

UNIVERSITY OF OXFORD

Geometric Quantity in Leibniz's *Analysis Situs*

Matthew McMillan

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Supervised by Prof Paul Lodge, Mansfield College, Oxford.

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Abstract

Leibniz's *Analysis Situs* was a programme to reform geometry. He wanted to provide a new characteristic, or artificial technical language, for geometric *quality* understood by some kind of contrast with geometric *quantity*. This thesis is a study of Leibniz's concept of geometric quantity aimed to help us understand the *Analysis Situs*.

The few works of critical scholarship on the *Analysis Situs* employ a naïve concept of geometric quantity as a simple distance relation predicable of abstract mathematical figures and their parts. It is supposed that Leibniz thought that quantities and qualities are mutually independent and jointly exhaustive classes of attributes of abstract figures, and it is supposed that Leibniz's plan with the *Analysis Situs* was to develop a characteristic for geometric quality in contradistinction to quantity. There is no indication that Leibniz was moving in the direction of projective geometry or modern topology, which in their own ways eliminate geometric quantity, so these scholars conclude that his project was a failure.

I offer a novel analysis of the logic of geometric quantity with key features: (1) Geometric quantity is a species of a more general metaphysical concept of quantity that is 'the affection of a composite relative to its parts'. (2) Geometric quantity is aimed at bodies or figures with indeterminate place and extension in an abstract homogeneous continuum, and place and extension are abstracted from the determinate 'situs' and 'complexion' of a prior phenomenal reality that I call the 'perceptual field'. (3) Since geometric quantity requires the abstraction *and* the concrete perceptual composite, it is not attributable to abstract figures.

This account shows Leibniz's concept of geometric quality to be close to our concept of *abstract structure*, where identity of quality (similarity) is just isomorphism. The separation between the abstraction and the perceptual field combines with the dependence of quantity on the perceptual field to yield a very advanced notion of *mathematical space*. The account also shows that critics were wrong to think geometric quantity is an attribute of an abstract figure and were wrong to think Leibniz aimed for a geometry without quantity.

Introduction

Context: quantity and the Analysis Situs. Leibniz developed the Analysis Situs in the period just after his mathematical education and invention of the infinitesimal calculus in Paris, and his work on the project in that period culminated in a long essay from 1679 called *Characteristica Geometrica*.¹ He wrote a letter to Christiaan Huygens that year advertising the programme laid out in the essay. The letter concisely conveyed the most important and fundamentally novel aspects of his project, at least to the degree that he thought they should be comprehensible and appealing to the intellectuals of his day. Huygens responded dismissively and Leibniz never published his essay, but he continued to work on the Analysis Situs and alluded to his work in later correspondence.

In his letter Leibniz proposed a project to reform geometry by giving long overdue attention to distinctively geometric *quality*. He explains how *quantity* has already received ample attention, relatively speaking, while quality has been neglected. Everyone knows that geometric figures contain quantities, but many seem to have forgotten that they have qualities too. Quantity has been provided an exemplary ‘characteristic’, or an artificial technical language, in arithmetic and algebra (determinate and indeterminate quantities, respectively); geometric quality has no such tailored language. For the purpose of improving the science of shapes and everything that depends on it, Leibniz wished to develop a specialised characteristic for geometric quality.

Quantity and quality are distinguished in a lengthy digression of the *Characteristica Geometrica*² as well as in nearly all of Leibniz’s later writings on the Analysis Situs. Quality³ is whatever may be discerned in a figure *taken singly*, while quantity or mag-

¹GM, V.141-179.

²GM, §§23-39.

³Leibniz uses ‘quality’ and ‘form’ interchangeably in the Analysis Situs; I use mainly ‘quality’ to avoid confusion.

nitude⁴ is only discerned in several figures by *simultaneous co-perception*. His favourite illustration for quantity is a pair of similar triangles of different sizes: they can be distinguished only if viewed together (immediate co-perception) or if a third ‘measuring rod’ can be moved between them (mediate co-perception). The quantity-quality distinction appears most clearly in the later *De Analysis Situs* from 1693:⁵

Besides quantity, figure in general includes also quality or form. And as those figures are *equal* whose magnitude is the same, so those are *similar* whose form is the same. The theory of similarities or of forms lies beyond mathematics and must be sought in metaphysics. Yet it has many uses in mathematics also, being of use even in the algebraic calculus itself. But *similarity* is seen best of all in the *situations* or figures of geometry. Thus a true geometric analysis ought not only consider equalities and proportions which are truly reducible to equalities but also similarities and, arising from the combination of equality and similarity, congruence.⁶

The distinction is cast in perceptual terms in the *Characteristica Geometrica*:

If we imagined that God shrunk everything that appears in us and around us in some room, keeping the same proportions, everything would appear in the same way, and we would not be able to discern the difference between the first and last state, unless of course we emerged from the sphere of things proportionally diminished, that is to say, from our room; then indeed by co-perception, bringing them before the unshrunk things, the distinction would appear. From this it is manifest indeed that *Magnitude* is that which may be distinguished in things only by co-perception, that is, either by immediate application, the things being either actually congruent or coincident, or else mediate, by the intervention of a measure, which now

⁴Leibniz uses ‘magnitude’ for ‘quantity’ but with a special connotation to be explained later.

⁵I follow De Risi’s dating of the relevant texts (De Risi, *Geometry and monadology*, 117-126).

⁶L, 254-55, *De Analysis Situs*, emphasis original.

to one and now to the other is applied, this sufficing for the things to be congruent, that is, to be able to be made congruent by action.⁷

These considerations motivate a natural research question that guides this thesis. How should we understand Leibniz's concept of quantity in geometry? More specifically, how can we understand it in a way that explains the role of the quantity-quality distinction in defining and realising the Analysis Situs? Leibniz portrays quality by a contrast with quantity, and this contrast played a critical role in the designation of his project, so understanding what he meant by quantity is a necessary and significant step toward understanding his project.

If geometric quantity illuminates geometric quality and through that the Analysis Situs, the Analysis Situs itself illuminates Leibniz's life programme. A characterisation of his programme is emerging from sustained attempts to find a real unity behind the plurality of his myriad scattered projects. Maria Antognazza argues in her recent study⁸ that much of Leibniz's labour can be understood as the outworking of a hope to develop a universal encyclopaedia of human science, and to provide each branch with a naturally fitted technical language that is a branch of the corresponding universal characteristic. Analysis Situs is the received title for the sector devoted to *shape and figure*, whether arising in geometry proper, or else in architecture, engineering, materials science, or anything involving spatial structures, and Leibniz meant to develop that branch together with a geometric characteristic. This new science received more sustained attention over the course of his life than any other branch of his grand project save perhaps logic.

