Characteristica Geometrica

Gerhardt, *Mathematical Writings* vol. V, pp. 141–68.
Echeverría, pp. 142–232.

G.W. Leibniz, 10 August 1679

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(1) Characters are certain things that express the relations other things have among themselves, and which are easier to handle than the others. And so every operation taking place in the characters corresponds to a certain proposition in the things, and we can often defer consideration of the things themselves until the very end. Once we have found in the characters what we sought, we will easily find the same among the real things, by the agreement originally set up between the things and the characters. In this way machines could be exhibited by their measurements, and solid bodies represented on a planar surface so that no point of the body lacks a corresponding point assigned on the planar surface according to the laws of perspective. And then if we perform some geometrical operation on the diagram on the planar surface, like working out graphically some scene, this operation may yield some other point of the diagram from which we may easily find a corresponding point on the real thing. And thus, we may solve in the plane problems concerning the measures of solid bodies.

(2) Now, the more exact the characters are, i.e., the more relations of things they exhibit, the more useful they can be, and if they exhibit all the relations among the real things, such as do the characters I use for Arithmetic, there is nothing in reality that could not be apprehended by the characters. The characters of Algebra are as successful as Arithmetic characters because they signify indefinite numbers. And since nothing in Geometry cannot be expressed by numbers, once a certain Scale of equal parts has been set, it happens that whatever can be handled by Geometry is susceptible of calculation.

(3) Now we certainly know that the same objects can be related in various ways in characters, and some ways are more convenient than others. Thus a writing surface with a hump could be used, with the method of perspective to mark out a body, but a planar surface is nonetheless superior. And nobody can fail to see that today's characters for numbers, which are called Arabic or Indian, are more suitable for calculation than the old Greek or Latin ones, although those too could be used for calculation. The same also enters Geometry, namely, that neither can Algebraic Characters express everything that should be considered in space (indeed, they presuppose Elements already discovered and demonstrated), nor can they signify directly the situs itself of points, but rather they trace things out very obliquely with magnitudes. Whence it is exceedingly difficult to express in calculation what is exhibited by a figure, and yet more difficult to effect in a figure what is discovered by calculation. So the constructions exhibited by calculation are for the most part marvellously distended and awkward, as I showed elsewhere in the case of the following problem: to find the triangle with given base, altitude, and vertex angle.

(4) Indeed, I notice that Geometers tend to attach certain descriptions to their figures which explain them, so that what can't be recognized from the figure alone, such as equality or proportionality of lines, at least may be understood from the attached words. Usually they go further and express in words even what is manifest from the figure itself, so that the

argument would be more strict, and nothing would depend on sense or even imagination, but everything would be done by logical reckoning; or else that the figures could be drawn from the description or be restored if by chance they are lost. And even if they don't attend to this exactly enough, still they give us certain traces of a Geometric Characteristic; as when Geometers say "the rectangle ABC", they understand in fact the one given by guiding AB over BC at right angles. When they say "AB equals BC equals AC", they express an equilateral triangle. When they say "of the three AB, BC, AC, some two equal the third", they designate that all three A, B, C lie in the same line.

(5) So I notice that only by this use of letters to denote points of a figure are we able to designate some of the figure's properties; I began to wonder whether we couldn't designate all of the relations of the points of a figure by the same letters, so that the whole figure might be exhibited in the characteristic, and we may discover by the mere arrangement and transposition of those letters what is barely (or not at all) apparent from the numerous lines we draw. Usually confusion arises with the figure from the multiplicity of drawn lines, particularly when we are still trying things, while on the contrary by means of characters we can try things without risk. But there is something greater behind this, for we can express with these characters the true definitions of everything that is handled by Geometry, and we can continue an analysis to the principles, the axioms and postulates of course, whereas Algebra for its part would not be adequate but would require propositions discovered through Geometry; when they try to render everything in terms of those two propositions — the one which adds two rectangles into one, and the other which compares similar triangles — most things have to be distorted from the natural order.

(6) Having once demonstrated the Elements in our characters, we shall easily see how to discover solutions to problems that simultaneously exhibit the constructions and demonstrations using lines. On the contrary, after Algebraists find the values of their unknowns, they still need to worry about the constructions, and after discovering the constructions, they still seek demonstrations in terms of lines. And so I am amazed that people haven't considered that, given the possibility of demonstrations and constructions in terms of lines that are devoid of all calculation, and much shorter, surely the invention also could be done in terms of lines? Logical regression must exist in linear no less than in algebraic Synthesis. Indeed, the reason we have not yet seized upon an analysis in terms of lines is doubtless nothing else than that Characters haven't been contrived by which the situs of points can be represented directly. For without characters it is difficult to move freely among a great multiplicity and confusion of objects.

(7) Wherefore, if we can once and for all represent exactly figures and bodies by letters, not only Geometry will be marvellously improved, but also optics, kinematics, mechanics, and in general whatever is subject to the imagination we could handle by a sure method, as it were by analysis. And with a marvelous art we will render the invention of machines in the future no more difficult than constructions for problems in Geometry. Yes, without any labor or cost, even very complex machines, and indeed natural objects, could be delineated without figures, that they may be transmitted to posterity and figures formed from their descriptions with the greatest precision whenever we wish. Much is lost with the current difficulty and cost of delineating figures, and people are deterred from describing things they have explored and that are of public use. Up to now the terms we have are indeed neither sufficiently exact nor sufficiently apt for the compilation of descriptions; which is evident, for instance, from the case of botanists, and also those who illustrate armaments and insignia. With characters we could easily denote the other qualities by which points, which are considered similar in Geometry, do differ. And then finally, at last, there will be hope of penetrating the secrets of nature, when we can tranquilly and securely deduce everything by a sure method from some given data, while another would wrest it from the same data by sheer power of innate intelligence and imagination.

(8) Now since no such thing has occurred to anybody, as far as I know, and no helps appear anywhere, I am forced to attack the matter afresh from the very beginning; the difficulty is more than anyone could believe without trying it themselves. And so I have attacked

the thing on more than ten different occasions, in different ways, which were all tolerable and had various advantages, but did not satisfy my scruples. Finally, after cutting out very much I arrived at the greatest simplicity, I realized, when I could demonstrate everything from the characters alone with no extra assumptions. But I lingered for a long time even after finding the true characteristic reckoning for this, since I saw that I should start from Elements that are easy in themselves and already familiar from elsewhere. Arranging them so carefully could be rather unpleasant; I proceeded nonetheless, overcoming the irritation, until finally I was led out to better things.

(9) Now we see that to treat everything in order, we must first consider Space, that is, pure absolute Extension. Pure, I say, from material and change, and absolute, meaning without limit and containing all extension. So all points are in the same space and can be related to each other. Whether this space is some actual thing, though distinct from material, or only a consistent impression or phenomenon, does not matter here.

(10) Next will be consideration of a Point, which is that among all things pertaining to Space or Extension which is most simple; as Space contains absolute Extension, so a point expresses that which is maximally limited in extent, namely, simple situs. From which it follows that a point is minimal, it lacks parts, and all points are congruent to each other (or are able to coincide), hence also are similar and, if I may say so, are equal.

(11) If two points are understood simultaneously to exist or rather to be perceived, their very relation to each other is thereby offered for consideration, which varies from one to another pair of points, namely the relation of place, or situs, which two points have to each other, by which their distance may be understood. That distance of the two is nothing but the smallest quantity of a path from one to the other. And if a pair of points $A \,.\, B$, preserving the situs among themselves, is congruent to, or can be overlaid on, another pair of points $C \,.\, D$ also preserving the situs among themselves, then the situs or rather distance of these two is the same as the distance of those two. For those things are congruent that are able to coincide one with the other without any change occurring within either of them. If the pairs $A \,.\, B$ and $C \,.\, D$ are coincident they have the same distance, hence also if they are congruent, since then, without any change within $A \,.\, B$ or within $C \,.\, D$, they could be returned to coincidence.

(12) Now a Path (by which we also defined distance) is nothing but a locus of continuous succession. And the path of a point is called a Curve. From this one can discern that the extremes of a curve are points, and that any part of a curve is a curve, or terminates in points. And then a path is a certain continuum, since any part has common extremes with the previous and subsequent parts. From this it follows, if I may add in passing, that if a certain curve is drawn out on some surface, one cannot cause another curve on the same surface to pass continuously between the two extremes of the first curve without intersecting the first curve.

(13) The path of a curve of this type, such that its points do not always follow one another in succession, is a Surface; and the path of a surface with points not always following each other, is a Body. A body, however, cannot be moved, without its points following each other (as must be demonstrated in its place), and for that reason does not produce a new dimension. It is thus apparent that there is no part of a body whose boundary is not a surface, and no part of a surface whose boundary is not a curve. It is also apparent that the extreme of a surface (or, similarly, of a body) closes on itself, i.e. it is a boundary.

