University of Oxford

Geometric Quantity in Leibniz's Analysis Situs

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Thesis submitted in partial fulfilment of the requirements for the degree of *Bachelor of Philosophy in Philosophy*.

Word count: 31,952 (authorised limit extension to 32,500).

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June 2017

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Abstract

Leibniz's *Analysis Situs* was a programme to reform geometry. He wanted to provide a new characteristic, or artificial technical language, for geometric *quality* understood by some kind of contrast with geometric *quantity*. This thesis is a study of Leibniz's concept of geometric quantity aimed to help us understand the Analysis Situs.

The few works of critical scholarship on the Analysis Situs employ a naïve concept of geometric quantity as a simple distance relation predicable of abstract mathematical figures and their parts. It is supposed that Leibniz thought that quantities and qualities are mutually independent and jointly exhaustive classes of attributes of abstract figures, and it is supposed that Leibniz's plan with the Analysis Situs was to develop a characteristic for geometric quality in contradistinction to quantity. There is no indication that Leibniz was moving in the direction of projective geometry or modern topology, which in their own ways eliminate geometric quantity, so these scholars conclude that his project was a failure.

I offer a novel analysis of the logic of geometric quantity with key features: (1) Geometric quantity is a species of a more general metaphysical concept of quantity that is 'the affection of a composite relative to its parts'. (2) Geometric quantity is aimed at bodies or figures with indeterminate place and extension in an abstract homogeneous continuum, and place and extension are abstracted from the determinate 'situs' and 'complexion' of a prior phenomenal reality that I call the 'perceptual field'. (3) Since geometric quantity requires the abstraction *and* the concrete perceptual composite, it is not attributable to abstract figures.

This account shows Leibniz's concept of geometric quality to be close to our concept of *abstract structure*, where identity of quality (similarity) is just isomorphism. The separation between the abstraction and the perceptual field combines with the dependence of quantity on the perceptual field to yield a very advanced notion of *mathematical space*. The account also shows that critics were wrong to think geometric quantity is an attribute of an abstract figure and were wrong to think Leibniz aimed for a geometry without quantity.

Introduction

Context: quantity and the Analysis Situs. Leibniz developed the Analysis Situs in the period just after his mathematical education and invention of the infinitesimal calculus in Paris, and his work on the project in that period culminated in a long essay from 1679 called *Characteristica Geometrica*.[1] He wrote a letter to Christiaan Huygens that year advertising the programme laid out in the essay. The letter concisely conveyed the most important and fundamentally novel aspects of his project, at least to the degree that he thought they should be comprehensible and appealing to the intellectuals of his day. Huygens responded dismissively and Leibniz never published his essay, but he continued to work on the Analysis Situs and alluded to his work in later correspondence.

In his letter Leibniz proposed a project to reform geometry by giving long overdue attention to distinctively geometric *quality*. He explains how *quantity* has already received ample attention, relatively speaking, while quality has been neglected. Everyone knows that geometric figures contain quantities, but many seem to have forgotten that they have qualities too. Quantity has been provided an exemplary 'characteristic', or an artificial technical language, in arithmetic and algebra (determinate and indeterminate quantities, respectively); geometric quality has no such tailored language. For the purpose of improving the science of shapes and everything that depends on it, Leibniz wished to develop a specialised characteristic for geometric quality.

Quantity and quality are distinguished in a lengthy digression of the *Characteristica Geometrica*[2] as well as in nearly all of Leibniz's later writings on the Analysis Situs. Quality[3] is whatever may be discerned in a figure *taken singly*, while quantity or mag-

[1] GM, V.141-179.

[2] GM, §§23-39.

