[4]In general, every point in a curve not returning onto itself is unique with its relation to two points taken in it. Remark by Leibniz.

quae Euclides sine demonstratione circa rectam assumsit.

concerning the line are demonstrated from our definition of a line.

(24) *Circulus* fit rectae circa unum punctum quiescens motu in plano. Extremum quiescens erit centrum, linea ab altero extremo descripta erit circumferentia.

(24) A *circle* is produced by the motion of a line around a single point remaining stationary in the plane. The stationary extremum is the center, and the curve described by the other extremum is the circumference.

(25) Itaque (fig. 71) omnia circumferentiae puncta, ut X, sese eodem modo habebunt ad centrum C, seu omnia X se habebunt ad C, ut A ad C. Quod ut calculo nostro exprimatur, si sit $X.C \simeq A.C$, erit $\bar{X}$ *circuli circumferentia*.

(25) Therefore (fig. 71) every point of the circumference, say X, will have the same relation to the center C, or i.e. every X will be to C as A is to C. As expressed in our calculus, this is: if $X.C \simeq A.C$, then $\bar{X}$ will be the *circumference of the circle*.

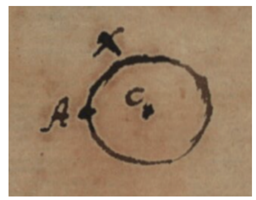

Figure 7

(26) Locus omnium punctorum eodem modo se ad duo puncta habentium est *recta*. Sint duo puncta C et D (fig. 72), sitque locus $\bar{X}$, cujus quodlibet punctum X eodem modo se habeat ad C quo ad D; dico $\bar{X}$ esse rectam. — Quod ut demonstretur, in loco $\bar{X}$ sumantur duo puncta A et B, ducatur recta $\bar{Z}$ per A et B; ea determinata est ex ipsis $A.B$ per § 20. Jam $A.B.C \simeq A.B.D$ ex hypothesi, quia A et B cadunt in $\bar{X}$; ergo (per axioma § 3) etiam recta per $A.B$ seu $\bar{Z}$ eodem modo se habebit ad C ut ad D, seu erit $\bar{Z}.C \simeq \bar{Z}.D$; jam etiam $X.C \simeq X.D$ ex hypothesi, ergo conjungendo $X.\bar{Z}.C \simeq X.\bar{Z}.D$. Ergo punctum X non potest esse ab uno latere rectae z, veluti (si placet) a latere D; ita enim se aliter haberet ad rectam $\bar{Z}$ et ad D, quam ad rectam $\bar{Z}$ et ad C, itaque necesse est X cadere in $\bar{Z}$ seu omne X erit Z, unde et $\bar{X}$ cadet in $\bar{Z}$, quod erat demonstrandum.

(26) The locus of all points having the same relation to two points is a *line*. Let C and D be two points (fig. 72), and the locus be $\bar{X}$, any point X of which has the same relation to C that it has to D; I claim that $\bar{X}$ is a line. In order to demonstrate this, let two points A and B be taken in the locus $\bar{X}$, and the line $\bar{Z}$ is drawn through A and B; it is determined from $A.B$ by §20. Now $A.B.C \simeq A.B.D$ by hypothesis, since A and B fall in $\bar{X}$; therefore (by an axiom in §3) the line through $A.B$ or i.e. $\bar{Z}$ also will have the same relation to C as to D, or i.e. $\bar{Z}.C \simeq \bar{Z}.D$; and now again $X.C \simeq X.D$ by hypothesis, and then by conjoining, $X.\bar{Z}.C \simeq X.\bar{Z}.D$. Therefore the point X cannot be on one side of the line $\bar{Z}$, such as (if you will) on the side of D; then indeed it would relate differently to the line $\bar{Z}$ and to D, than to the line $\bar{Z}$ and to C; and so necessarily X falls in $\bar{Z}$ or i.e. every X will be Z, whence also $\bar{X}$ falls in $\bar{Z}$, which is what we wanted to prove.

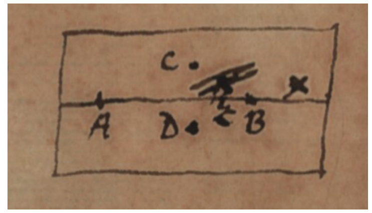

Figure 8

(27) Hic ergo specimen calculi habuimus non inelegans ad praescriptum § 4. Nempe quia $X.\bar{Z} \simeq X.\bar{Z}$, quod est identicum, et $X.C \simeq X.D$ ex hypothesi, et $\bar{Z}.C\ \bar{Z}.D$, quod probatum dedimus ex natura rectae, ex his binionibus omnibus singulatim respective congruentibus sequitur congruere et conflatas inde terniones seu conjungendo esse $\underbrace{X \smile \bar{Z} \smile C} \simeq \underbrace{X \smile \bar{Z} \smile D}$.