(14) Now, given two points, the simplest possible path through one and the other in turn is eo ipso already determined: otherwise their distance, and indeed their situs, would not be determined. But this curve, which is determined simply by the two points through which it passes — not surprisingly, in order to offer for consideration the thing itself that was posited as passing through two given points, we will call it a straight line. And however far it is drawn out, it will be called one and the same line. From this it follows that the same two points cannot have two distinct lines in common, unless the two lines coincide after being drawn out far enough. Consequently, two lines cannot have a common segment (otherwise they would have in common the two extremes of this segment), neither can they

enclose a space or (i.e.) create a boundary that closes on itself; otherwise one line diverging from the other would return to it and thus they would meet in two points.

A part of a line, also, is a line, for it is determined by those two points alone by which alone the whole is determined. Determined, I say, meaning that all of its points that are being considered, or rather traversed, are offered by consideration of those two points alone. From this it is apparent that if $A.B.C$ and $A.B.D$ are congruent, and $A.B.C$ are declared to lie in a line, then C and D coincide. Or if a point is unique in having the relation to two points that it has, then all three will lie in one line. On the contrary, if there are more than two points related in the same way to three or more given points, then those three must lie on the same line, and the others outside it; the reason is that things related in the same way to the determiners eo ipso are related in the same way to the things determined, so three or more points in the same line can be supplied instead of two. However, I require more than two points having the same relation (for if there are just two, each of which has the same relations to the three, then we can conclude only that the three lie in the same plane, not that they lie on the same line).

A line is uniform, moreover, because of its simplicity; that is, it has parts similar to the whole. And a line is similar to any other line, since a part of one is congruent to the other, while a part is similar to the whole. And for a straight line a concave and a convex side cannot be distinguished, or rather, the line does not have two dissimilar sides, or what amounts to the same thing: if two points are assumed outside the line that relate in the same way to the extrema of the line (or any two points on the line), then they relate in the same way to the whole line, or to any point on the line, no matter which side of the line, in the end, those two points outside the line are on. The reason for this is that when things relate in the same way to the points that determine some extension, they must necessarily relate in the same way to the whole extension.

Finally, a line is the minimal [path] from one point to another point, and accordingly the distance of the points is the quantity of the line they intercept. For the minimal line is certainly determined in magnitude solely by the two points; but it is also determined in position, for neither can there be in space, in the absolute sense, multiple minimal paths from a point to a point (as in a spherical surface there are many minima from one pole to the other). For if it is absolutely minimal, the extremes cannot be drawn apart while maintaining the quantity of the line, and therefore neither can the extrema of its parts (for the parts also are necessarily minimal between their own extremes) while preserving the quantities of the individual parts, and therefore neither while preserving the quantity of the whole.

Now if the two extremes of the line remain unmoved, but the line itself is transformed, it is necessary that some of its points be drawn apart from each other. And so if the extremes of a line are fixed, and the minimal quantity between the two points is preserved, it cannot be transformed in any way, and so one cannot give two incongruent, dissimilar minima between two points. In this way, any two paths between two points that are both minimal must be congruent to each other. Now one such minimum is a straight line (as shown above), so therefore the other minimum must be a line. And two lines between two points coincide. And so a path between two points is not minimal unless it be unique.

(15) Now this is the simplest method of generating a line. If a solid body is itself moved, though some two of its points are fixed and unmoved, then all stationary points of the body lie on the line passing through the two fixed points. Clearly those points have the same place determined from the two given fixed points, or rather they cannot be moved with the whole solid existing and the two points remaining fixed; whereas other points outside the line can change their place while preserving the same relation to the two fixed points. One inconvenience here is that the line described in this way is not permanent.

A straight line can be generated in another way, if we are given a curve which is flexible but cannot be extended to a greater length. For if its extremes are drawn apart as much as possible, the flexible curve will be transformed into a straight line.

In the same way, the properties of a plane or a Circle or a Triangle can be deduced from the established definitions by a certain natural order of thought. For we have examined the straight line just as a model.

(16) All of this can be followed easily enough in one's head, even though no figures were drawn (except by imagining), and neither were any characters put to use other than words. But since in long, drawn-out argumentation, words (such as I have until now tended to adopt) are not sufficiently exact, and the imagination is not sufficiently adept, Geometers have until now used figures. Except that drawing figures is often difficult, and the delay allows the best thoughts to dissipate, and sometimes the figure gets muddled on account of the multiplicity of points and lines, particularly when we are still trying things out and investigating; so I thought that characters could fruitfully be employed in the following way.

(17) Space itself, or rather extension (that is, a continuum whose parts exist simultaneously), cannot conveniently be designated any other way, that I can see, except by points. This is because we are proposing to exactly express drawings of figures, and in these we observe nothing but points and certain continuous tracks from one point to another, in which we could take infinitely many points arbitrarily. Therefore we will express determinate points simply by letters, such as A, and likewise B.

(18) Continuous tracks, on the other hand, we will express through certain indefinite or arbitrary points, a certain order being assumed, while making clear nevertheless that other points between, beyond, or before these could always be taken. Thus ${}_3b_6b_9b$ will signify for us an entire track, each point of which will be called b, and in which we have chosen arbitrarily two parts, the extremes of one of which are ${}_3b$, ${}_6b$, and the extremes of the other are the points ${}_6b$, ${}_9b$.

From this it is clear that these two parts are contiguous [continuous], since they have a common point ${}_6b$ and a division between them was made arbitrarily. This trace, in which the common extremes of two parts is nothing but a point, is called a Curve, and it can also be represented as the motion of a point b, which runs along a certain path, or which is understood to leave traces, as many as there are distinct points e.g. ${}_3b$, ${}_6b$, ${}_9b$. Hence a curve may be called the path of a point. A path, moreover, is a locus of continuous succession. It may be designated also in the following abbreviated fashion: Curve ${}_{\overline{y}}b$, designating by the letter ${}_{\overline{y}}$ (or another) some arbitrarily chosen ordinal numerals taken together. Whereas when we write yb without the mark over y, we understand each point of the curve ${}_{\overline{y}}b$, taken distributively.

In the same way, one can make certain tracks with parts that are connected by curves, or which we understand to be described by the motion of a curve such that its points do not follow in succession, but rather come into new positions. This track, or path of a curve, is called a surface. In Figure 6, let us suppose, of course, that the curve called (above) ${}_3b_6b_9b$ moves, and that its first position is named ${}_{3_3}b_{3_6}b_{3_9}b$, and its other, subsequent is ${}_{6_3}b_{6_6}b_{6_9}b$, and yet another [is called] ${}_{9_3}b_{9_6}b_{9_9}b$, and the resulting surface ${}_{3_3}b_{3_6}b_{3_9}b,{}_{6_3}b_{6_6}b_{6_9}b,{}_{9_3}b_{9_6}b_{9_9}b$ we will then designate with the abbreviation ${}_{\overline{zy}}b$.

(19) It is apparent here also that just as motion of the curve ${}_{\overline{y}}b$ following the points ${}_{\overline{z}}b$ describes the surface ${}_{\overline{zy}}b$, so again motion of the curve ${}_{\overline{z}b}$ following the points ${}_{\overline{y}b}$ describes the same surface ${}_{\overline{yz}}b$. Now ${}_{zy}b$ will stand for each place of the point b, not collectively but rather distributively; and ${}_{z\overline{y}}b$ stands for some curve ${}_{\overline{y}}b$ in the surface ${}_{\overline{zy}}b$, all of them taken non-collectively, again, but rather distributively.

(20) Nor does the form matter of the curves themselves that are moved; or even those [curves] according to which the motion occurs, or rather, those which are described by one of the points of the moving curve: see Figure 7.

It can also happen that during the motion, the moving curve itself changes form, as the curve ${}_{\overline{z}}b$ in Figure 7 already mentioned. This could be understood more clearly by thinking of the surface described by a bow, if the whole bow moved while firing, for example if it fell to the ground. The moving curve could even lose certain parts during the motion, which are separated from it either in reality or in the mind, as is shown in Figure 8.

It can happen also that one or more points in the moving curve (e.g. ${}_3b$) is stationary during the motion, and its positions, [although] expressed as if distinct (e.g. as ${}_{3_3}b_{6_3}b_{9_3}b$), actually coincide with each other, as one realizes by considering Figure 9. But all these variations and many others also can be designated by characters, as will be shown in its place.