[3] Leibniz uses 'quality' and 'form' interchangeably in the Analysis Situs; I use mainly 'quality' to avoid confusion.

nitude[4] is only discerned in several figures by *simultaneous co-perception*. His favourite illustration for quantity is a pair of similar triangles of different sizes: they can be distinguished only if viewed together (immediate co-perception) or if a third 'measuring rod' can be moved between them (mediate co-perception). The quantity-quality distinction appears most clearly in the later *De Analysis Situs* from 1693:[5]

> Besides quantity, figure in general includes also quality or form. And as those figures are *equal* whose magnitude is the same, so those are *similar* whose form is the same. The theory of similarities or of forms lies beyond mathematics and must be sought in metaphysics. Yet it has many uses in mathematics also, being of use even in the algebraic calculus itself. But *similarity* is seen best of all in the *situations* or figures of geometry. Thus a true geometric analysis ought not only consider equalities and proportions which are truly reducible to equalities but also similarities and, arising from the combination of equality and similarity, congruence.[6]

The distinction is cast in perceptual terms in the *Characteristica Geometrica*:

> If we imagined that God shrunk everything that appears in us and around us in some room, keeping the same proportions, everything would appear in the same way, and we would not be able to discern the difference between the first and last state, unless of course we emerged from the sphere of things proportionally diminshed, that is to say, from our room; then indeed by co-perception, bringing them before the unshrunk things, the distinction would appear. From this it is manifest indeed that *Magnitude* is that which may be distinguished in things only by co-perception, that is, either by immediate application, the things being either actually congruent or coincident, or else mediate, by the intervention of a measure, which now

[4]Leibniz uses 'magnitude' for 'quantity' but with a special connotation to be explained later.
[5]I follow De Risi's dating of the relevant texts (De Risi, *Geometry and monadology*, 117-126).
[6]L, 254-55, *De Analysis Situs*, emphasis original.

> to one and now to the other is applied, this sufficing for the things to be congruent, that is, to be able to be made congruent by action.[7]

These considerations motivate a natural research question that guides this thesis. How should we understand Leibniz's concept of quantity in geometry? More specifically, how can we understand it in a way that explains the role of the quantity-quality distinction in defining and realising the Analysis Situs? Leibniz portrays quality by a contrast with quantity, and this contrast played a critical role in the designation of his project, so understanding what he meant by quantity is a necessary and significant step toward understanding his project.

If geometric quantity illuminates geometric quality and through that the Analysis Situs, the Analysis Situs itself illuminates Leibniz's life programme. A characterisation of his programme is emerging from sustained attempts to find a real unity behind the plurality of his myriad scattered projects. Maria Antognazza argues in her recent study[8] that much of Leibniz's labour can be understood as the outworking of a hope to develop a universal encyclopaedia of human science, and to provide each branch with a naturally fitted technical language that is a branch of the corresponding universal characteristic. Analysis Situs is the received title for the sector devoted to *shape and figure*, whether arising in geometry proper, or else in architecture, engineering, materials science, or anything involving spatial structures, and Leibniz meant to develop that branch together with a geometric characteristic. This new science received more sustained attention over the course of his life than any other branch of his grand project save perhaps logic.

[7]'Si vero fingeremus, DEUM omnia in nobis ac circa nos in aliquo cubiculo apparentia proportione eadem servata minuere, omnia eodem modo apparerent neque a nobis prior status a posteriore posset discerni, nisi sphaera rerum proportionaliter imminutarum, cubiculo scilicet nostro egrederemur; tunc enim comperceptione illa cum rebus non imminutis oblata discrimen appareret. Hinc manifestum est etiam, *Magnitudinem* esse illud ipsum quod in rebus distingui potest sola comperceptione, id est applicatione vel immediata, sive congruentia actuali sive coincidentia, vel mediata, nempe interventu mensurae, quae nunc uni nunc alteri applicatur, unde sufficit res esse congruas, id est actu congruere posse.' (GM, 152-4, my translation.)

[8]Antognazza, *Leibniz*.