⁷'Si vero fingeremus, DEUM omnia in nobis ac circa nos in aliquo cubiculo apparentia proportionem eadem servata minuere, omnia eodem modo apparent neque a nobis prior status a posteriore posset discerni, nisi sphaera rerum proportionaliter imminutarum, cubiculo scilicet nostro egrederemur; tunc enim comperceptione illa cum rebus non imminutis oblata discrimen appareret. Hinc manifestum est etiam, *Magnitudinem* esse illud ipsum quod in rebus distingui potest sola comperceptione, id est applicatione vel immediata, sive congruentia actuali sive coincidentia, vel mediata, nempe interventu mensurae, quae nunc uni nunc alteri applicatur, unde sufficit res esse congruas, id est actu congruere posse.' (GM, 152-4, my translation.)

⁸Antognazza, *Leibniz*.

The Analysis Situs was developed *primarily as* a piece of his encyclopaedia after Leibniz left Paris. Now Leibniz thought his Infinitesimal Calculus was as successful as it was because it constituted a very effective and natural characteristic for a useful subject matter, namely continuous quantity. Undoubtedly it received time and attention comparable to what he gave the Analysis Situs. But, in contrast to the Analysis Situs, it was inspired by engagement with concrete problems (and mathematicians) in Paris. The Analysis Situs is therefore the more authentic developed direct exhibition of his programme. So if anything about that programme can be learned from a detailed case study, the Analysis Situs is the case to study.

Interpretive questions. The concepts of quantity and quality cast shadows in the formal abstract system of pure mathematics. That this is so for quality is the point of the *De Analysis Situs* passage above, and ‘a true geometric analysis ought not *only* consider equalities’ implies the same for quantity. Importantly, these shadows are cast by full-bodied concepts properly found in the trunk of the encyclopaedia, in metaphysics. And the rays delineating these shadows are mediated by a phenomenal layer, since apparently we can define magnitude as that ‘distinguished in things only by co-perception’.

An image of Leibniz’s quantity-quality distinction can be found, approached, and analysed in each of these three domains, in metaphysics, phenomenology, and mathematics. This means interpretive questions about the distinction and its role in framing Leibniz’s project come in two kinds: first-order questions about the image in a single domain, and second-order questions concerning the relationships between images in different domains. Examples of the former questions would be those concerning the definition and application of quantity in mathematical practice; the latter are illustrated by questions about the relation between mathematical and metaphysical definitions of quantity.

One second-order question is of particular interest for us. We can ask whether the

image of quantity in geometry can be understood without invoking links to the other domains, to metaphysics and phenomenology. That is to say, we can ask whether the image of quantity in geometry is relevantly *autonomous*.

It might be reasonable to ask this question if sustained attempts to understand the role of quantity in the foundation of Leibniz's geometric project that restrict attention to the abstract mathematical domain fail to make sense of his claims or fail to yield an analysis able to resist the most obvious objections. The cardinal hypothesis of the present work presupposes (1) that extant attempts to understand Leibniz's Analysis Situs have failed in just this regard, putting us in a context that justifies the second-order question whether geometric quantity is autonomous, and it supposes (2) that these efforts failed because geometric quantity in fact is *not* autonomous, but instead it can only be understood in connection with the perceptual domain. In a relevant sense the same is not true of geometric quality.

Traditional approaches to understanding quantity in the Analysis Situs have largely recognised these three domains, have recognised that the concept of quantity makes an appearance in each, and have even recognised that the various images are (meant to be) images of the same thing.⁹ The most basic second-order question has been asked too, the question whether the images cohere. But I do not believe adequate attention has been given to the possibility that the images simply are not autonomous, meaning (specifically, for us) the possibility that the mathematical image of quantity *in geometry* is necessarily incomplete.

Approaches that take for granted the autonomy of geometric quantity have failed to explain and failed to support Leibniz's distinction and the project it designated. They have failed to explain the way Leibniz uses co-perception in working out mathematical

⁹De Risi divides his book (De Risi, *Geometry and monadology*) into three parts for these three domains. The division between geometry and phenomenology has made it difficult to see that the concept of quantity in (the abstract part of) geometry is incomplete. Cox, in her dissertation (Cox, "Leibniz's philosophy of space"), appears to be the first to make these divisions explicit in a work on the Analysis Situs. Couturat didn't divide these domains in (Couturat, *La logique de Leibniz*) but equally didn't draw the phenomenal into geometric quantity, and that led to the misreading we discuss later.

demonstrations, and the contexts in which he both denies the possibility of defining a unit of length while also maintaining either that quantity can be known through co-perception or that quantity consists in the repetitions that may be discerned in a thing. They also failed to provide interpretations that resist seemingly obvious objections. One popular objection is that Leibniz's geometry was inescapably metrical (if not necessarily Euclidean), and this precludes any possibility of developing a distinctively qualitative characteristic for geometry. Other objections are that Leibniz never managed to define quantity at all, or that his definitions are manifestly circular.¹⁰ These failings motivate a reassessment of autonomy.

Geometric quantity depends, I propose, on the perceptual domain as follows: the *concept* of geometric quantity can be defined abstractly, as well as certain abstract or 'extrinsic' quantities (numbers or ratios), and used in mathematics, but a concrete 'intrinsic' geometric quantity, such as the length of a *particular* line segment, cannot be defined. The reason it cannot be defined is that the abstraction of a line is a 'stepping away', to invoke etymology, from the very information given in perception that is necessary to define a quantity. Geometry trades in the abstractions of place and extent, and it is in the nature of places and extents that they cannot be distinguished from each other by internal conceptual constituents.

Given the change in presupposition that I advocate, the original interpretive challenges must be faced anew. We need a detailed exposition of the concept of geometric quantity that draws explicitly on more than the abstract mathematical domain, and that explains the role Leibniz thinks the definition has in the foundation of his project. That is what I hope to provide in this thesis.