(21) Now, as the Track that is called a Surface is described by the motion of a curve, so also the Track that is called a Solid, or body, is described by the motion of a surface (such that its parts, or points, do not everywhere follow each other in succession). This may be satisfactorily understood from one example (Figure 10): If the (linear) curve ${}_{\overline{z}3}b$ (that is, ${}_{3_3}b\,{}_{6_3}b\,{}_{9_3}b$) remains fixed in the (rectangular) surface ${}_{\overline{zy}}b$, that is

$$\begin{pmatrix} {}_{3_3}b & {}_{6_3}b & {}_{9_3}b \\ {}_{3_6}b & {}_{6_6}b & {}_{9_6}b \\ {}_{3_9}b & {}_{6_9}b & {}_{9_9}b \end{pmatrix},$$

and this same surface is moved, then its motion will describe the solid

$$\begin{pmatrix} 3_{3_3}b & 3_{3_6}b & 3_{3_9}b & 3_{6_3}b & 3_{6_6}b & 3_{6_9}b & 3_{9_3}b & 3_{9_6}b & 3_{9_9}b \\ 6_{3_3}b & 6_{3_6}b & 6_{3_9}b & 6_{6_3}b & 6_{6_6}b & 6_{6_9}b & 6_{9_3}b & 6_{9_6}b & 6_{9_9}b \\ 9_{3_3}b & 9_{3_6}b & 9_{3_9}b & 9_{6_3}b & 9_{6_6}b & 9_{6_9}b & 9_{9_3}b & 9_{9_6}b & 9_{9_9}b \end{pmatrix}$$

One should note here that because the line ${}_{\overline{z}3}b$ is fixed, the points ${}_{3_{3_3}}b$, ${}_{6_{3_3}}b$, and ${}_{9_{3_3}}b$ coincide (thus their position is given as just ${}_{3_3}b$), and so also the points ${}_{3_{6_3}}b$, ${}_{6_{6_3}}b$, ${}_{9_{6_3}}b$, from which we also have just ${}_{6_3}b$; and lastly in the same way the points ${}_{3_{9_3}}b$, ${}_{6_{9_3}}b$, ${}_{9_{9_3}}b$ coincide and are expressed simply by ${}_{9_3}b$. Now this solid we will express by abbreviation in this way: ${}_{\overline{xzy}}b$ and one of its surfaces or a locus of ${}_{\overline{zy}}b$ we will express in this way: ${}_{x\overline{zy}}b$ (thus exhibiting a section of this cylindrical region, or rather of this solid, made by a plane through the axis). We can also take one of its surfaces ${}_{\overline{xy}}b$ in this way, ${}_{z\overline{xy}}b$ (thus exhibiting a section of this cylindrical region according to the base, that is, a section by a plane parallel to the base); and in the same way ${}_{y\overline{xz}}b$ (thus exhibiting a section of this cylindrical region through another cylinder having a common axis with this one). Likewise other sections of the same Figure can be discerned, since one can devise infinitely many methods of generating it by motion or else of resolving it into parts according to some fixed law. Clearly all the other variations in production or resolution of a surface that we indicated a little earlier are even more relevant for solids. And finally we must demonstrate in its own place that another dimension higher than a solid, or rather the track described by such a motion of this solid that its points don't everywhere follow after one another, cannot be obtained.

(22) Again the tracks themselves, or rather the loci of certain indefinite points, are determined by certain fixed points, or else by certain Laws, according to which the other indefinite points come in order into consideration from the few fixed points, and the track itself can be generated, or rather described.

Before explaining this, let us clarify certain symbols that will be used subsequently. First, then, it can happen that two or more names of distinct form really refer to just one thing, or locus, that is, a point or line or another track, and thus they are said to be the same or to coincide. So, if there are two lines AB and CD, and points A and C are one and the same, we will designate it thus: $A \infty C$, that is, A and C coincide.

This will be especially useful for designating extremal points (and other extremes) shared by distinct Tracks. For indeed the same point, or rather extremum, has its designations according to the one track and according to the other. If we say $A.B \infty C.D$, the sense will be at once $A \infty C$ and $B \infty D$. Likewise for more than two; and the same order should be observed on each side of the statement.

(23) Now if the two do not in fact coincide, that is, they do not simultaneously occupy the same place, but nevertheless can be placed onto each other, and without any change in either one considered in itself, the one can be substituted into the place of the other; then those two are said to be congruent, as $A.B$ and $C.D$ in Figure 11. And so it happens with $A.B \gamma C.D$; in that manner $A.B \gamma C.D$ in Figure 12, that is, while preserving the situs between A and B and likewise the situs between C and D, $C.D$ can be placed onto $A.B$, i.e., one can simultaneously place C onto A and D onto B.

(24) If two extensions are not congruent, but can nonetheless be brought to congruence without any change of mass or *quantity*, that is, retaining all of the same points, even if a certain transmutation or transposition is needed of the parts or points, then they are said to be *equal*. Thus in Figure 13 the square $ABCD$ and the isosceles triangle EFG having base EG double the side AB of the square, are equal: translate FHG into EGF, since $EGF \gamma FHG$. So EFG is made equal to $EHFG$. Now $EHFG \gamma ABCD$, thus EFG equals $ABCD$. Hence more generally if $a \gamma c$ and $b \gamma d$ then we would have $a + b \sqcap$ or equals $c + d$. Even further: if $a \gamma e, b \gamma f, c \gamma g, d \gamma h$, we have: $a + b - c + d \sqcap e + f - \gamma + h$. Or if two sums are made out of certain parts by one and the same manner of addition or subtraction, and the parts of the one are congruent to the parts of the other coming together in the same way to comprise the whole, each part of the one sum congruent to the corresponding one of the other sum, in order; then the two resulting sums, which won't always be congruent, nevertheless will be equal. And so the argument from congruence to equality is established by the definition of equality. Alternatively, things are equal when they have the same magnitude. Of course the magnitude is the multiplicity of parts congruent to a certain thing, or measure, as in Figure 14 if we have two magnitudes a and b, and we are given a third thing c which is *twice* a + *thrice* b. Then clearly its magnitude is expressed by the multiplicity of parts congruent either to a or to b, and so things that can be brought to congruence by no addition or subtraction are necessarily equals.

(25) Now to return to this matter more deeply, we should explain what is part and whole, what is homogeneum, what is magnitude, and what ratio. *Part* is nothing but what is required immediately by the whole, yet distinct from it (or so one cannot be predicated of the other), directly occurring with its corequisites. Thus AB is required by AC, that is, if it were not for AC there would be no AB; they are also distinct, since of course AC is not AB; in another way we have that rational is a requirement of man, but since man is rational, therefore man and rational (which is a requirement of man) are the same; even if they differ in expression, in re they do come together. The part is immediately required; indeed, the connection between AB and BC does not depend upon any inference or causal connection; rather it is apparent per se from the hypothesis of the assumed whole. It is, moreover, directly occurring with its corequisites, coming together always according to a fixed mode of consideration. For instance, things we view as Beings, and also things we view as objects of thought, like for instance God, man, virtue, we can consider as parts of the single whole that they comprise. Thus are excluded requisites immediate and indeed distinct, such as rationality in the abstract, which is a requisite of man and distinct; for certainly man is not rationality, viewed here albeit not as coming together with man, but as an attribute: in general one cannot reasonably deny that from these two, man and rationality, one whole could be formed, of which they are the parts. But rationality is not part of man, it being required for man obliquely, or, that is, it doesn't come together in a certain respect with other things which are additionally required of man. But these are more metaphysical [matters], and are not brought out except to oblige those desiring an intimate understanding of the concepts. Most simply defined: Parts are requisites of an individual insofar as they do come together with it.

(26) *Number* is that which has the same relation to unity as that between part and whole or whole and part; in this way fractions and even irrationals are included.

(27) The *Magnitude* of a thing (distinctly conceived) is the number (or composite of numbers) of parts congruent to a certain fixed thing (which is assumed as a measure). So if I know there is a line segment that is equal to twice the side, thrice the diagonal of a certain sufficiently known rectangle, so that I can return to the same thing at will, then I am said to know the magnitude, which would be two parts congruent to the side, three parts congruent to the diagonal.

Although the same thing can be expressed in various ways, according to the various distinct measures we are allowed to take, nonetheless the magnitude always comes out the same, since by another resolution of the measures themselves, the same thing is always reached in the end; and thus various measures already involve that very number coming out the same by resolution. In this way the number three quarters is one and the same with six eighths if the quarter is decomposed into two parts. Such is Magnitude distinctly conceived.

Otherwise *magnitude* is the attribute of a thing through which one can tell whether some given thing is a part of it, or another homogeneum pertaining to the thing, and indeed such that it remains the same if the configuration of the parts is changed.

Or indeed Magnitude is the attribute which remains the same, provided the homogenea pertaining to it remain the same or congruent ones are substituted for them. By 'homogenea pertaining to some thing' I understand not only parts, but also extrema and minima, or points. Namely a curve is created by a certain continuous repetition or motion of a point. Often however a thing is transformed so that there is not a single part of the new figure congruent to a part of the previous one.

