De Calculo Situum

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G.W. Leibniz, 1715-16

Tr. David Jekel, Matthew McMillan

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(1) In the Calculus of Magnitudes, ⟨not only do we form those Magnitudes when⟩ we add, multiply, square [in se ducimus], and take their reciprocals, but we also convey them in ratios, and other relations, progressions, and finally Majorizations, Minorizations, and Equations. Just so in Situs we form Extensions by Sections and Motions, and then we compare and observe in them, besides Magnitudes, also Similarity, Congruence (when Equality and Similarity concur), Coincidence, and thus Determination. Indeed, a thing is determined to which some other must coincide when the same conditions are set.

(2) And the doctrine of Magnitude has its Axioms, for instance the Whole is greater than its part. What is greater than a greater is greater than a lesser. If you add equals to equals, they come out equals; and others of that kind. Just so the Doctrine of Situs has its own Axioms of this sort:

If Similarity, Congruence, Coincidence are in the Determiners, they are also in those determined, and conversely, if they are in those Determined, they will also be in the simplest Determiners.

For example, let us suppose that only a unique straight Line can be drawn from a point to a point; it follows that all Lines are similar to each other, since nothing is needed for the determination of the Line from A. to B. except for A, B to be taken, and for the other LM, just for the situs of the points L, M to be taken. The situs of two points, indeed, is always similar to the situs of another two, since no difference can be ascribed beyond the magnitude of the entire distance, but now the magnitude is something in relation to a third thing [aliquid ad tertium relatum]. ⟨Nonetheless, the Situs of two points clearly won't be the same as the Situs of another two points, unless indeed they are placed so that some continuous Extension that can be attached between the Termini of the one situs could in fact be attached between the Termini of the other situs.⟩

Things are similar, in fact, which are indiscernible when both are seen separately, so that nothing could be taken in one to which something similar could not be taken in the other, abstracting everywhere from some determined Magnitude, excepting the magnitude of Angles, which should be referred to the doctrine of situs, not in fact the doctrine of Magnitude.

Since, therefore, we have proved that every situs of two points is similar, then indeed those determined, or all Straight Lines, will be Similar.

(3) On the other hand, not all Triangles determined by the situs of three points are similar to each other. And indeed, ABC are not similarly related as LMN. The Distance AB can, for instance, have another ratio to the Distance BC than the Distance LM to the distance MN, so that dissimilitude arises in those determining. From this it is also clear that in two Straight lines, three points could be chosen with situs dissimilar to three other points.

For, reciprocally, similarity from determination holds only for those purely determining, but indeed not for those that are more than determining.

Likewise, although a Circle is determined by three given points of the periphery, and least of all should it be denied that all Circles are similar to each other, nonetheless here the Inference from the similarity of the determined to the similarity of the determiners is not valid, since the three given points of the Periphery determine more than the Circle itself, namely a fixed Angle in a segment, and three parts of the periphery having a determined ratio to the whole Circle.

But conversely, if two Circles are determined by two given Chords and by equal Angles in the segments created above the Chords, then finally the Circles are not only similar, but also similarly determined. This question, however, is not indeed about such determination, but merely about the first and simplest determiners, which, when the things determined come out similar, must also be similar.

If it actually happens that dissimilar determiners nevertheless give similar determined things, that fact itself is certain evidence that this determination is not the simplest, but there is another simpler one.

(4) As we reduce the general Logic or Mathesis of Magnitudes to calculation, making use especially of ratios and equations, just so a certain calculus of situs can be established with similarities and congruences. The letters, further, in the Calculus of Magnitudes typically designate the Magnitudes themselves. In the Calculus of Situs they can designate points and loca. Hence if $YA \simeq B.A.$, the locus of all Y is a spherical surface.