In the next section I sketch what I take to be (an approximation of) the minimal coherent whole needed to make sense of Leibniz's concept of geometric quantity. For clarity I save the textual defence and discussion for the thesis body, indicating sources here in very broad terms in footnotes. Not every detail is essential for my main claims,

¹⁰These objections are treated in detail in Chapter 3.

and not every detail is defended in the thesis; I won't insist on details that I don't defend.

Geometric quantity: a sketch. Geometric quantity is a species of metaphysical quantity. The general features of the logic of geometric quantity are shared by all applications of the metaphysical notion. Quantity is the relative 'affection' of a composite whole to its parts, with special reference to a certain part taken as the measure or unit: quantity is expressed by the number representing the whole as a repetition of this unit. The parts within a composite whole may have their own parts ('complexion') and relations to each other ('relative situs') or to the whole ('absolute situs'). This general metaphysical concept should be set against the general metaphysical concepts of quality and relation. *Quality* is an absolute affection of a thing, *quantity* is the relative affection of a composite thing to its parts, and *relation* is the relative affection of one thing to another.¹¹

The logic of quantity presupposes a composite whole, and the composite whole for geometric quantity is what I shall call the *perceptual field*. The perceptual field is not precisely that given in perception. Perception is the expression of the universe (the many) in the perceiving monad (the one). Some monads are capable of an intellectual act ('recognition') which carves that given in perception into *regions*, or the parts of the composite perceptual field. Every monad expresses in itself every other monad to some degree of clarity. But not every monad can recognise regions as parts of a whole, and accordingly not every monad has a perceptual field. It isn't clear whether worms or amoebae have perceptual fields. Potatoes do not; humans do.¹²

Geometric quantity, viz. length, area, or volume, involves a mode of repetition of the chosen measure that presupposes the abstractions of place and extent. *Place* is an abstract relation between one body and another (or all the others) and is a relative (or

¹¹This metaphysical backdrop is drawn from *De Arte Combinatoria* (L, 76-7). That piece is very early (1666) but I believe the 'situs' of later mathematical works is just the mathematical expression of this metaphysical 'situs'.

¹²The perceptual field, meaning some more determinate composite than the abstractions of bodies in space, is an essential posit of my analysis and is not *quite* explicitly found in Leibniz.

absolute) relation. One place could be occupied by any of various bodies and each body could occupy any place without internal change; this is what it means for place to be an abstract relation. Since a body has no logically privileged place, *place is indeterminate*. Likewise *extension* is an abstract relation, but that of a body to its parts: the same parts could compose each of various bodies and the same body could be composed of parts in various ways. Since a body has no privileged composition, *composition is indeterminate*. These two indeterminacies are responsible for the *homogeneity* and *continuity* of space, respectively.¹³

In the most general possible terms, place is abstracted from the perceptual field by ignoring the confused infinite detail inside a region in a pretence that everything lies outside. Extension abstracts from the perceptual field by ignoring the confused infinite detail outside a region in a pretence that everything lies inside. The extreme of place is a point, the extreme of extension is space. Space is the totality of place.¹⁴

The perceptual field is still inherently perspectival, and is conceptually prior to the notions of place and extension. The abstractions of place and extent are attempts to transcend perspective: the same body can preserve place and size while the situs and complexio of the corresponding perceptual regions change. The abstractions of place and extent facilitate a general description of the world, a description according to a ‘view from nowhere’. But they are abstractions.

Since place and extent are abstract relations, they are ideal. Indeed insofar as they are indeterminate, which by nature they are, they are not particulars. Any choice (or labelling, naming, conceiving, determining) of size or position must have a *reason* since every act of the intellect has a reason, but since no position or composition is more

¹³The connection between indeterminacy, possibility, abstract relations, and the natures of place and extent are drawn mainly from Leibniz’s correspondences with Arnauld, De Volder, and Clarke. Note ‘homogeneous’ is used here with the modern sense for space, not Leibniz’s sense as explained in (§1.4, p. 28), while ‘continuity’ is meant with Leibniz’s technical sense.

¹⁴This internal-external dialectic is a synthesis of remarks in writings on the Analysis Situs (absolute place is a simple point, absolute extent is the space of all place; Cf. §§9-10 of the *Characteristica Geometrica*) and §47 of the fifth letter to Clarke (‘I will here show how men come to form to themselves the notion of space’).

natural than another (homogeneity and continuity), that is, since place and extent are abstract relations, no reason for any such choice could be given. Since the parts of abstract space are indeterminate, there can be no ‘certain part’ taken for measure or unit, and since the places are indeterminate, one could not count the parts of space even if their sizes were determinate. Therefore an abstract figure, which is a synthesis of the abstractions of place and extent, cannot be ascribed a geometric quantity.

And neither can a region in the perceptual field be ascribed a geometric quantity. A particular region has a particular situs and complexion which in its infinite detail is ineluctably perspectival: its situs and complexion are *fully determinate*. Regions are fully distinguished by their ‘herenesses’ and ‘therenesses’. There is therefore no possibility of the repetition that is required to define quantity.

Without the abstraction there cannot be repetition, but without the perceptual field repetition cannot be marked. Geometric quantity is neither entirely real nor completely ideal. It must, therefore, live somewhere *between* the perceptual field and the ‘external’ abstract world of ‘bodies in space’.

Through the particular intellectual acts of abstraction, the places and extents of abstract figures are *grounded* in the perceptual field. The abstract is never given (and can’t be given) alone; it always arises from a pretence about concrete particulars, namely particular parts of the perceptual field. The possibility of abstraction is consequently accompanied by the possibility of returning to the detail and determinacy of perception. Abstraction is pretence, not deception.

Geometric quantity, finally, invokes the determinate particularity of the complex parts of the perceptual field in order to specify a unit or measure and to distinguish the parts to be counted by that measure. It invokes the abstraction of place to pretend the measure exists in multiple places, and the abstraction of extent to pretend the measure’s repetitions in these places have one size. Given all this, geometric quantity is expressed by the number appropriate to the repetition. In this sense geometric quantity lives between the abstraction and its perceptual ground: the mind must exercise an ability to

consider an abstraction, to consider it *qua* abstraction, and to consider the perceptual field *of which* it is an abstraction, all at the same time.

Despite the indeterminacy and ideality of figures in space, they may be considered to have a certain kind of nature. There are some things one may say about a figure that do not depend on its particular perceptual ground. A triangle, for instance, always has three sides and three vertices. A right triangle, moreover, has side *ratios* satisfying the Pythagorean theorem. All this suggests that it is useful to say there is a kind of properly abstract *geometric quality* that can be shared by figures grounded in distinct perceptual fields.