I define magnitude in another way below, as that by which two similar things can be distinguished, or that in things which is only discerned by co-perception. But all this comes back to the same thing.

(28) The *ratio of A to B* is nothing but the number by which the magnitude of the same A is expressed if the magnitude of the same B is held to be the unit. From this it is clear that Magnitude differs from ratio as concrete number from abstract number; magnitude is indeed the number of things, of course of parts; ratio is really the number of unities. It is also clear that the magnitude of a thing remains the same, no matter what measure we want to assume through which to express it; the ratio becomes one thing or another if we assume one or another measure. And it is clear (from the definition of division) that if the number expressing the magnitude of A is divided by the other number expressing the magnitude of B (provided the same measure or unity is applied to both), the number that is the ratio of one to the other will result.

(29) Equals are those of which the magnitude is the same, or which without loss or addition can be brought to congruence.

A thing is called *Lesser* which is equal to a part of the other, and indeed that having a part equal to the other is called *Greater*.

Hence the part is less than the whole of which it is a part; of course it is equal to itself. We will moreover use these symbols:

$a \sqcap b$	a equals b
$a \ulcorner b$	a greater than b
$a \urcorner b$	a lesser than b

If a part of one is equal to the whole of the other, the remaining parts of the greater we call the difference. The magnitude, moreover, of the whole is the *sum* of the magnitudes of the parts, or of others equal to its parts.

(30) If two things are homogeneous (or if we can assume some parts in the one equal to parts of the other, and the same can always be done also with the remainder) and there is not a difference between them, i.e. if neither a is greater than b, nor b greater than a, they are necessarily equal. Indeed, let them be transmuted into congruents as far as possible, then certainly either something superfluous will remain in one, or they will become congruent, and therefore they are equal. And so this inference is valid here:

$$a \text{ not } \ulcorner b,\ a \text{ not } \urcorner b, \text{ therefore } a \sqcap b.$$

(31) *Similars* are those which cannot be distinguished by considering them individually in themselves, as two equilateral triangles (in fig. 16). There is not an attribute, not a property we can discover in one which we cannot also find in the other. And calling one of them a, the other b, we notate similarity $a \sim b$. If however they are simultaneously perceived, the difference appears immediately that one is greater than the other. The same thing can happen even if they are not perceived together, only assuming something as an intermediary or a measure that can be applied first to the one, or something in it, and noting how it (or a part of it) is congruent to the measure or its part, then later applying the same measure

to the other. Therefore I customarily say that similars cannot be distinguished except by co-perception.

But you might say that you could still properly distinguish two unequal equilateral triangles if you were to see them in succession.

But of course I am speaking here about discernment, that the mind would note something in one that does not appear in the other, and not about sense and imagination. The reason, more specifically, that the eyes may distinguish two similar yet unequal things is apparent: namely that we retain images of the things perceived before, which, applied to the images of the newly perceived thing, show the differences of these two images by co-perception. And the very back of the eye, the greater or lesser part of which is occupied by the image, serves as a measure.

Finally we are accustomed to always perceiving together other things which we also perceived with the ones perceived previously; from this we note the difference without difficulty, referring the last perceived thing back to them, just as we referred the previous perceived to the same ones.

If we imagined God to shrink everything that appears in us and around us in some room, keeping the same proportions, everything would appear the same, and we could not distinguish the first state from the last, unless of course we emerged from the sphere of things proportionally diminished, that is to say, from our room; then indeed by co-perception, by bringing them before the unshrunk things, the difference would appear. From this it is manifest indeed that Magnitude is that which may be distinguished in things only by co-perception, that is, either by immediate application, the things being either actually congruent or coincident, or else by mediate [application], by the intervention of a measure, which is applied now to one and now to the other, this sufficing for the things to be congruent, that is, to be able to be made coincident by action.

(32) From this one can understand, moreover, that things simultaneously equal and similar are congruent. And similarity we shall denote by this sign: $\sim$, thus $a \sim b$, i.e. a is similar to b. See Figure 17. From this we have an inference: $a \sim b$ *and* $a \sqcap b$. Thus $a\gamma b$.

(33) We also have other inferences:

$$a \between b. \ \textit{therefore}\ a \sqcap b$$
$$a \between b. \ \textit{therefore}\ a \sim b.$$
$$a \infty b. \ \textit{therefore}\ a \between b.$$
$$\ldots\ldots \ \textit{therefore}\ a \sqcap b.$$
$$\ldots\ldots \ \textit{therefore}\ a \sim b.$$

(34) For instance, those which really coincide are certainly congruent; and those congruent are certainly similar, and likewise equal. From this we see that there are three modes, or as it were degrees, of distinguishing things provided with extension and not otherwise differing in their qualities.

The greatest is when they are dissimilar, so for instance when observed each individually in itself, they are easily distinguished by the difference of properties that are observed in them: thus an isosceles triangle is easily distinguished from a scalene, even if they are not seen simultaneously. Indeed if someone required me to see whether some proffered triangle be isosceles or scalene, I wouldn't need to assume anything outside of it, I would simply compare its sides to each other.

But on the contrary, if I were asked to choose the greater of two equilateral triangles, I would need to do so by co-location of the triangles, or by co-perception, as I have explained that no other notable mark of difference could be assigned to them individually.

If in fact two things are not only similar, but also equal, that is if they are congruent, then they cannot be distinguished even if seen simultaneously, except by their locations; that is unless I assume something extra outside of them, and I observe them to have a different situs to the third assumed thing. Finally, if both are simultaneously in the same place, I now have nothing more by which to distinguish them.

And this is the true Analysis of reasoning that we have about these matters, the ignorance of which has led to the fact that a true geometric characteristic has not been established until now. It may be understood from these things, finally, that as magnitude is reckoned when things are understood to be congruent or reducible to congruence, so ratio is reckoned by similarity, that is, when things are reduced to similarity, for then everything will necessarily become proportional.

(35) From these descriptions of coincidents, congruents, equals, and similars, certain inferences may be derived.

Of course, those which are equal, similar, congruent, or coincident to the same thing, are also to each other. Thus:

$a \infty b$	and	$b \infty c$	therefore	$a \infty c$
a ♉ b	–	b ♉ c	–	a ♉ c
$a \sim b$	–	$b \sim c$	–	$a \sim c$
$a \sqcap b$	–	$b \sqcap c$	–	$a \sqcap c$

But these inferences are not valid:

$$a \text{ not } \infty b \text{ and } b \text{ not } \infty c, \text{ therefore } a \text{ not } \infty c,$$

just as in Logic nothing follows from pure negatives.

(36) If coincidents are conjoined to coincidents, or to the same thing, then coincidents will result, as

$$a \infty c \text{ and } b \infty d, \text{ therefore } a.b \infty c.d.$$

But with congruents, this does not follow, because, for example, if $A.B.C.D.$ are points, then of course we always have A ♉ C and B ♉ D; for instance no matter what point is congruent to which. But evidently you cannot therefore say that $A.B$ ♉ $C.D$, i.e. that you can bring A to coincide with C and simultaneously B with D while preserving both the situs $A.B$ and the situs $C.D$. Although, conversely, supposing $A.B$ ♉ $C.D$, it would follow that A ♉ C and B ♉ D, from the meaning of our characters, and moreover it is true if $A.B.C.D$ are allowed to be magnitudes and not just points. On the contrary if they are congruent to each other, from this arises equality, thus: $a + b - c \sqcap d + e - f$ be supposed, then $a\gamma d$ and $b\gamma e$ and $c\gamma f$, since congruents are always equals.

(37) Certainly if congruents are added to or subtracted from congruents in a similar manner, congruents will result. The reason for this is that when congruents are added to congruents in a similar manner, then in fact similars are added to similars in a similar manner (since congruents are similars), therefore they will be similars. They are also, moreover, equals (for congruents added to congruents make equals); now things similar and equal are congruent. Therefore, *if congruents are added to congruents in a similar manner, they will make congruents.* Likewise if they are subtracted.

(38) Whether, moreover, some things are treated similarly, may be understood from our characteristic and the manner in which each thing is described or determined; if no difference can be noted in each one considered singly, then certainly everything will necessarily always come out similar. The noteworthy point is that if things are similar according to one manner of determination (distinct cognition, description), they will also be similar according to another manner. For each manner of determination involves the whole nature of a thing.