(28) Hinc si $X.C \simeq X.D$, erit $\bar{X}$ *recta*, quae congruentia permagnae est utilitatis in calculo nostro. Et vicissim si $\bar{X}$ sit recta, oportet existere puncta qualia C et D, ut locum habeat congruentia.

(29) Fieri nequit, ut recta eodem modo se habeat ad tria plani puncta seu ut sit $X.C \simeq X.D \simeq X.E$ (fig. 73). Nam si hoc esset, foret conjungendo $X.C.E \simeq X.D.E$, ergo E non potest esse ab alterutro latere. Sed idem E non potest esse in $\bar{X}$, ita enim etiam C et D forent in x, et hoc amplius, coinciderent cum E, alioqui punctum aliquod rectae (nempe ipsum E) se aliter haberet ad C et ad D quam ad E, ergo punctum E praeter C et D nuspiam reperiri potest.

(30) Circulus circulo non occurrit in pluribus quam duobus punctis. Sint duae circumferentiae circulares $\bar{X}$ et $\bar{Z}$ (fig. 74), dico eas non posse secare nisi in duobus punctis, velut L et M. Nempe ipsius $\bar{X}$ centrum sit A, ipsius $\bar{Z}$ centrum sit B. Quia jam L est X et M est X, erit $L.A \simeq M.A$, et quia L est

(27) Here we have a not inelegant sample of the calculus following the rule of §4. Indeed, since $X.\bar{Z} \simeq X.\bar{Z}$, which is an identity, and $X.C \simeq X.D$ by hypothesis, and $\bar{Z}.C \simeq \bar{Z}.D$, which we proved from the nature of the line, from all these congruent doubles considered individually, it follows that the triples fused from them are also congruent, or that by conjoining, $\underbrace{X \smile \bar{Z} \smile C} \simeq \underbrace{X \smile \bar{Z} \smile D}$.

(28) Hence if $X.C \simeq X.D$, then $\bar{X}$ will be a *line*, which congruence is of the utmost utility in our calculus. And conversely, if $\bar{X}$ is a line, there should exist points of such kind as C and D, so that a congruence has [that] locus.

(29) It cannot happen that a line has the same relations to three points of the plane, or i.e. that $X.C \simeq X.D \simeq X.E$ (fig. 73). For if this were so, then by conjoining, also $X.C.E \simeq X.D.E$ would hold, and thus E cannot be to either side. But the same E cannot be in $\bar{X}$, for indeed then also C and D would be in $\bar{X}$, and moreover, they would coincide with E, otherwise some point of the line (of course, E itself) would relate differently to C and to D than to E, therefore a point E cannot be found in addition to C and D.

(30) A circle does not intersect a circle in more than two points. If two circular circumferences be $\bar{X}$ and $\bar{Z}$ (fig. 74), I say they cannot cross except in two points, such as L and M. Indeed, let the center of $\bar{X}$ be A, the center of $\bar{Z}$ be B. Since now L is X and M is X, then $L.A \simeq M.A$, and since L is Z and M is Z, then $L.B \simeq M.B$, both of

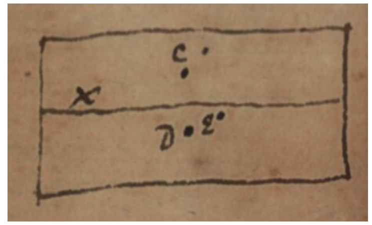

Figure 9

Z et M est Z, erit $L.B \simeq M.B$, utrumque ex natura circumferentiarum, quibus puncta sunt communia per § 25. Ergo conjungendo $L.A.B \simeq M.A.B$. Sit $\bar{Y}$ recta per $A.B$, utique etiam $L.\bar{Y} \simeq M.\bar{Y}$ per axiom. § 3; sed si daretur praeterea aliud duabus circumferentiis commune punctum N, haberemus $L.\bar{Y} \simeq M.\bar{Y} \simeq N.\bar{Y}$, seu recta $\bar{Y}$ se eodem modo haberet ad tria puncta L, M, N, quod fieri nequit per praecedentem. Hinc sequitur, tribus datis punctis circumferentiam, cui insint, esse determinatam, cum pluribus simul inesse non possint.

them from the nature of the circumferences to which the points are common, by §25. Then by conjoining, $L.A.B \simeq M.A.B$. Suppose $\bar{Y}$ is the line through $A.B$, so that now $L.\bar{Y} \simeq M.\bar{Y}$ by the axiom of §3; but if there is some additional point N common to the two circumferences, we would have $L.\bar{Y} \simeq M.\bar{Y} \simeq N.\bar{Y}$, or i.e. the line $\bar{Y}$ would have the same relation to the three points L, M, N, which cannot happen by the preceding. Hence it follows that by three given points a circumference is determined, in which they in-are; since they cannot simultaneously in-be in multiple [circumferences].