The Analysis Situs was developed *primarily as* a piece of his encyclopaedia after Leibniz left Paris. Now Leibniz thought his Infinitesimal Calculus was as successful as it was because it constituted a very effective and natural characteristic for a useful subject matter, namely continuous quantity. Undoubtedly it received time and attention comparable to what he gave the Analysis Situs. But, in contrast to the Analysis Situs, it was inspired by engagement with concrete problems (and mathematicians) in Paris. The Analysis Situs is therefore the more authentic developed direct exhibition of his programme. So if anything about that programme can be learned from a detailed case study, the Analysis Situs is the case to study.

Interpretive questions. The concepts of quantity and quality cast shadows in the formal abstract system of pure mathematics. That this is so for quality is the point of the *De Analysis Situs* passage above, and 'a true geometric analysis ought not *only* consider equalities' implies the same for quantity. Importantly, these shadows are cast by full-bodied concepts properly found in the trunk of the encyclopaedia, in metaphysics. And the rays delineating these shadows are mediated by a phenomenal layer, since apparently we can define magnitude as that 'distinguished in things only by co-perception'.

An image of Leibniz's quantity-quality distinction can be found, approached, and analysed in each of these three domains, in metaphysics, phenomenology, and mathematics. This means interpretive questions about the distinction and its role in framing Leibniz's project come in two kinds: first-order questions about the image in a single domain, and second-order questions concerning the relationships between images in different domains. Examples of the former questions would be those concerning the definition and application of quantity in mathematical practice; the latter are illustrated by questions about the relation between mathematical and metaphysical definitions of quantity.

One second-order question is of particular interest for us. We can ask whether the image of quantity in geometry can be understood without invoking links to the other domains, to metaphysics and phenomenology. That is to say, we can ask whether the image of quantity in geometry is relevantly *autonomous*.

It might be reasonable to ask this question if sustained attempts to understand the role of quantity in the foundation of Leibniz's geometric project that restrict attention to the abstract mathematical domain fail to make sense of his claims or fail to yield an analysis able to resist the most obvious objections. The cardinal hypothesis of the present work presupposes (1) that extant attempts to understand Leibniz's Analysis Situs have failed in just this regard, putting us in a context that justifies the second-order question whether geometric quantity is autonomous, and it supposes (2) that these efforts failed because geometric quantity in fact is *not* autonomous, but instead it can only be understood in connection with the perceptual domain. In a relevant sense the same is not true of geometric quality.

Traditional approaches to understanding quantity in the Analysis Situs have largely recognised these three domains, have recognised that the concept of quantity makes an appearance in each, and have even recognised that the various images are (meant to be) images of the same thing.[9] The most basic second-order question has been asked too, the question whether the images cohere. But I do not believe adequate attention has been given to the possibility that the images simply are not autonomous, meaning (specifically, for us) the possibility that the mathematical image of quantity *in geometry* is necessarily incomplete.

Approaches that take for granted the autonomy of geometric quantity have failed to explain and failed to support Leibniz's distinction and the project it designated. They have failed to explain the way Leibniz uses co-perception in working out mathematical

[9] De Risi divides his book (De Risi, *Geometry and monadology*) into three parts for these three domains. The division between geometry and phenomenology has made it difficult to see that the concept of quantity in (the abstract part of) geometry is incomplete. Cox, in her dissertation (Cox, "Leibniz's philosophy of space"), appears to be the first to make these divisions explicit in a work on the Analysis Situs. Couturat didn't divide these domains in (Couturat, *La logique de Leibniz*) but equally didn't draw the phenomenal into geometric quantity, and that led to the misreading we discuss later.

demonstrations, and the contexts in which he both denies the possibility of defining a unit of length while also maintaining either that quantity can be known through co-perception or that quantity consists in the repetitions that may be discerned in a thing. They also failed to provide interpretations that resist seemingly obvious objections. One popular objection is that Leibniz's geometry was inescapably metrical (if not necessarily Euclidean), and this precludes any possibility of developing a distinctively qualitative characteristic for geometry. Other objections are that Leibniz never managed to define quantity at all, or that his definitions are manifestly circular.[10] These failings motivate a reassessment of autonomy.