(39) The axioms that were used by Euclid, that if equals are added to equals then equals will result, and others of the sort, are easily demonstrated from the fact that equals have the same magnitude, that is, they can be substituted [for each other while] preserving magnitude. If we have $a \sqcap c$ *and* $b \sqcap d$, *then* $+a + b \sqcap c + d$ results; that is, if we write $a + b$, and then substitute equals $c.\ d.$ in place of $a.\ b.$, this substitution will preserve magnitude. And accordingly, those yielding $+c + d$ will have the same magnitude as the first yielding $+a + b$. But this pertains rather to the Algebraic calculus, and has been explained sufficiently, so we will not delay over the rules of magnitude, ratio, and proportion. I will undertake mainly to explain that which involves situs.

(40) I return now to the things that were interrupted in §22, and first to points, and from there I will consider Tracks. Every point is congruent to any point and is therefore equal (if I may speak thus) and similar:

$$A \between B, A \sqcap B, A \sim B.$$

(41) $A.B \between C.D$ means that $A \between C$ and simultaneously $B \between D.$, preserving the situs of $A.B$ and $C.D$. [[$A.B \between A.Y$ is the proposition signifying that, positing two points $A.B$, some third Y can be found (which I denote by this letter since it is indefinite) such that $A.Y.$ and $A.B.$ can be brought onto each other while preserving the situs, namely simultaneously A onto A and B onto Y. Then A remains on A (that is A remains where it is), and B on Y.]]

(42) The sense of the proposition $A.B \between B.A$ is that, supposing two points $A.B.$, you can permute their locations, placing A in the place of B and vice versa, while retaining the same situs between them. This is manifest because the relation of situs that they have pertains to both in the same way, and without assuming external things, no difference can be found after they are permuted. [insert figures 18 and 19] [[The proposition $A.B \between C.Y$ signifies that, given three points A. This is manifest because the relation of situs that they have pertains to both in the same way, and without assuming external things, no difference can be found after they are permuted. From this it is clear that since the relation of situs that they have pertains to both in the same way, one cannot tell any difference after they are permuted without assuming some external things. [insert figures 18 and 19] [[The proposition $A.B \between C.Y$ signifies that, given three points $A.$ $B.$ $C.$ one can discover a fourth $Y.$ having the same situs to one of them $C.$ as the situs of the remaining two $A.B.$ among themselves, or such that A can be brought to congruence with C and B with Y while simultaneously preserving the situs AB and CY. From this it follows that $A.B \between A.Y$ if we suppose $C \infty A$. The reason for this, moreover, arises from the nature of space, in which nothing can be taken which cannot be taken again in [exactly][1] the same way, so that they differ only in place. The same thing is demonstrated by motion thus: translate simultaneously $A.B.$ while preserving situs, and when A is incident upon the place of C, then of course B will be incident upon the place of some point Y. In the same way the proposition $A.B. \between X.Y$ is demonstrated.]]

(43) The proposition $A.B \between X.Y$ signifies that, given two points A and B, another pair X and Y can be found having the same situs among themselves as those two, or such that they can be simultaneously made congruent to those two while preserving the situs in both pairs. It is demonstrated from this: that $L.M$ can be moved, preserving the situs among themselves, and returned to the first $A.B$ and from there to $X.Y.$, and of course $3L.3M. \between 6L.6M$. If $A \infty 3L.$, $B \infty 3M.$, $X \infty 6L.$, $Y \infty 6M.$, then $A.B \between X.Y$ will result. Furthermore, nothing prevents $X \infty A$: from which $A.B \between A.Y$ follows. Given now rather $X \infty C$, $A.B \between C.Y$ comes out.

(44) *If $A.B \between D.E$ and $B.C \between E.F$ and $A.C \between D.F$ then we will have $A.B.C \between D.E.F$.* See fig. 20.

For $A.B \between D.E$ signifies simply $A \between D$ and $B \between E$ simultaneously, preserving the situs $A.B$ and $D.E$. In the same way, $B \between E$ and $C \between F$ follow from $B.C \between E.F$, preserving the situs $B.C$ and $E.F$; and $A \between D$ and $C \between F$ follow from $A.C \between D.F$ preserving the situs $A.C$ and $D.F$. We have therefore simultaneously $A.B.C \between D.E.F$, preserving the situs $A.B$ and $B.C$ and $A.C.$, and likewise the situs $D.E$ and $E.F$ and $D.F$; whereas on the contrary, from $A.B \between D.E$ and $B.C \between E.F$ alone, it would follow that simultaneously $A \between D$ and $B \between E$ and $C \between F$, but only the situs $A.B.$ and $B.C.$ and likewise $D.E$ and $E.F.$ would be preserved. However, $A.C$ and $D.F.$ are not expressed as preserved unless we add $A.C \between D.F$.

Thus, we now have the principle for producing reasoning for even more points.

(45) *If $A.B \between B.C \between A.C$ then $A.B.C \between B.A.C.$*, or in any other order. For if we assign to congruents $A.B$ and $(B).(A)$ some congruents C and (C) in congruent fashion, then

[1] The Latin word here in Echeverria's edition is *plano* which means 'plane' or 'plain'. However, it is possible that Leibniz meant the Latin *plane*, which means 'plainly'.

since $A.C$ ♉ $(B).(C)$ and $B.C$ ♉ $(A).(C)$, we will get congruents $A.B.C$ ♉ $(B).(A).(C)$, or $A.B.C$ ♉ $B.A.C$ by what precedes. Parentheses are added just to avoid confusion from the repetition. From this it is clear what it is to be assigned in congruent fashion, namely, when all combinations from one side of the statement are congruent to all from the other side. From which it is clear that if $A.B$ ♉ $B.C$ ♉ $A.C$, then we would have

$A.B.C$ ♉ $A.C.B$ ♉ $B.C.A$ ♉ $B.A.C$ ♉ $C.A.B$ ♉ $C.B.A.$

(46) *If $A.B.C$ ♉ $A.C.B$ then $A.B$ ♉ $A.C$ (only) follows,* (or the triangle is isosceles), namely it follows:

$A.B$ ♉ $A.C$ $B.C$ ♉ $C.B$ $A.C$ ♉ $A.B$ of which $B.C$ ♉ $C.B.$ is clear per se , and the remaining two $A.B$ ♉ $A.C.$ and $A.C$ ♉ $A.B.$ reduce to the same, and thus we deduce this one $A.B$ ♉ $A.C.$

(47) *If $A.B.C$ ♉ $B.C.A,$ then $A.B$ ♉ $B.C$ ♉ $A.C$ follows* (or the triangle is equilateral). For then $A.B$ ♉ $B.C$, $B.C$ ♉ $C.A$.

(48) *If $A.B.C$ ♉ $A.C.B$ and $B.C.A$ ♉ $B.A.C$ then $A.B$ ♉ $B.C$ ♉ $A.C$ will result.* For $A.B.C$ ♉ $A.C.B$ yields $A.B$ ♉ $A.C.$ and in the same way, $B.C.A$ ♉ $B.A.C.$ yields $B.C$ ♉ $B.A.$, or $A.B.$ ♉ $B.C.$ (and therefore, whenever in a transposition in the order of the points, one of the three keeps the same place in either order, and the former situs is congruent to the latter, from this alone the triangle can be proved to be isosceles; but if none of the points keeps its place, and still the former situs is congruent to the latter, the Triangle is equilateral).

(49) *If we have $A.B$ ♉ $B.C$ ♉ $C.D$ ♉ $D.A$ and $A.C$ ♉ $B.D$ then*

$A.B.C.D$ ♉ $B.C.D.A$ ♉ $C.D.A.B$ ♉ $D.A.B.C$
♉ $D.C.B.A$ ♉ $A.D.C.B$ ♉ $B.A.D.C$ ♉ $C.B.A.D.$

will result.

This is easy to show from the preceding, as are many others of the kind. It will be enough to demonstrate them when we need them. Now we have adequately given the principle of discovering these by calculation alone, without inspecting a figure.

(50) If three points $A.B.C.$ are said to be situated in alignment, then positing $A.B.C$ ♉ $A.B.Y$ will yield $C \infty Y$. This proposition is the definition of points which are said to be situated in alignment.

Inspect fig. 24 where C has some situs to A and B; take now some point Y having the same situs to $A.B$; if it can be taken distinct from C, then $A.B.C$ are not situated in alignment, but if it necessarily coincides with C, they are said to be situated in alignment.