That abstract figures have certain qualities means only that some things can be said about figures independently of their particular perceptual grounds. This in turn, combined with the variety of possible perceptual fields (whether actual or imagined), entails a *separation* of content between the abstraction and its perceptual field. Perhaps the most important idea here is this separation of mental content into layers, the abstract layer grounded in the more concrete phenomenal layer. For with this all the ingredients are available to fashion a robust concept of the notion of *mathematical space*. The ‘isomorphism structure’ of a mathematical space, with figures as parts, is given by its abstract geometric form, and the ‘particular’ space is determined by the (real or imagined) perceptual ground.

Geometric quantity must be relative in the practice of geometry to a mathematical space since it is relative in the real world to the perceptual field. If two mathematical spaces are different (by fiat – by imagining their perceptual fields to be different), they and their parts cannot be compared in quantity, but they may nonetheless be comparable in quality or abstract structure.

Thesis plan. I hope to accomplish three things in this thesis, in three chapters. In the first chapter I discuss a series of texts explaining various facets of the most general metaphysical definition of quantity. In the second chapter I turn to the geometric

context and aim to establish my main hypothesis – that quantity in geometry draws from a more concrete perceptual field as well as from mathematical abstractions – and I hope to provide the beginnings of a plausible story about how exactly this works. In the final chapter I show how the foregoing analysis, primarily the main hypothesis, can serve as a resource from which Leibniz’s project is easily defended against otherwise powerful objections.

Attention to the most general metaphysical concept of quantity in the first chapter sets the stage for the main story of geometric quantity in the second chapter. Showing the full structure of the concept, and seeing the geometric species as an image or shadow in a restricted domain, will set in clear relief certain pieces of the concept that are missing from the restricted domain. Two pieces missing from abstract geometry are (1) the requisite composite whose ‘affection to its parts’ yields quantity, and (2) a determinate measure or unit by repetition of which quantity is expressed. (These are provided by the perceptual field.) A convenient consequence of this approach is that it removes some temptation to think the variety of ideas and definitions Leibniz considers represents oscillation between different conceptions of quantity. The variety can be seen instead either as aspects of a single complex concept in a particular domain, or the result of a struggle to express in that domain a concept he understood well in general metaphysical terms. In the first chapter several notions related to quantity will also be discussed, most importantly abstract quantity and ratio, since these play an important role in the last chapter and conclusion.

The second chapter presents the main business. My goal is to propose a well-motivated theory of Leibniz’s geometric quantity that draws from the perceptual domain where necessary. I postulate a single entity at the conceptual layer of perception – the perceptual field – that satisfies three independent interpretive needs. In the first place, the perceptual field serves as the prerequisite part-whole composite: quantity is always the affection of some composite to its parts. Regions of the perceptual field also provide the more determinate concrete parts from which bodies in space are abstrac-

ted, things Leibniz seems to presuppose in his description of the abstraction process to De Volder and to Clarke. That these regions are more determinate also means a *choice of unit* can be grounded at this level, whereas no such choice has sufficient reason in the abstract domain alone. Finally, the perceptual field plays the all-important role of explaining the notion of ‘co-perception’, since figures are ‘co-perceived’ precisely when they are parts of the same perceptual field.

There is a series of criticisms that if correct would decimate Leibniz’s project, showing it a pipe dream founded on simple mistakes. Leibniz doesn’t seem to have given up, or thought it a failure, let alone a disaster, after working off and on for some 36 years on the project. If we find the project so easy to dismantle, it seems appropriate to ask whether we’ve properly understood it. In the last chapter I aim to show that these criticisms are themselves founded on mistakes.

A remaining desideratum is to clarify the role of the quantity-quality distinction in the foundation and definition of the Analysis Situs. In the conclusion I hope to use the criticisms in the last chapter as pivot points, as it were, from which to suggest that the distinction did not have the role traditionally assumed. I suggest that the quantity-quality distinction does not pose constraints on the abstract structure of geometry at all, and wasn’t meant to (except that the abstract structure is supposed to be abstracted from perception). The purpose of the distinction was to *point the way* toward geometric quality, to manifest the fact that there is such a thing, in order to gain support for his programme. Lastly, at the very end, I suggest some directions for future work.

Chapter 1

Quantity: a thing and its parts

The cornerstone of my analysis of geometric quantity is an analysis of Leibniz's more general metaphysical concept of quantity. I'll begin with his earliest work introducing quantity, continue with a discussion of the concept in some later writings, and then turn to an extended commentary on the related notions of *homogeneity*, *ratio*, and *number*. These notions reappear with important roles in the geometric context. I hope it becomes apparent from the discussion that Leibniz maintained the general contours of his concept for most of his life. The next chapter will show how geometric quantity can be understood in terms of the more general concept explained in this chapter.

1.1 Earliest definition

In *De Arte Combinatoria* of 1666, among the earliest of his writings, Leibniz distinguished quality, quantity, and relation. Quality is ascribed to a thing *qua* whole, quantity to a thing *qua* composition of parts, and relation to a thing *qua* distinct from another given thing. The parts of a composite whole have situs, which is either an 'affection' of a part relative to another or relative to the whole itself. The parts have complexion when they are composed of further parts.

[§1] Metaphysics, to begin at the top, deals with being and with the affections of being as well. Just as the affections of a natural body are not themselves bodies, however, so the affections of a being are not themselves beings.

[§2] An affection (or mode) of a being, moreover, is either something absolute, which is called *quality*, or something relative, and this latter is either the affection of a thing relative to its part if such is had,¹ that is, *quantity*, or that of one thing relative to another, *relation*. But if we speak more accurately and assume a part to be different from the whole, the quantity of a thing is also a relation to its part.²

[§3] Therefore, it is obvious that neither quality nor quantity nor relation is a being; it is their treatment in a signate actuality that belongs to metaphysics.

[§4] Furthermore, every relation is either one of *union* or one of *harmony*. In *union* the things between which there is this relation are called *parts*, and taken together with their union, a *whole*. This happens whenever we take many things simultaneously as *one*. By *one* we mean whatever we think of in one intellectual act, or at once. For example, we often grasp a number, however large, all at once in a kind of blind thought, namely, when we read figures on paper which not even the age of Methuselah would suffice to count explicitly.