Geometric quantity depends, I propose, on the perceptual domain as follows: the *concept* of geometric quantity can be defined abstractly, as well as certain abstract or 'extrinsic' quantities (numbers or ratios), and used in mathematics, but a concrete 'intrinsic' geometric quantity, such as the length of a *particular* line segment, cannot be defined. The reason it cannot be defined is that the abstraction of a line is a 'stepping away', to invoke etymology, from the very information given in perception that is necessary to define a quantity. Geometry trades in the abstractions of place and extent, and it is in the nature of places and extents that they cannot be distinguished from each other by internal conceptual constituents.

Given the change in presupposition that I advocate, the original interpretive challenges must be faced anew. We need a detailed exposition of the concept of geometric quantity that draws explicitly on more than the abstract mathematical domain, and that explains the role Leibniz thinks the definition has in the foundation of his project. That is what I hope to provide in this thesis.

In the next section I sketch what I take to be (an approximation of) the minimal coherent whole needed to make sense of Leibniz's concept of geometric quantity. For clarity I save the textual defence and discussion for the thesis body, indicating sources here in very broad terms in footnotes. Not every detail is essential for my main claims,

[10]These objections are treated in detail in Chapter 3.

and not every detail is defended in the thesis; I won't insist on details that I don't defend.

Geometric quantity: a sketch. Geometric quantity is a species of metaphysical quantity. The general features of the logic of geometric quantity are shared by all applications of the metaphysical notion. Quantity is the relative 'affection' of a composite whole to its parts, with special reference to a certain part taken as the measure or unit: quantity is expressed by the number representing the whole as a repetition of this unit. The parts within a composite whole may have their own parts ('complexion') and relations to each other ('relative situs') or to the whole ('absolute situs'). This general metaphysical concept should be set against the general metaphysical concepts of quality and relation. *Quality* is an absolute affection of a thing, *quantity* is the relative affection of a composite thing to its parts, and *relation* is the relative affection of one thing to another.[11]

The logic of quantity presupposes a composite whole, and the composite whole for geometric quantity is what I shall call the *perceptual field*. The perceptual field is not precisely that given in perception. Perception is the expression of the universe (the many) in the perceiving monad (the one). Some monads are capable of an intellectual act ('recognition') which carves that given in perception into *regions*, or the parts of the composite perceptual field. Every monad expresses in itself every other monad to some degree of clarity. But not every monad can recognise regions as parts of a whole, and accordingly not every monad has a perceptual field. It isn't clear whether worms or amoebae have perceptual fields. Potatoes do not; humans do.[12]

Geometric quantity, viz. length, area, or volume, involves a mode of repetition of the chosen measure that presupposes the abstractions of place and extent. *Place* is an abstract relation between one body and another (or all the others) and is a relative (or

[11]This metaphysical backdrop is drawn from *De Arte Combinatoria* (L, 76-7). That piece is very early (1666) but I believe the 'situs' of later mathematical works is just the mathematical expression of this metaphysical 'situs'.

[12]The perceptual field, meaning some more determinate composite than the abstractions of bodies in space, is an essential posit of my analysis and is not *quite* explicitly found in Leibniz.

absolute) relation. One place could be occupied by any of various bodies and each body could occupy any place without internal change; this is what it means for place to be an abstract relation. Since a body has no logically privileged place, *place is indeterminate*. Likewise *extension* is an abstract relation, but that of a body to its parts: the same parts could compose each of various bodies and the same body could be composed of parts in various ways. Since a body has no privileged composition, *composition is indeterminate*. These two indeterminacies are responsible for the *homogeneity* and *continuity* of space, respectively.[13]

In the most general possible terms, place is abstracted from the perceptual field by ignoring the confused infinite detail inside a region in a pretence that everything lies outside. Extension abstracts from the perceptual field by ignoring the confused infinite detail outside a region in a pretence that everything lies inside. The extreme of place is a point, the extreme of extension is space. Space is the totality of place.[14]

The perceptual field is still inherently perspectival, and is conceptually prior to the notions of place and extension. The abstractions of place and extent are attempts to transcend perspective: the same body can preserve place and size while the situs and complexion of the corresponding perceptual regions change. The abstractions of place and extent facilitate a general description of the world, a description according to a 'view from nowhere'. But they are abstractions.