(51) With two given points we may always assume a third situated in alignment, or i.e. if $A.B.Y$ ♉ $A.B.X$ then $Y \infty X$ will result. For namely, with two given points $A.B$, we may always assume a third Y, which can be moved while preserving its situs with those, themselves unmoved. But the path which its own motion describes can be ever smaller and smaller, as further and further points Y are assumed, until at last we might take one such that the space of its motion disappears, and then the three points would be in a line. Much better to write thus: $A.B.3Y$ ♉ $A.B.6Y$ yields $3Y \infty 6Y$, that is, we can assign some Y which cannot be moved or change place while preserving its situs to $A.B$. I seem to be able to show this in another way: suppose there be some extension, which is moved while preserving the situs of its points among themselves and to two unmoved points taken within it. For if anyone denies that it can be moved, he therefore grants that its points cannot be moved while preserving their situs to the two assumed points, whence they are, by definition, situated in alignment. But there is no reason why only these points $A.B$ can be taken as fixed during the same motion, and not indeed also others, or that is to say, there is no reason why the points of the extension, which is moved with these two fixed, should preserve only the situs to these two fixed points, and not indeed also to others; for the situs which $A.B$ possess among themselves makes no difference, and therefore one could take some Y in place of B possessing another situs to A than B does. Indeed, whatever can be taken as fixed, will be fixed provided the motion of the extension remains the same. And because the motion of the extension is determined by the two points we take as fixed, or it

is determined which of its points are moved and which are not, hence by the two assumed fixed points, many others are determined which cannot be moved while preserving their situs to the same, or i.e. which lie in alignment.

(52) *If* $E.A.B \between F.A.B.$ *and* $E.B.C \between F.B.C.$ *then* $E.A.C \between F.A.C.$, see fig. 25.

For by $E.A.B \between F.A.B$ we have $E.A \between F.A$, and by $E.B.C \between F.B.C$ we have $E.C \between F.C$. Now if $E.A \between F.A$ and $E.C \between F.C$ then $\overbrace{E \frown A \frown C} \between \overbrace{E \frown A \frown C}$ results by prop. 44 (that is, $E.A \between F.A$ et $E.C \between F.C$ et $A.C \between A.C.$) therefore if $E.A.B \between F.A.B$ and $E.B.C \between F.B.C$ then $E.A.C \between F.A.C$. QED

(53) Hence now $E.A.B.C \between F.A.B.C$ *follows supposing* $E.A.B \between F.A.B$ *and* $E.B.C \between F.B.C$. For indeed $E.A.C \between F.A.C$ by the preceding; so we have: $E.A.B \between F.A.B$ and $E.A.C \between F.A.C.$ and $E.B.C \between F.B.C$ and $A.B.C \between A.B.C$, that is we have everything which may be derived from this: $E.A.B.C \between F.A.B.C.$; so now we have also $E.A.B.C \between F.A.B.C.$ (evidently this is an excellent way of returning back to the antecedent from all its consequences exhausting the complete nature of the antecedent).

(54) *If* $E.A \between F.A$*,* $E.B \between F.B$*,* $E.C \between F.C$ *then*

$$E.\overbrace{A \frown B \frown C} \between F.A \frown B \frown C;$$

for combinations remaining on each side that need to be compared, $A.B$ and $B.C$ and $A.C$, coincide on both sides.

(55) Suppose we are given $A.B.X \between A.B.Y$, or rather given two points $A.B$, two others $X.Y$ may be found, so that X and Y have the same relation to $A.B$. See fig. 26.

Then indeed $A.X \between A.Y$ and $B.Z \between B.V$ can be found, by prop. 43. Setting $Z \infty X$ (this can be done by prop. 43, or Z can be given, or rather assuming X, since $A.B \between A.V$) and likewise setting $A.X \between B.X$ (for also in $A.X \between B.Y$, X can be given since $A.C \between A.Y$ is given by prop. 43) then $V.B(\between B.Z \between B.X \between A.X) \between A.Y$. Therefore $V.B.X \between Y.A.X \between X.B.V$, which contains everything determined up to now. Thus we can set $V \infty Y$; this is not prevented by anything determined already. Then $Y.B.X \between Y.A.X$. Thus $Y.B \between Y.A$, $B.X \between A.X$. By reversing, $Y.B.X \between X.B.Y$. So $Y.B \between X.B$. Then $Y.B \between X.B \between Y.A \between A.X$. Therefore $A.B.X \between A.B.Y$.

(56) If three points $E.F.G$ (taken distributively) have the same relation to three points $A.B.C$ taken collectively, the former three will be in the same arc of a circle, the latter three will lie in the same line, or i.e. in alignment. It is desirable to make note of this proposition; the reason will become apparent from what follows.

(57) If $E.A.B.C \between F.A.B.C \between G.A.B.C.$ and E *not* ∞F, E *not* ∞G, F *not* ∞G, then we say *any points A. B. C. are situated in alignment*, or are in the same line.

(58) The point C may be omitted. If $E.A.B \between F.A.B \between G.A.B$ then the points $E.F.G$ are in the *same plane*.

(59) By this supposition the points $E.$ $F.$ $G.$ will be in the same *arc of a circle*.

(60) Between any two congruents, infinitely many other congruents can be taken, for it is only through congruents that the one can be transferred into the place of the other while preserving its form.

(61) Hence from any point to any other point a curve can be drawn. For a point is congruent to a point.

(62) Hence a curve can be drawn from any point through any point.

(63) A curve can be drawn that passes through any given point.

(64) In the same way it is shown that some surface could pass through any given curves. For if they are congruent, clearly the generating curve could be in all of them in succession. If they are not congruent, clearly the generating curve can be augmented, diminished, and transformed during its motion, so that when it arrives there it will be congruent.

(65) Each thing can be put in space, preserving its form; or rather to anything existing in space, an infinity of other congruent things can be assigned.

(66) Each thing can be moved, preserving its form, in an infinity of ways.

(67) Thus each thing can be moved, preserving its form, such that it is incident on a given point.

More generally: each thing can thus be moved, preserving its form, such that it is incident on another thing to which something congruent in it can be designated. For one congruent thing can be transferred to the place of another; and nothing prevents that in which the congruent thing to be transferred sits from being transferred simultaneously with it, since there is no reason for separation: what can be fitted to one of the congruents can also be fitted in a similar manner to the other of the congruents.

(68) $A \Taurus B$, that is, any other is congruent to the assumed point.

(69) $A.B \Taurus B.A$ as above.

(70) $A.B \Taurus X.Y$. In the same way.

$A.B.C \Taurus X.Y.Z$ and $A.B.C.D \Taurus X.Y.Z.\Omega$ and so on. This is just the fact that any points can be moved while preserving the situs among themselves. Their situs among themselves, moreover, can be understood to be preserved if it is set as the extremes of a certain rigid curve of whatever kind.

(71) $A.B \Taurus C.X$, $A.B.C \Taurus D.X.Y$, etc.

Here is just the fact that any points, such as $A.B.C$, can be moved while preserving the situs among themselves, so that one of them, A, is incident on some given point D, the two remaining being incident on some other $X\ Y$.

(72) If $A.B.C$ is not $\Taurus\ A.B.Y$ unless $C \infty Y$ then the points $A\ B\ C$ are said to be *situated in alignment* (see fig. 24); or, C *will be situated in alignment with* $A\ B$ if it be unique having that situs to $A.B.$. But whether such a situs of points can be found, will require investigation later. A curve all the points of which are situated in alignment, is called *straight.* If indeed $A.B.zY \Taurus A.B.zX$ and thus $zY \infty zX$, then $\overline{z}Y$ will be $(\infty \overline{z}X)$ a *straight line*; that is if the point Y is so moved that it always maintains a situs to the points $A.B$ that applies to itself alone, or is determined and least varying or moving, then it describes a straight line.

(73) If $A.B.C \Taurus A.B.D$, then $\Taurus\ A.B.zY$, see fig. 29, for then $C \infty 3Y.$, and $D \infty 6Y$, and of course C and D are places of the motion of Y such that it keeps the same situs to $A.B$, between which there will necessarily be other indefinite ones, designated by zY. The curve $\overline{z}Y$ is called a *Circle.* It should be noted, moreover, that this description of a circular Curve is prior to the one Euclid gave; Euclid, indeed, requires the straight line and the plane. But ours proceeds whatsoever rigid curve is assumed, just taking two points in it that are unmoved while the same curve is moved, or at least some point in it; for this point keeps the same situs to the two assumed points, since they are all in the rigid path. So this point describes a circular curve by this definition of ours. But if someone denies that such a point could be found on the rigid curve that is moved while the two given ones remain fixed, it will be necessary by the preceding definition (prop. 72) that all the points of the rigid Curve are situated in alignment, that is, it will be necessary that a straight Curve is given. In this way it must be granted that either a straight line is possible, or a circle. Later on, we will deduce the one from the other being admitted. It should be noted here in passing that, as appears in its own place, an arc of a circle can pass through any three given points, hence from three given points we can discover one bearing the same relation to those three, namely such that $X.C \Taurus X.D \Taurus X.E$, and it can be done multiple times, or i.e. various X can be discovered for the same $C.D.E$, with every X falling in one straight line passing through the center of the circle and forming right angles with its plane.