[§5] The concept of *unity* is abstracted from the concept of one being, and the whole itself, abstracted from unities, or the totality, is called *number*. *Quantity* is therefore the number of parts. Hence quantity and number obviously coincide in the thing itself, but quantity is sometimes interpreted extrinsically, as it were, in a relation or ratio to another quantity, to aid us, namely, when the number of parts is unknown. . . .

[§8] Furthermore, the whole itself (and thus number or totality) can be broken up into parts, smaller wholes, as it were. This is the basis of

¹Loemker: 'to its parts if it has any'. The Latin is 'ad partem suam, si habet' (GM, V.12; G, IV.35). The whole is expressed quantitatively as a repetition of this (singular) part. The term should also be consonant with the subsequent two occurrences of 'part' in the singular.

²Or 'to a part' or 'to the part' since Latin lacks articles.

complexions, provided you understand that there are common parts in the different smaller wholes themselves. For example, let the wholes be ABC ; then AB , BC , and AC will be smaller wholes, its parts. And the disposition of the smallest parts, or of the parts assumed to be smallest (that is, the unities) in relation to each other and to the whole can itself also be varied. Such a disposition is called *situs*.

[§9] So there arise two kinds of *variation*: *complexion and situs*. And viewed in themselves, both complexion and situs belong to metaphysics, or to the science of whole and parts. If we look at their variability, however, that is, at the quantity of variation, we must turn to numbers and to arithmetic. I am inclined to think that the science of complexions pertains more to pure arithmetic, and that of situs to an arithmetic of figure. For so we understand unities to produce a line. ...³

Several points warrant further explanation in this context. Notice first from §2 that quality, quantity, and relation are all ‘affections’ of an individual being, but that quantity and relation are affections of a being *relative to* another being. In quantity the other is part of the same being, in relation it isn’t.

What is the ‘affection of a thing relative to its part’? Let me suggest the following representation (speculative for now), in order to guide discussion:

To express a composite thing **quantitatively** is to express the whole *qua* repetition of a certain part taken as measure or unit.

Notice first that this is a characterisation of the *expression* of quantity. According to Leibniz (only later, admittedly), something is said to express another when the relations between its parts correspond to the relations between the parts of the other. Often only a fraction of some things and their relations are expressed. I frame the concept

³L, 76-7.

of quantity in terms of expression in order to remain neutral regarding the meaning of ‘*qua* repetition of a part’, specifically to remain neutral as to whether it is actually possible for anything to be repeated (that is, for there to be individual things differing only numerically). If it isn’t possible in some context, as it isn’t for monads, then ‘expression *qua* repetition’ involves a *pretence* that something is repeated, and this pretence is effected by choosing not to express whatever detail would suffice to distinguish the things qualitatively.

Observe also that an expression of quantity thus construed presupposes a ‘certain part’ taken as measure, or even explicitly indicates that part, and that some method of counting (locating repetition) is required before a quantitative expression can be given or understood. A method of counting necessarily involves abstraction, or a grouping together of different things under the same universal, or a pretence that different parts are the same part seen over and over.

In §5 Leibniz writes that quantity is the ‘number of parts’, which suggests that ‘repetition’ cast as a pretence is out of place since parts needn’t look the same to be numbered. But notice that he is defining *number*, which is the abstraction of the totality of unities, and unities are simples. Therefore he can say that quantity is the number of parts because the parts are taken as numerically distinct simple unities. So ‘repetition’ isn’t in fact out of place.

The ‘intellectual act’ of §4 is worth emphasising. Leibniz thinks that composite wholes are generated⁴ by *acts of taking things together*. Let us call them acts of synthesis. It follows that a determination of quantity presupposes an act of synthesis, since such an act is required for the very possibility of a composite whole. His discovery that ‘co-perception’ lies at the foundation of geometric quantity is, to be argued later, the discovery of the relevant synthesis for geometry. The synthesis is what I call the perceptual field, in virtue of which one simultaneously apprehends a whole with dis-

⁴It is admittedly not clear from the passage whether some or all composites are generated by intellectual acts. The phrase ‘this happens whenever’ in §4 can be read either way. My argument does not turn on this question.

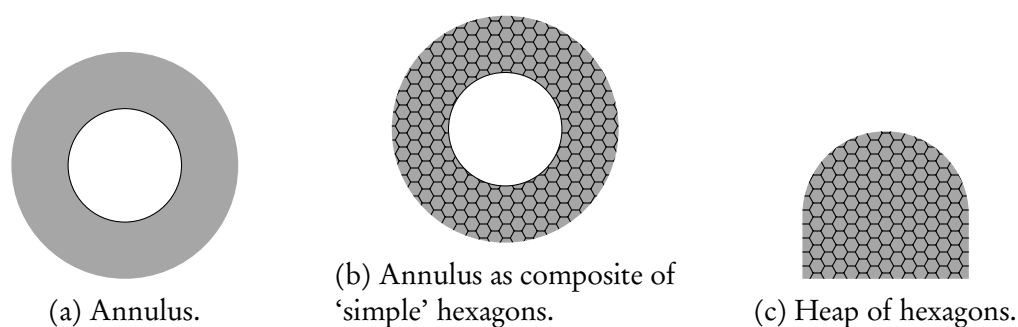


Figure 1.1: Leibniz's quantity and situs.

tinguishable parts or 'regions'.

Observe from §8 that the part-whole relation is denoted *complexion*, while the pure part-part relation is called *situs*. Situs is most clearly presented in terms of the smallest parts since they involve only part-part relations and no part-whole relations.

A concrete illustration of these terms may help to track further discussion. Let us consider solid 2-dimensional geometric shapes with boundaries. Take for example a ring or annulus, as in Fig. 1.1 (a). Imagine (b) carving the shape into a large number of repetitions of a simple shape: here hexagons are chosen for their tiling properties. Pretend that one of these small hexagons has no parts. Let 'hg' (singular) and 'hgs' (plural) denote these hexagons. Then to express the quantity of the ring we simply express the ring as (say) '57 hgs'. Supposing the heap (c) to have the same number, its quantity is expressed the same way.

Situs, on the other hand, is the arrangement of those same hexagons with respect to each other. There is a method of constructing the annulus, or the heap, out of hexagons. That method, or technique, 'building plan' we might say—a plan which *takes the hexagons as simples* and is concerned only with their arrangement—is the situs.