In this Co-signification, $B.A.$ signifies the situs of a point $B.$ to a point A, but $\simeq$ is the sign of congruence. The sense of this Co-signification is therefore that whatever indeterminate Y has the situs to the determined point A that B has to $A.$; from which B also is understood to be among those Y or i.e. in the same spherical surface. But if I had put $Y.A. \simeq B.C.$, then B would not need to be put in the spherical surface. ⟨But now $Y.A. \simeq B.A.$ remains.⟩

(5) By positing now yet another sphere $ZL. \simeq ML.$ and considering these two spherical surfaces to intersect, and the loca [sic] of mutual concurrence to be called $V.$, each $V.$ will be simultaneously $Y.$ and $Z.$, so I could write $V.A. \simeq BA$ and $V.L. \simeq ML.$ Moreover, B can be assumed coincident with M (which is then signified $B \infty M$), let us call it F, determined among these $V.$, and $V.A. \simeq F.A.$ and $V.L. \simeq F.L.$ will hold, hence by composing, $V.A.L. \simeq F.A.L.$ holds, from which it follows that the Curve in which two spherical surfaces intersect is of

such a nature that any point V on it would have the same situs to two given ones $A.L.$ as the fixed one $F.$ (which is thus one of these V) to the same points $A.L.$

(6) We could state the same thing again thus: Some [extension] $A.G.L.$, whose two points $A.$ and $L.$ are stationary, would describe by its motion the kind of curve $V.V.V.$ that two spherical surfaces form by their intersection, that is a Circle, since when a rigid Extension is placed so that some point such as G preserves its situs to the points $A.L.$ that are stationary during the continuous motion of the extension, thereupon any Trace of the revolved $G.$ retains the same situs to the two fixed points $A.$ and $L.$, not otherwise than what we wrote above: $V.A.L \simeq F.A.L.$

(7) Any points which are actually stationary during the stated Motion together with the points A and L, by that fact, since they are stationary, must be unique with their situs to $A.$ and $L.$ For if they are moved, the same situs to A and $L.$ could be exhibited in multiple places, if in fact all of their traces have the same situs to $A.$ and $L.$ Now actually these points are their own traces; that is, they describe Circles indefinitely small, vanishing into points. So a Straight Line is produced, of which this is the Expression: Setting some indeterminate point $R.$ on it, $R.A.L.$ is said to be Unique, or i.e. if $R.AL \simeq (R)A.L.$, then $R \infty (R)$.

(8) Hence clearly two Straight Lines do not pass through the same two points such as ABC and ABS, for if in a Rotation of the Plane, with the points $A.$ and $B.$ fixed, the whole plane is moved, the rotation will make whatever was once above or closer to another external thing at the beginning of the rotation, to become afterwards, with the face flipped around, lower or further from the external thing in the beginning of the rotation. But, if both the Lines ASB and ACB were Straight, made by rotation on Fixed points $A.$ and $B.$, then both must be stationary, by the nature of a Straight Line just shown. If both are stationary, S always remains above the extension ACB and never falls lower, which is contrary to the Nature of Rotation.

(9) Hence, we immediately deduce that Straight lines are similar to each other, and have part similar to the whole, and moreover that the Straight Line is the simplest, since it requires nothing other than the extremes for its whole determination, and so is also the minimum between extremes, and in what follows it can be taken for the distance of points. It will be taken for distance, since with the Termini unmoved, the distance of the Termini must be unmoved. If therefore another Curve between $A.$ and B besides the Straight one is assumed for distance, then it too remains unmoved during rotation of the Plane with the points $A.$ and $B.$ being Fixed, besides, indeed, the Straight Line $AB.$ [that is] unmoved during the same rotation by §7. Therefore, two distinct Curves simultaneously unmoved during this rotation would be given, which is absurd by §7.

It will be shortest, since if another shorter one reaches from $A.$ to $B.$, a Curve ⟨or extension⟩, it follows that the distance is greater than itself, which is absurd. If another, equal, is given, such as if ASB were indeed not Straight, but nevertheless equal to the line ABC, then the distances $AS + SB.$ must not be greater than $A.B.$, since they cannot be greater than the co-terminal curves $AS+SB$ (which are set equal to AB) by the nature of the shortest. But Euclid demonstrated that $AS+SB$ is greater than $AB.$, relying on no implicitly assumed principles [beyond] this (there are not two Shortest between the same termini), but by reasoning from the pure situs of angles. Therefore, clearly our assertion is also true, that there are not two shortest between the same Termini.

(10) Perhaps this thing from Euclid can still be demonstrated from fewer things, namely:

Dissimilar Arcs cannot be cut out by equal Chords in the same Circle. Something to be considered as established per se from the nature of similars.