(74) Suppose there is some curve $\overline{z}Y$, see fig. 29, in which numerous points $3Y.6Y.9Y.12Y$ etc. can be taken, such that $3Y.6Y \Taurus 6Y.9Y \Taurus 9Y.12Y$ etc. For in general, if there is a curve sufficiently small, with one extreme in another curve, the former can be moved, with its extreme point common to both curves being fixed, so that the other extreme meets the latter curve; therefore by this motion a part is cut off, and then another by the new discovered point remaining fixed, and so on. But now I observe that this is not actually necessary; it suffices that one curve is attached in multiple ways to the same curve by its extremes, such that several parts of the same curve are assigned with extremes that are congruent to the extremes of the others, as in fig. 30 the rigid curve LM is attached by its extremes $L.$ and

M. to the same curve zY, whether in $1Y.2Y$ or $3Y.4Y$ or $5Y.6Y$, which coincide with $1L.1M$ and $2L.2M$ and $3L.3M$. For if $L.M$ could once be attached to $\overline{z}Y$, it can be attached in an infinite number of ways, if the subsequent attachments were to differ arbitrarily little. Now let two congruent curves be drawn from L and M, that drawn from L having the same relation to L as that drawn from M has to M, and which meet each other in X, and $1L.X.1M$ ♉ $2L.X.2M$, and so on; that is, those drawn from $1L$ and $1M$ are produced so that they do not meet each other before the place where those from $2Y.2M$, drawn in the same way, meet each other. From this it is clear that the point X has the same relation to all those assigned Y, and indeed if the curve is such that there is a point of this kind, having the same relation to all of its points, it is discovered in this way. If, moreover, the curve is circular, as here, it's enough to discover some point having the same relation to three points of the circular curve; then it will have the same relation to all the others. The reason for this is that, from three given points $C.D.E$ (see fig. 29), supposing $C.D$ ♉ $D.E$, some definite point is determined by the method written a little earlier for fig. 30, and accordingly, taking any other three points in the circle congruent to the first three, drawing congruent curves in the same way that meet each other, one will necessarily arrive at the same X. From here we see that since points X could be discovered from three given points $D.C.E$ in various ways, according as the congruent curves are drawn in one or another manner, or they converge faster or slower, clearly other points X can certainly be found, and all will fall on one line.

(75) However, we can obtain the same thing more simply without a circle. Suppose we have three points $A.B.C$, such that $A.B$ ♉ $B.C$ ♉ $A.C$, and points X are found such that $A.X$ ♉ $B.X$ ♉ $C.X$, as many as you like, or what amounts to the same: a point X is moved such that any of its places, say zX, has the same relation to $A.B.C$, that is such that $A.3X$ ♉ $B.3X$ ♉ $C.3X$, then the points zX will be put in alignment, or rather $\overline{z}X$ will be a straight line. And thus it appears what Euclid means when he says that a straight Line lies between its points by equality, that is, it doesn't lurch to any side, or during its motion bear any relation to the point A otherwise than to B or C. Hence, moreover, one also has a method of finding the points X of the line $\overline{z}X$. Evidently, if from A any curve whatever is drawn having the same relation to B and C, and likewise another through B that is congruent and congruently placed, that is, so that a point of this corresponding to a point of that would have the same relation to $B.A.C$ as the point of that to $A.B.C$, and these curves are drawn out until they meet, then they will necessarily meet in a point X which has the same relation to $A.B.C$. And now if such a curve would be drawn through the point C, congruent and congruently to the former ones, this one too would meet in the same point X. Hence, we can find however many points of this kind, and furthermore a straight line can be described by the points.

(76) Let us resume, anyhow: one can infer a (rigid) line drawn from any point to any other point.

(77) The situs of A and B among themselves is signified by $A.B$; that is, some rigid track through A and B. It is enough for us that this track be a line. Thus $A.B.C$ signifies another rigid track through A. B. C.

(78) No matter what may be put in space, it can be moved, whether a point, or a line, or some track, or anything to which another congruent thing can be assigned in extension. Hence A ♉ X, $A.B.$ ♉ $X.Y$, $A.B.C$ ♉ $X.Y.Z$ or $A.B.C$ ♉ $\omega X.Y.Z$.

(79) Given two distinct things in extension, one can be set stationary, the other set in motion.

(80) If some [points] of those that are in a rigid track are moved, the rigid track itself is moved.

(81) Every track can be moved such that a given one of its points falls on another given one, $A.B.C.$ ♉ $D.X.Y$.

(82) Every track can be moved keeping one of its points fixed, $A.B.C$ ♉ $A.X.Y$.

(83) The *Straight Line* is a track which cannot be moved while two points in it remain stationary; or if a certain track is moved with two of its points remaining unmoved, and if moreover some other points are supposed to remain stationary, all of these points are

said to be situated in alignment, or rather they fall in a track which is called a straight line. Or rather if $A.B.C$ ♉ $A.B.Y$, then necessarily $C \infty Y$, that is if some point C is found situated in alignment with the points $A.B$, then the track $A.B.C$ (or $A.C.B$) cannot be moved while $A.B$ remain unmoved, such that C is transferred onto Y, and that the track $A.B.Y$ is congruent to the former one $A.B.C$. Or what is the same: another point Y cannot be found beyond the one C, which has the same situs to the fixed points $A.B$ that C has, rather, it is necessary that if such a point Y is taken, it must coincide with the very same C, or i.e. $Y \infty C$ must hold. Hence we can say that the point C is the unique instance having its situs to $A.B$. And a point which is thus moved, so that it maintains a situs unique of its kind to two fixed points, is moved in a line. Certainly if $A.B.Y$ ♉ $A.B.X$ and thus $Y \infty X$, then $\overline{\varpi}X$ $(\infty\overline{\varpi}Y)$ will be a straight line. Whether, moreover, two such points situated in alignment are given, and whether they form a track, and whether this track is a line, should not be assumed, but rather demonstrated. But certainly the path of a point so moved will be a *straight line*. If it passes through all points of this kind, indeed the locus of all points that are in alignment with two points, it will be no other track than a straight line.

(84) If two points $A.B$ remain unmoved in a track $A.C.B$, and the track itself is moved, the curve, which the point C describes by its motion, is called a *circle*. But whether indeed some track could be moved while two points remain unmoved should not be assumed but needs to be settled by a demonstration. [If] $A.C.B$ ♉ $A.Y.B$ then $\overline{\varpi}Y$ is called a *circular* curve, and if there are however many points $C.D.E.F.$ and $A.B.C$ ♉ $A.B.D$ ♉ $A.B.E$ ♉ $A.B.F$, then they are said to be in one and the same circle. This definition of the circular curve does not presuppose a straight line and plane to exist, which the definition of Euclid does.

(85) The locus of all points which have the same relation to A as to B, will be called a *plane*. Or if $A.Y$ ♉ $B.Y$, then Y is a *plane*.

(86) Hence if $A.C.B$ ♉ $A.Y.B.$, and if $A.C$ ♉ $C.B.$ (and thus also $A.Y$ ♉ $B.Y.$), then the curve $\overline{\varpi}Y$ will be circular in a plane. Whether every circle is planar needs to be settled later.

(87) If $A.B$ ♉ $B.C$ ♉ $A.C$ and $A.Y.$ ♉ $B.Y.$ ♉ $C.Y.$ then $\overline{\varpi}Y$ is a straight line.

(88) If $A.Y$ ♉ $A.(Y)$. then Y is a *spherical* surface. (♉ signifies congruence, ∞ coincidence. When I say $A.B$ ♉ $A.Y.$, I could indeed say that the distance AB is equal to the distance AY, but since later on when three or more are employed, as $A.B.C$ ♉ $A.B.Y$, we don't just want the triangle ABC to be equal to the triangle ABY, but also to be similar, that is to be congruent, it is thus preferable to use the sign ♉.)

(89) If Y ♉ (Y) then the locus of all Y, or rather $\overline{Y}$, is absolute extension, i.e. *Space*. For the locus of all points congruent to each other is the locus of all points in the universe. Indeed, all points are congruent.

(90) The same if Y ♉ A; for (from the signification of the characters) if Y ♉ A then (Y) ♉ A. Therefore $Y.$ ♉ (Y). The locus of all points Y congruent to a given point A is indeed unbounded space itself, for indeed all points are congruent to any given one.

(91) Next: $A.Y$ ♉ $A.(Y)$. The locus of all Y, or rather $\overline{Y}$, is called a *Sphere*, which is the locus of all existing points with its same situs to the given point. The given point, moreover, is called the *center*.