Quantity and situs are both expressed by choosing a 'measure' that is or can be taken as a simple part. Quantity is the number of repetitions of this part required to fill the whole, situs is the arrangement of these repetitions relative to each other or to the whole. Expressions of quantity and situs both, then, presuppose or take as given this measure, ignoring its further complexion.

1.2 Consonance with later writings

The idea that quantity involves a composite and is the affection of the whole relative to its parts continues into Leibniz's later writings.

In the *Characteristica Geometrica* of 1679, the main work on the Analysis Situs which he summarised for Huygens, Leibniz introduced the definition of geometric quantity as 'that known only by co-perception'. It is not obvious *a priori* how this definition is related to the general metaphysical notion, and certainly not clear that it is a species concept or even compatible. It is significant, then, that Leibniz reiterates the more general definition shortly after completing the *Characteristica Geometrica*. This can be seen in a short essay called *De Quantitate*, dated by De Risi to 1682:

Quantity is that which belongs to a thing insofar as it has all its parts, or because of which it can be said to be equal, greater, or less than another thing (anything homogeneous) or can be compared. The quantity of a thing e.g. of the area ABCD (fig. 7 [omitted]) is expressed by a number, for example, four, granted that another thing such as the square foot AEFG is taken for the primary measure or real unity.⁵

And in the same year, possibly the very same period, *De Magnitudine et Mensura*:

*Magnitude*⁶ is that in a thing which is expressed by the number of parts which are congruent to a given thing called a *measure*.⁷

Written shortly after intense work in the foundations of geometry, these statements characterise quantity in terms of a measure and congruence. But any expression of

⁵'*Quantitas* est id quod rei competit, quatenus habet omnes suas partes, sive ob quod alteri (homogeneae cuicunque) aequalis major aut minor dici sive comparari potest. *Quantitas* rei ex.gr. areae ABCD (fig. 7) exprimitur per numerum ex.gr. quaternarium, posito aliam rem ut pedem quadratum AEFG sumturo esse pro *Mensura* primaria seu *Unitate* reali.' (GM, VII.30, my translation.)

⁶Again, I assume 'magnitude' and 'quantity' are relevantly synonymous. This is confirmed by his (admittedly much later) '*Quantitas* seu *Magnitudo* est' of the *Initia* (GM, VII.18ff). A subtle difference of connotation is drawn out below.

⁷'*Magnitudo* est, quod in re exprimitur per numerum partium, congruentium rei datae, quae *Mensura* appellatur.' (GM, VII.35, my translation.)

quantity requires a part chosen as unit and a mode of repetition. In geometry the part is called a measure, and the mode of repetition is by congruence (defined through translation—see below).

Twenty years later, in a letter to Sophia in 1702 called *On what is independent of sense and matter*, Leibniz says that mathematical ideas ‘always include magnitude, or multitude of parts’.⁸ This indicates that Leibniz hadn’t abandoned the metaphysical definition even long after developing his project of *Analysis Situs*.

We have thus far contributed the term ‘repetition’ without explicit textual support, but this word is not hard to find. In the *Specimen Geometriae luciferae* of c. 1695 he wrote that magnitude (quantity) is given by counting repetitions of the measure:

Nevertheless the magnitude of each other thing can be known by taking one fixed thing as a measure, which is applied to other things by counting the number of repetitions.⁹

And then in the *Analysis geometrica propria* of c. 1698 Leibniz made a very general statement about the nature of quantity:

Magnitude is given even in things lacking situs, as in number, proportion, time, velocity, indeed in anything in which there are parts that may be evaluated by number or [*seu*] repetition. ... But situs adds [*superaddit*] a certain form [*formam*] to magnitude or [*vel*] multitude of parts, as in numerous shapes.¹⁰

The statement is key for my thesis that geometric quantity is a species of quantity, and that quantity always involves a composite thing whose composition of parts can be

⁸L, 548.

⁹‘Una tamen re fixa sumta, tanquam mensura, hujus applicatione ad alias res adhibitisque repetitionum numeris magnitudo quoque aliarum cognosci potest.’ (GM, VII.266, my translation.)

¹⁰Magnitudo datur etiam in illis quae situm non habent, ut numerus, proportio, tempus, velocitas, ubicunque scilicet partes existunt, quarum numero seu repetitione aestimatio fieri potest. ... Sed situs magnitudini vel partium multitudini formam quandam superaddit, ut in numeris figuratis. ... Nunc autem ad explicandam rem situs non nisi *congruentia* utemur, sepositis in alium locum *similitudine* et *motu*. (V.172 GM, my translation.)

expressed by repetition of a unit.

This statement also clarifies the contrast between magnitude and situs, an issue pertinent to an objection treated in Chapter 3: given a decomposition of a thing as a collection of parts (recall the small hexagons), magnitude is simply the number of those parts, and situs is something like their arrangement. Here Leibniz asserts that situs *adds* ('*superaddit*') to magnitude the arrangement data of shape or form. Situs *contains* magnitude in the following sense: insofar as form is conceived as the arrangement of certain simple parts, and magnitude is the number of those same simples, form contains magnitude because it is not possible to have two *different numbers* of parts in the *same arrangement*. The number cannot be changed without changing the arrangement. Form obviously goes *beyond* quantity since many different shapes can have the same number of parts. Given a fixed decomposition of a figure into simple parts (a big 'given', as will become more clear), quantity is just an *aspect* or *partial determination* of situs.

1.3 Quantity as clear or distinct, concrete or abstract

Leibniz defines and discusses quantity in a lengthy digression in the *Characteristica Geometrica*. His analysis there is of central importance for understanding his conception of quantity in connection with his programme for geometry.

(26) *Number* is that which has the relation to unity as that between part and whole or whole and part, in which way fractions and even irrationals are understood.

(27[a]) The *Magnitude* of a thing distinctly known is the number (or composite of numbers) of parts congruent to a certain fixed thing (which is assumed as a measure). So if I know there is a line segment that doubled is equal to a side, and tripled to the diagonal of a certain sufficiently known rectangle, so that I can return to the same thing at will, then I am said to

know the magnitude, of which the side has two congruent parts and the diagonal three congruent parts.

[(27b)] Various assumed measures are allowed, by which the same thing is in various ways expressed, nevertheless the same magnitude always results, since by decomposition¹¹ of the same measure again we always come to the same thing; and thus various measures already involve that very same number advancing by its decomposition. In this way the number three quarters is one and the same with six eighths if the quarter is decomposed into two parts. Such is Magnitude distinctly conceived.