Thus, Diameter AB is greater than Chord AD, for Chord AD cuts out an Arc dissimilar to half of the Circle AB (otherwise it goes from A to B, contra §8). Therefore, by the principle proposed, $AD = AB$ will not hold. But neither will $AD \Gamma AB$ hold, since $CA + CD = AB$, [or] twice the Radius [equals] twice the Radius. Thus, by composing, we would have $AD \Gamma CA + CD$, the Shortest [path] being greater than another interposed between the same Termini, which is absurd. Since therefore the Chord AD would be neither equal to the Diameter, nor greater, clearly the Diameter is greater than any Chord. Hence follows [a third thing,] that the two sides of Isosceles Triangle AMN are greater than the third. For, drawing a Circle with Center A passing through M and N, $AM+AN$ is equal to the Diameter, or twice the Radius, but MN will just become a Chord of the Circle. Therefore, as proven a little before, $AM + AN \Gamma MN$.

Finally, I say, in any Triangle at all, two sides [jointly] are greater than that remaining: $DE + DF \Gamma EF$. For by cutting out $DX = DE$, Therefore $DE + DX \Gamma EX$, as in the Isosceles Triangle shown. I add XF on either side. Therefore $DE + DX + XF \Gamma EX + XF$. That is, $DE + DF \Gamma EX + XF (\aleph)$. Then, either $DE + DF$ is smaller than the shortest EF, which is absurd by §9, or else equal (and likewise by what I showed at the letter $\aleph$, $EF \Gamma EX + XF$, the Shortest [greater than] another interposed, an absurdity) or else finally $DE+DF$ is greater than EF, what we sought to demonstrate.

(11) As the Straight Line is the locus of all points unique with their situs to two points, so the Plane is the locus of all points unique with their situs to three points, whence clearly a Plane is obtained by assuming two intersecting straight lines. Indeed, let there be a Straight Line through $A.L.$ and another through $A.M.$ We have three points $A.L.M.$, and not only are all the points of the Line through AL determined and all [points] of the Line through AM, but also all distances from any point of the one Line to any point of the other line, and so any point on whatever of these distances (which indeed are Straight Lines) is determined, or i.e. is unique with its situs to $A.L.M.$

(12) Now, let all points of the Line through $A.L.$ be called Y, and all points of the Line through $A.M.$ be termed $Z.$, so $A.L.Y.$ is unique and $A.M.Z.$ unique. Letting one of the Y be H, and one of the Z be N, then $A.L.H.$ will be unique and $A.M.N.$ unique. Let another locus be taken, any point V of which is unique with its situs to $H.N.$ But this $H.$ is unique to $A.L.$ and this $N.$ is unique to $A.M.$ Therefore $V.$ will be unique to $A.L.A.M.$ For the Determiners may be substituted for the Determined in Determinations. Since, therefore, $V.$ is unique

to $A.L.A.M.$ and the repetition of its $A.$ is redundant, we infer from this that $V.$ is unique even to $A.L.M.$, that is, all points $V.$ are in the same plane with $A.L.M.$ because the Plane is the locus of all points Unique with their situs to three Fixed points.

(13) It follows that two planes intersect in a Straight Line. Let $X.$ be Unique to $A.B.C.$ and $Y.$ unique to $L.M.N.$ All common points, actually of either Plane, are called $Z.$ so that points Z are unique with their situs to $A.B.C.$ as well as to $L.M.N.$ Therefore all Z will be $X.$ as well as $Y.$ Draw out the distances $LM.$, $LN.$, and $MN.$ until the plane through $A.B.C.$ meets [them] in $\lambda., \mu.$, and $\nu.$, which must happen because every plane divides the whole of space, and a common section continues to Infinity. Likewise, every Straight Line continues to infinity. It is therefore necessary that it reaches to that Plane, or i.e. to a common section.

(14) But lest an objection be raised that perhaps one of the Distances $L.M.N.$ is Parallel to the section, two of the points $\lambda.$ and $\nu.$ suffice for us. But if all three did fall on the section, then the third will nevertheless be determined from the two of them determined, otherwise if the three were mutually indeterminate, they would determine a Plane in the same intersection of Planes, which is absurd, since in this way the intersection would also be a Plane. Therefore $Z.\lambda.\nu.$ becomes unique, that is, all points $Z.$ fall in a Straight Line. Hence since two Lines cannot cut across each other except in a unique point, the Intersection of three Planes will be a point.