Since place and extent are abstract relations, they are ideal. Indeed insofar as they are indeterminate, which by nature they are, they are not particulars. Any choice (or labelling, naming, conceiving, determining) of size or position must have a *reason* since every act of the intellect has a reason, but since no position or composition is more

[13]The connection between indeterminacy, possibility, abstract relations, and the natures of place and extent are drawn mainly from Leibniz's correspondences with Arnauld, De Volder, and Clarke. Note 'homogeneous' is used here with the modern sense for space, not Leibniz's sense as explained in (§1.4, p. 28), while 'continuity' is meant with Leibniz's technical sense.

[14]This internal-external dialectic is a synthesis of remarks in writings on the Analysis Situs (absolute place is a simple point, absolute extent is the space of all place; Cf. §§9-10 of the *Characteristica Geometrica*) and §47 of the fifth letter to Clarke ('I will here show how men come to form to themselves the notion of space').

natural than another (homogeneity and continuity), that is, since place and extent are abstract relations, no reason for any such choice could be given. Since the parts of abstract space are indeterminate, there can be no 'certain part' taken for measure or unit, and since the places are indeterminate, one could not count the parts of space even if their sizes were determinate. Therefore an abstract figure, which is a synthesis of the abstractions of place and extent, cannot be ascribed a geometric quantity.

And neither can a region in the perceptual field be ascribed a geometric quantity. A particular region has a particular situs and complexion which in its infinite detail is ineluctably perspectival: its situs and complexion are *fully determinate*. Regions are fully distinguished by their 'herenesses' and 'therenesses'. There is therefore no possibility of the repetition that is required to define quantity.

Without the abstraction there cannot be repetition, but without the perceptual field repetition cannot be marked. Geometric quantity is neither entirely real nor completely ideal. It must, therefore, live somewhere *between* the perceptual field and the 'external' abstract world of 'bodies in space'.

Through the particular intellectual acts of abstraction, the places and extents of abstract figures are *grounded* in the perceptual field. The abstract is never given (and can't be given) alone; it always arises from a pretence about concrete particulars, namely particular parts of the perceptual field. The possibility of abstraction is consequently accompanied by the possibility of returning to the detail and determinacy of perception. Abstraction is pretence, not deception.

Geometric quantity, finally, invokes the determinate particularity of the complex parts of the perceptual field in order to specify a unit or measure and to distinguish the parts to be counted by that measure. It invokes the abstraction of place to pretend the measure exists in multiple places, and the abstraction of extent to pretend the measure's repetitions in these places have one size. Given all this, geometric quantity is expressed by the number appropriate to the repetition. In this sense geometric quantity lives between the abstraction and its perceptual ground: the mind must exercise an ability to consider an abstraction, to consider it *qua* abstraction, and to consider the perceptual field *of which* it is an abstraction, all at the same time.

Despite the indeterminacy and ideality of figures in space, they may be considered to have a certain kind of nature. There are some things one may say about a figure that do not depend on its particular perceptual ground. A triangle, for instance, always has three sides and three vertices. A right triangle, moreover, has side *ratios* satisfying the Pythagorean theorem. All this suggests that it is useful to say there is a kind of properly abstract *geometric quality* that can be shared by figures grounded in distinct perceptual fields.