(92) The same if $A.B$ ♉ $A.Y$. For then also $A.B$ ♉ $A.(Y)$ and also thus $A.Y$ ♉ $A.Y$, where we note that B is from the collection of Y, or rather is bY. If indeed Y covers all points having that situs to A that B has, then certainly it covers B, which certainly has the situs to A that it has. The sphere is the locus of all existing points of a given situs to a given point (that is, of a situs congruent to the situs of a given point).

(93) If $A.Y$ ♉ $B.Y$, the locus of all Y, or rather $\overline{Y}$, is called a *plane*, or the locus of points Y, each having the same relation to one point A of two given points as it has to the other B, is a plane. It should be noted that the expression of this Locus could not be converted to another in which there are simultaneously Y and (Y).

(94) If $A.B$ ♉ $C.Y$ then $\overline{Y}$ will be a *sphere*. For then

$A.B$ ♉ $C.Y$ ♉ $C.dY$;

if ${}_dY \infty D$, then $C.D \between C.Y$, and so the locus will be a sphere by prop. 91.

(95) Now Y and (Y) signify any point of some locus and any other besides the first. If zY signifies whatever point of a locus, or rather all points of a locus taken distributively, then the same is signified by Y placed absolutely. Then ${}_dY$ signifies some one specific point of the locus; $\overline{Y}$ signifies all points of the locus taken collectively, or the whole locus. If the locus is a Curve, I signify this by $\overline{\omega Y}$. If it is a surface, likewise by $\overline{\omega\psi Y}$. If a solid, likewise: $\overline{\omega\psi\phi Y}$.

(96) If $A.B.C \between A.B.Y$ (or if $A.B.Y \between A.B.(Y)$) then the locus of all Y, i.e. $\overline{Y}$, is called a *Circle*, that is, if the situs to two given points is the same for many points (or it is given), the Locus will be a *Circle*.

(97) If $A.Y \between B.Y \between C.Y$, then the locus of all Y, i.e. $\overline{Y}$, is called a *straight line*.

(98) If $A.B.C.Y \between A.B.D.Y$, then $\overline{Y}$ will be a *Plane*, or rather if $C.D$ are two points having the same relation to the three $A.B.Y$, these three will be in the same plane, and if from two of these given ones, $A.B$, a third Y is sought, the locus of all Y will be a plane. From this it is clear that $A.B$ are included under Y. It needs to be demonstrated here that this locus coincides with the other one from Prop. 93. So then: $C.Y \between D.Y$ (1) is the locus for a plane by Prop. [93]. If ${}_3Y \infty A$ and ${}_7Y \infty B$, then $C.A \between D.A$ (2) and $C.B \between D.B$ (3). Therefore: . Namely 1 is clear by itself, 2 through (3) and 3 through (1) and 4 through (2) and 5 by itself and 6 by itself.

(99) If $A.Y \between B.Y \between C.Y$, the locus $\overline{Y}$ will be a *point*, or Y will satisfy it uniquely, or $Y \between (Y)$. This proposition should be demonstrated.[2]

(100) Thus we have the loca for a point, for a straight Line, for a Circle, for a Plane, and for a Sphere, expressed with wonderful simplicity only through congruences, but this is partly truth, partly a possibility, and it needs to be demonstrated that ours coincide with other definitions.

(101) If any track or extension whatever is moved while one point turns out to be fixed, any other point on it is moved in a Sphere. Let the extension, furthermore, be rigid, or i.e. [its] parts maintain the same situs. We will then have a way of finding however many points of a Sphere. Indeed, $A.B \between A.Y.$ can be given if the track passes through two points $A.B$. Certainly we have the power to draw some track (whether it be a curve, a surface, or a solid) through two given points, and to move the track with one point fixed.

(102) If two congruent tracks pass *congruently* through two given points, that is, such that corresponding points in the two tracks have congruent situses to the two given points, each to its own, and they are moved, or indeed enlarged congruently if needed, until they meet one another, then the places at which their corresponding points meet one another will be points of the plane which we defined to have the same relation in any of its points to the two given points. I claim, moreover, that they can be moved congruently, or drawn out congruent or congruently, until they meet.

(103) If I section a sphere by a sphere, or by a plane, we will have a circle; if a plane by a plane; we will have a line; if a line by a line, a point. But it needs to be shown that these sections can be made, and a track exists of the points common to a Sphere and a Sphere, or a plane and a plane, or a plane and a Sphere. If a sphere is tangent to a plane or a Sphere, the locus is a point, when the circle becomes small and vanishes.

(104) All the other common definitions of a straight line are not yet sufficiently perfect, since it could be doubted, since till now a demonstration was always needed, that such a line is possible. But it seems that this should be placed among ¡the easiest [consequences]¿, and so we need a definition such that it immediately appears that the line is possible. If you define [it as] the minimum, indeed here one can doubt whether a minimum from a point to a point exists. If you define it [as] that the points of which cannot be drawn apart further,

[2]The proposition is not plausible as written. Maybe Leibniz meant to add a third congruence "$\between D.Y$" in the first line, which would determine 0, 1, or 2 possibilities for Y provided that the given points $A.\ B.\ C.\ D$ are not coplanar. Alternatively, perhaps he means to refer to the unique point in the plane of $A.B.C$ on the line passing through the circumcircle of this triangle at right angles to the plane. Among the points on this line, the one in that plane is unique in lacking a distinct partner point reflected in the plane.

you presuppose distance, or a minimal path. To draw apart is indeed to acquire greater distance. ¡Equ…¿ it is one thing to exhibit a material straight line, or to draw it, another thing to comprehend it mentally.

(105) When two points are perceived simultaneously, eo ipso their very situs to each other is perceived. But any two situses between two points are similar, and thus they are distinguished only by coperception, or i.e. by magnitude. Therefore the difference of situs is the magnitude of a certain extension, when two points are perceived simultaneously and by that fact a certain extension is perceived.

(106) The *Straight Line* is the extension which is perceived eo ipso when two points are simultaneously perceived.

(107) Perceiving one point, and then returning to another perceived separately, no difference can be noted.

(108) If $A.B$ are simultaneously perceived, and then again $C.D$ simultaneously perceived, the difference is of situs. Nevertheless, what is perceived when $A.B$ are perceived and what is perceived when $C.D$ are perceived, are similar: indeed, clearly nothing can be noted in each at a time that cannot be noted in the other; therefore what is perceived when $A.B$ are perceived and what is perceived when $C.D$ are perceived, if they can be distinguished from each other they are distinguished only by magnitude, and so $A.B$ being perceived simultaneously, something having magnitude is perceived simultaneously. When two things are simultaneously perceived to be in space, eo ipso a path from the place of one to the other is perceived. But two points are congruent. And so what is perceived, the two points being simultaneously perceived, is a Curve, or the path of a point. Clearly then this path is such that, whether you give the line from A to B, or from B to A, the place where A will be is congruent to the place where B will be and congruently placed. That is, as the place of a point coming from A will be to A, so the place of a point coming [from] B will be to B, and also, as the place of a point coming from B will be to A, so the corresponding place of a point coming from A will be to B. When we conceive of two points as simultaneously existing, and we examine the reason why we say they exist simultaneously, we will imagine them to be simultaneously perceived, or at least able to be simultaneously perceived. When we perceive something as though existing, eo ipso we perceive it to be in space, that is, infinitely many others to exist which in no way can be distinguished from it. Or what is the same, it can be moved, or can be as much in one place as another, and since it cannot be simultaneously in several places, nor moved in an instant, therefore we perceive the locus as a continuum. But since it is still indefinite to where it is moved, it can be moved in many ways which cannot be distinguished from each other. Hence, if in addition we put another thing congruent to the former, and eo ipso the one is imagined as able to the reach the place of the other, then the mind is determined to some definite motion. Although it can be done in several ways, nevertheless a unique one is determined, for considering which we don't need to assume any thing besides the two things already set. That is, from the two congruent ones set in space, a path from one to the other is set, connecting them. But the simplest position is the point. For even if you set some other, since one of the various motions of one to the other is determined according to which the points corresponding to each other are moved determinately, clearly the mind eventually settles upon the consideration of two points, for they are evidently congruent per se.

An *extension* is continuous. Parts can be made in an extension. In an extension one can make parts that exist simultaneously. In an extension parts can be made in unlimited ways. A part of an extension is extended. In one extension there could exist many parts. In one extension there could exist infinitely many. In one extension there could exist infinitely many similars. In one extension there could exist infinitely many congruents. If something exists in an extension, and it is congruent to it, it coincides with it. If something exists in an extension and it is not congruent with it, there could exist infinitely many in the same extension which do not coincide with the former, but nonetheless they are congruent. Two moveable things that are taken in an extension could be congruent to each other, or could be located in various times so that the former condition could not be distinguished from the latter. The locus itself of the extension is extended. The locus of the extension is congruent to an extension. The locus is immovable.