[(27c)] Otherwise *magnitude* is the attribute of a thing through which one can tell whether some given thing is a part of it, or another thing is homogeneous to it, and as such remains if the constitution of the parts among themselves is allowed to be changed.

[(27d)] Or indeed Magnitude is the attribute which remains the same in the homogeneous things pertaining to it, or rather remains the same through substitution of congruents. I understand by ‘a thing homogeneous to some thing’ however not only parts but also extrema and minima, or points. Namely by continuous repetition of certain points, or by motion, a line is created. Often however a thing is transformed so that there is no unique part of the new figure congruent to a part of the previous one.

[(27e)] I define magnitude in another way below, as that by which two similar things can be discerned, or that in things which is only discerned by co-perception. But all this comes back to the same thing.

...

(31) ... From this it is manifest indeed that *Magnitude* is that which may be distinguished in things only by co-perception, that is, either by immediate application, the things being either actually congruent or coincident, or

¹¹Latin ‘resolutis’.

else mediate, by the intervention of a measure, which now to one and now to the other is applied, this sufficing for the things to be congruent, that is, to be able to be made congruent by action.¹²

The most striking and important feature of this passage is the expression of a conception of quantity in line with the basic metaphysical definition from above (27a-d) *immediately before* the seemingly different conception in (27e) and in (31). The first is the ‘expression through number according to a measure’, the second ‘that which may be discerned only by co-perception’. More striking still, he seems to equate them: ‘Sed omnia haec eodem recidunt’. We will return to this in the next chapter.

Leibniz distinguishes ‘distinct’ magnitude (27a-b) from magnitude ‘otherwise’ (27c-d). A thing is ‘expressed’ by a measure (27b). If a fixed measure is assumed (presupposed or given in the expression of magnitude), the magnitude is represented by a number, namely the number of parts congruent to the measure. Knowledge of quantity exhibited through number is *distinct*: a concrete mark or sign is provided in virtue of which the magnitude can be recalled, considered, and compared with others.

¹²“(26) *Numerus* est, cuius ad unitatem relatio est quae inter partem et totum vel totum et partem, quare fractos etiam et surdos comprehendo. (27) *Magnitudo* Rei distincte cognita est numerus (vel compositum ex numeris) partium rei cuidam certae (quae pro mensura assumitur) congruentium. Ut si sciam esse lineam, quae bis aequetur lateri, ter aequetur diagonali cuiusdam quadrati certi mihi quae satis cogniti, ut ad ipsum cum lubet recurrere possim, eius lineae magnitudo mihi cognita esse dicitur, quae erit binarius partium iteri congruentium, ternarius partium diagonali congruentium. Diversis autem licet assumtis mensuris, quibus eadem res diversimode exprimitur, tamen semper eadem prodit magnitudo, quia ipsis mensuris iterum resolutis ad idem denique semper devenitur; adeoque mensurae diversae illum ipsum numerum eundem resolutione prodeuntem jam involvunt. Quemadmodum unus idemque est numerus tres quartae, et sex octavae, si quartam iterum in duas partes resolves. Atque talis est Magnitudo distincte cognita. Alioquin *magnitudo* est attributum rei, per quod cognosci potest, utrum aliqua res proposita sit illius pars, vel aliud homogeneous ad rem pertinens et quidem tale ut maneat, licet partium habitudo inter se mutetur. Vel etiam Magnitudo est attributum, quod iisdem manentibus homogeneis ad rem pertinentibus aut substitutis congruis, manet idem. Homogenea autem ad rem aliquam pertinentia intelligo non partes solum, sed et extrema atque minima sive puncta. Nam puncti repetitione quadam continua, sive motu, fit linea. Saepe autem res ita transmutantur, ut ne unica quidem pars figurae posterioris, prioris parti congruat. Aliter Magnitudinem infra definio, ut sit id quo duae res similes discerni possunt, sive quod in rebus sola comperceptione discernitur. Sed omnia haec eodem recidunt. ... (31) ... Hinc manifestum est etiam, Magnitudinem esse illud ipsum quod in rebus distingui potest sola comperceptione, id est applicatione vel immediata, sive congruentia actuali sive coincidentia, vel mediata, nempe interventu mensurae, quae nunc uni nunc alteri applicatur, unde sufficit res esse congruas, id est actu congruere posse.” (GM, 152-4, my translation.)

In (27c) we get a general metaphysical definition of quantity as it may be known clearly but confusedly. Clear knowledge of quantity comes from an ability to recognise the *fact of parthood* for each part in the whole. This is consistent with his characterisation of ‘clear and distinct ideas’ in 1684.¹³ Clear knowledge only requires an ability to recognise a thing, whereas for distinct knowledge one must have marks or signs through which a thing is known by its constitution. Returning to our hexagon analogy, magnitude can be known confusedly but clearly if one has the ability to recognize whether a figure has more or fewer hexagons than another, perhaps without the ability to give their number.¹⁴

In (27d) he moves back to the geometric context with substitution of congruents. In geometry the mode of repetition of the measure is translation in space. I understand ‘by action’ in (31) to mean ‘by motion’, that is, by translation. I take it that in (27c) by ‘the constitution of the parts among themselves is allowed to be changed’ he means that the situs of the parts is changed but the parts *qua* parts of that whole are not changed. Presumably ‘changed’ means changed by motion.

He notes in (27b) that it is possible to decompose a figure using various different parts chosen as measures. Can *any* part be taken as a measure? Are there more or less natural measures? If a figure had a simple simplest part would it thereby have a ‘natural’ measure and accordingly a determinate quantity? He does not discuss the matter in this passage.

¹³L, 291, *Meditations on Knowledge, Truth, and Ideas*, 1684.