That abstract figures have certain qualities means only that some things can be said about figures independently of their particular perceptual grounds. This in turn, combined with the variety of possible perceptual fields (whether actual or imagined), entails a *separation* of content between the abstraction and its perceptual field. Perhaps the most important idea here is this separation of mental content into layers, the abstract layer grounded in the more concrete phenomenal layer. For with this all the ingredients are available to fashion a robust concept of the notion of *mathematical space*. The 'isomorphism structure' of a mathematical space, with figures as parts, is given by its abstract geometric form, and the 'particular' space is determined by the (real or imagined) perceptual ground.

Geometric quantity must be relative in the practice of geometry to a mathematical space since it is relative in the real world to the perceptual field. If two mathematical spaces are different (by fiat – by imagining their perceptual fields to be different), they and their parts cannot be compared in quantity, but they may nonetheless be comparable in quality or abstract structure.

Thesis plan. I hope to accomplish three things in this thesis, in three chapters. In the first chapter I discuss a series of texts explaining various facets of the most general metaphysical definition of quantity. In the second chapter I turn to the geometric context and aim to establish my main hypothesis – that quantity in geometry draws from a more concrete perceptual field as well as from mathematical abstractions – and I hope to provide the beginnings of a plausible story about how exactly this works. In the final chapter I show how the foregoing analysis, primarily the main hypothesis, can serve as a resource from which Leibniz's project is easily defended against otherwise powerful objections.

Attention to the most general metaphysical concept of quantity in the first chapter sets the stage for the main story of geometric quantity in the second chapter. Showing the full structure of the concept, and seeing the geometric species as an image or shadow in a restricted domain, will set in clear relief certain pieces of the concept that are missing from the restricted domain. Two pieces missing from abstract geometry are (1) the requisite composite whose 'affection to its parts' yields quantity, and (2) a determinate measure or unit by repetition of which quantity is expressed. (These are provided by the perceptual field.) A convenient consequence of this approach is that it removes some temptation to think the variety of ideas and definitions Leibniz considers represents oscillation between different conceptions of quantity. The variety can be seen instead either as aspects of a single complex concept in a particular domain, or the result of a struggle to express in that domain a concept he understood well in general metaphysical terms. In the first chapter several notions related to quantity will also be discussed, most importantly abstract quantity and ratio, since these play an important role in the last chapter and conclusion.

The second chapter presents the main business. My goal is to propose a well-motivated theory of Leibniz's geometric quantity that draws from the perceptual domain where necessary. I postulate a single entity at the conceptual layer of perception – the perceptual field – that satisfies three independent interpretive needs. In the first place, the perceptual field serves as the prerequisite part-whole composite: quantity is always the affection of some composite to its parts. Regions of the perceptual field also provide the more determinate concrete parts from which bodies in space are abstracted, things Leibniz seems to presuppose in his description of the abstraction process to De Volder and to Clarke. That these regions are more determinate also means a *choice of unit* can be grounded at this level, whereas no such choice has sufficient reason in the abstract domain alone. Finally, the perceptual field plays the all-important role of explaining the notion of 'co-perception', since figures are 'co-perceived' precisely when they are parts of the same perceptual field.

There is a series of criticisms that if correct would decimate Leibniz's project, showing it a pipe dream founded on simple mistakes. Leibniz doesn't seem to have given up, or thought it a failure, let alone a disaster, after working off and on for some 36 years on the project. If we find the project so easy to dismantle, it seems appropriate to ask whether we've properly understood it. In the last chapter I aim to show that these criticisms are themselves founded on mistakes.

A remaining desideratum is to clarify the role of the quantity-quality distinction in the foundation and definition of the Analysis Situs. In the conclusion I hope to use the criticisms in the last chapter as pivot points, as it were, from which to suggest that the distinction did not have the role traditionally assumed. I suggest that the quantity-quality distinction does not pose constraints on the abstract structure of geometry at all, and wasn't meant to (except that the abstract structure is supposed to be abstracted from perception). The purpose of the distinction was to *point the way* toward geometric quality, to manifest the fact that there is such a thing, in order to gain support for his programme. Lastly, at the very end, I suggest some directions for future work.