(15) We should look at what happens if three spherical surfaces intersect each other, where the locus of intersection cannot be an extension. Indeed, neither is the intersection of two Curves an Extension. It is easily shown, moreover, that countless circles pass through two points, although sometimes a Circle could touch a circle in just one point, also when they are not even in the same plane, although then they would not be tangent. To be sure, it is clear that a circle is determined by three points. For from two points $A.$ and B a Line is determined all points of which bear the same relation to these two points, among which indeed is the Center of the Circle. A similar locus of points bearing the same relation to B and C (among which the Center should likewise be) falls in a Line determined by points B and $C.$ Therefore the Center of the Circle is on both of these Lines, or, that is, on their Intersection. Therefore the intersection of both Lines is a point of the same relation to ($B.C.B.A.$ and since it is superfluous to repeat B, to) $B.C.A.$, which point assuredly should be the Center of the Circle through $A.B.C.$ But we defined above the Circumference of a Circle [to be] the locus of points bearing the same relation to two Fixed points. Hence the Circle will be the Locus of points bearing the same relation to any point $X.$ of the Line through AB, determined things being substituted for Determiners.

(16) Let us take three points in the Circumference of this Circle and the plane passing through them, which the Line through AB meets in a Point which is $C.$ Therefore the Circumference is the locus of points bearing the same relation to $C.$, and it ought to be shown that all points of the Periphery fall in this Plane drawn through the three points of the Periphery itself. This will be accomplished if it is shown that the Plane is the locus of all points bearing

the same relation to a certain two points. Certainly the line is the locus of all points bearing the same relation to a certain three points. Suppose there are points $A.B.C.$. The intersections of any Two Spheres around A and B. will fall in a Plane. Likewise of any two spheres around A and C. From this, since here it suffices for determining, the Consequence is that the Plane from the intersections of spheres around A and B and the Plane from the intersections of Spheres around B. and C. or around A. and C. determine the same Line having the same relation to whatever points of this plane to which the impact of the line in this plane has the same relation.

(17) In a Plane we can also understand the Line as the locus of all points having the same relation to just two points A. and B.. And then all equal Circumferences around A. and B. will cross in this locus or i.e. in this Straight Line. Here the manner of determining a locus is different than earlier. It is one thing, that is, to say the locus of all points having the same relation to two points A. and B. is a Line, another thing to say that the locus of all points having the same relation to A. as to B. is a Plane. For in the former, the property is expressed thus: $A.B.C. \simeq A.B.Y.$ in a solid. The locus of all [such] Y. [is] a Line; but in the latter the property is expressed thus: $A.Y. \simeq B.Y.$; the locus of all [such] Y. will be a Plane. However, if all Y. are in the same plane as AB and among them $A.Y \simeq B.Y.$ is supposed, the locus of all Y. will be a Straight Line.

From $A.B.C. \simeq A.B.Y.$ it follows that $A.C. \simeq A.Y.$ and $B.C. \simeq B.Y.$, whence it is established that Y. falls in a Sphere with Center A. and Radius AC. and in a Sphere with center B. and radius $B.C.$

(18) Yet again, from the Contacts of Spheres [being] in one point, it follows that there is a locus of those Unique [with situs] to two points; or conversely, from this follows the Contact of Spheres [being] in one point. Likewise for the Contacts of Circles in a Plane. $FA \simeq FB \simeq LA \simeq LB$. thus $GA \simeq GB \simeq MA \simeq MB$.; certainly the circle described with center A and radius AE, when E is below the Line and A. above the Line, will cross it twice, in F and L; these points of the crossings approach each other continuously, with F. passing to $G.H.$ etc. and L to $M.N.$ etc. Where they meet each other, moreover, there they will coalesce to one in D., and there it will be the Contact of two Circles. Hence if A and B are those to which every point of the line FL has the same relation, then D will be unique with its situs to them and will fall in the Line through $A.B$. It also appears [videtur] to follow that these Lines do not cross each other except in one point.