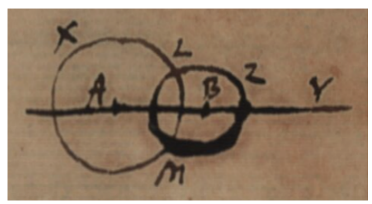

Figure 10

(31) Si circulus circulum tangat (fig. 75), punctum contactus in eadem est recta cum centris. Centra sunto A et B, et punctum contactus C, ubi scilicet duo puncta occursus coalescunt. Itaque (per praecedentem) circulus circulo praeterea non occurrit, alioqui forent puncta occursus tria. Punctum ergo contactus duobus circulis solum commune est. Ajo, id esse in recta per $A.B$. Quod patebit per § 19, si ostendatur, esse suae ad $A.B$ relationis unicum. Esto aliud, si fieri potest,

(31) If a circle is tangent to a circle (fig. 75), the point of contact is in the same line as the centers. Let the centers be A and B, and the point of contact C, where of course two points of intersection merge. Thus (by the preceding) the circle does not intersect the circle in addition [to C], otherwise there will be three intersection points. Therefore a single point of contact is common to the two circles. I claim it is in the line through $A.B$. This is apparent from §19, if it is shown to be unique of its relation to $A.B$. Let there be another, if it can happen, F, and

F et debet esse $F.A.B \simeq C.A.B$; ergo divellendo et $F.A \simeq C.A$, itemque $F.B \simeq C.B$; ergo F cadet in ambas circumferentias, atque adeo vel coincidet cum C, vel C non erit solum commune, quod est absurdum.

$F.A.B \simeq C.A.B$ should hold; therefore by disjoining, both $F.A \simeq C.A$, and likewise $F.B \simeq C.B$; therefore F falls in both circumferences, and thus either they coincide with C, or C is not alone in common, which is absurd.

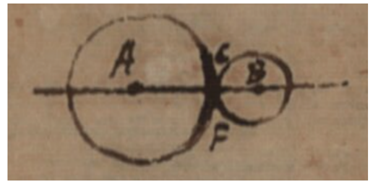

Figure 11

(32) Recta et circulus non possunt sibi occurrere in plus quam duobus punctis. Sint L et M (fig. 76) in recta $\bar{X}$, itemque in circumferentia $\bar{Z}$ circa C, dico praeter L et M non posse dari punctum N. Sumatur D eodem modo se habens ad rectam $\bar{X}$, ut C ab altera parte per § 17. Ob circulum est $L.C \simeq M.C \simeq N.C$, ergo quia puncta L, M, N sunt in recta eodem modo se habente ad D, quo ad C, etiam erit $L.D \simeq M.D \simeq N.D$; ergo conjungendo $L.C.D \simeq M.C.D \simeq N.C.D$. Sit $\bar{Y}$ recta per $C.D$, ergo (per axiom. § 3.) fiet $L.\bar{Y} \simeq M.\bar{Y} \simeq N.\bar{Y}$, seu recta $\bar{Y}$ se eodem modo habebit ad tria puncta L, M, N, quod fieri nequit per § 29.

(32) A line and a circle cannot intersect each other in more than two points. If L and M (fig. 76) are in line $\bar{X}$, and likewise in circumference $\bar{Z}$ around C, I say that there could not be another point N besides L and M. Let D be taken with the same relation to the line $\bar{X}$ as C on the other side, by §17. On account of the circle, $L.C \simeq M.C \simeq N.C$ holds, and therefore since the points L, M, N in the line have the same relation to D as to C, then also $L.D \simeq M.D \simeq N.D$; therefore by conjoining, $L.C.D \simeq M.C.D \simeq N.C.D$. Let $\bar{Y}$ be the line through $C.D$; then (by the axiom of §3) $L.\bar{Y} \simeq M.\bar{Y} \simeq N.\bar{Y}$ will hold, or i.e. the line $\bar{Y}$ will have the same relation to the three points L, M, N, which cannot happen by §29.

Atque ita fundamentalia rectae et circuli exposuimus, quomodo scilicet occurrere sibi possint haec loca: recta rectae, circulus circulo, recta circulo, quorum occursibus caetera determinantur. Unde consequens est, caetera quoque calculo nostro tractari posse.

And so we have expressed the fundamentals of the line and the circle, that is to say, how these loca can intersect: a line with a line, a circle with a circle, a line with a circle, by the intersections of which others are determined. The consequence of this is that the others can also be handled by our calculus.

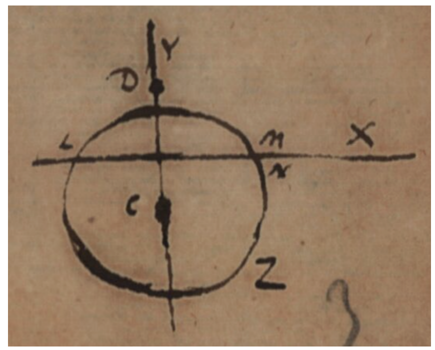

Figure 12