¹⁴Leibniz illustrates almost exactly this idea in the *New Essays*: ‘... a mathematician may have precise knowledge of the nature of nine- and ten-sided figures, because he has means for constructing and studying them, yet not be able to tell one from the other on sight. The fact is that a labourer or an engineer, perhaps knowing little enough of the nature of the figures, may have an advantage over a great geometrician in being able to tell them apart just by looking and without counting; just as there are porters and pedlars who will say what their loads weigh, to within a pound - the world’s ablest expert in statics could not do as well. It is true that this empiric’s kind of knowledge, gained through long practice, can greatly facilitate swift action such as the engineer often needs in emergencies where any delay would put him in danger. Still, this *clear image* that one may have of a regular ten-sided figure or of a 99-pound weight – this accurate sense that one may have of them – consists merely in a *confused idea*: it does not serve to reveal the nature and properties of the figure or the weight; that requires a *distinct idea*. The point of this example is to bring out the difference between ideas, or rather between ideas and images.’ (NE, II.29, p. 261)

Observe also in (27b) ‘the same magnitude’. A magnitude is not identical with its expression via a measure and number, but the same magnitude may be expressed with different measures using different numbers. (This is the ‘concrete’ sense of magnitude discussed very shortly.) It is possible to relate magnitudes by relating their measures with a third measure that divides them both.

Between (27e) and (31) is a section on *similarity*, where he introduces his definition through perception, and illustrates it with a thought experiment involving God shrinking us and everything around us by the same proportion. I will discuss similarity in more detail later. For now suffice it that Leibniz juxtaposed these definitions, and *clearly thought they were relevantly equivalent*. It is not possible that immediately after presenting such a thorough analysis of magnitude consistent with his original metaphysical notion, he would present another that ‘comes back to the same thing’ if he thought the two definitions were simply distinct and possibly unconnected. No, it must be that magnitude as ‘that which may be distinguished in things only by co-perception’ *just is* the magnitude he has been discussing all along, but applied to geometry.

Another distinction of occasional importance is that between concrete and abstract quantity:

Quantities signified by letters or numbers or other signs are either abstract or concrete. Abstract are the numbers, or even ratios, which (as said above) I take as numbers of fractions. Concrete quantities can be lines, figures, solids, times, motions, men, sounds, lights, and finally everything in which we can discern repetition of the same measure; these many things [are treated] elsewhere, so that the general application of the calculus to Geometry, Dynamics, Astronomy, Physics, and other sciences appears better.¹⁵

¹⁵Quantitates quae notantur per literas vel numeros vel alias notas, sunt vel abstractae, vel concretae. Abstractae sunt numeri, vel etiam rationes, quas ipsas (quemadmodum supra dictum) ut numeros fractos concipio. Quantitates concretae possunt esse lineae, figurae, solida, tempora, motus, vires, soni, lux, et omnia denique, in quibus ejusdem mensurae repetitio intelligi potest; de quibus alias pluribus, ut ap-

As I understand him, a concrete quantity is any composite whole that can be expressed as a repetition of a certain part. In this way ‘we can discern repetition of the same measure’. It follows that a concrete quantity is just a thing, but only things of this special sort are quantities. Calling something a ‘quantity’ calls attention to the fact that it is a composite expressible by a measure and number.

Abstract quantity, on the other hand, is abstracted from concrete quantities. All concrete quantities instantiate abstract quantity. Abstract quantity is number.¹⁶ A number n is a whole abstracted from an aggregate of unities, considered as repetition of those unities:

$$n = \sum \overbrace{1 + 1 + \cdots + 1}^{n\text{-times}}.$$

It is important to remember from this that Leibniz does not always use ‘quantity’ to signify the number-unit pair. That is quantity in the abstract, combined with a specification of concrete unit; it does not express any situs. Sometimes Leibniz means by ‘a quantity’ a concrete thing which in its concreteness also has parts with situs.

To summarise these distinctions: The number 5 is an abstract quantity, with measure the number 1. ‘Five eggs’ or ‘five amps’ are both quantitative expressions that may be used to express the quantities of concrete things (e.g. a particular basket of eggs or current through a wire). Five eggs considered as a composite whole is a concrete number, since we can discern in it a unit (one egg) and repetition. Five amps, or a line segment, are also concrete quantities, since by supplying a unit (an amp, a metre) we can express the wholes as repetitions of these units.

This last point is important in geometry because sometimes the denotation of ‘a certain quantity’ is a particular figure or part of a figure, also having situs, and sometimes instead it is the quantitative expression (‘5 inches’) that can be used to express the same feature of any number of different figures.

plicatio Calculi generalis ad Geometriam, Dynamicen, Astronomiam, Physicam et alias scientias melius appareat. (GM, VII.61, my translation.)

¹⁶Cf. §5 of the quotation from *De Arte Combinatoria*, p. 16.

1.4 Homogeneity

Things are *homogeneous* when they are comparable in quantity, meaning that it is possible to express them both as repetitions of the same measure. From *De Magnitudine et Mensura*:

Homogeneous things are those whose magnitude can be expressed by numbers assuming the same measure as a unity for everything.¹⁷

Homogeneous things are made of the same parts or the same kind of parts, so that a pretence that all parts are the same as the measure (for the estimation of quantity by repetition) is sufficiently grounded. Returning to our illustration with hexagons, the homogeneous things are all those things constructed from the same type of hexagon. In geometry, where a determinate choice of measure must be grounded in the determinate parts of a perceptual field, the homogeneity requirement implies that things are comparable in quantity only when they are abstracted from the same perceptual field, since only then is it meaningful to talk of the ‘same’ measure applied to each.

We saw Leibniz mention ((27d) on p. 22 above) that ‘not only parts but also extrema’ pertain to homogeneous things. This can be read as an attempt to maintain the basic idea while dealing with some apparent counterexamples. As he immediately notes, a line is composed as the *repetition* of a point (by continuous motion – this repetition must be infinite). Yet a line and a point certainly are not homogeneous; indeed he often reiterates that a point is not even (properly speaking) a part of a line. Presumably to solve this problem he eventually introduced another definition, recorded anyway by 1715, in the *Initia rerum mathematicarum metaphysica*:

Homogeneous things are those to which one can give equals that are similar between themselves.¹⁸

¹⁷‘Homogenea sunt, quorum magnitudines numeris exprimi possunt eandem assumendo pro omnibus mensuram tauquam unitatem.’ (GM, VII.36, my translation.)

¹⁸‘*Homogenea* sunt quibus dari possunt aequalia similia inter se.’ (GM, VII.19, my translation.)

The transformation from equal to equal preserves the parts, and the comparison of similars allows the change in part-decomposition without altering the genus of the figure in any way. A point and a line are not similar and are not equal to similar things so they cannot be homogeneous.

This second definition has the advantage of being clearer on the nature of the decompositions that are permitted in the same genus. The first simply says things ‘which can be’ decomposed by the same measure. In the second, the fact that only *similars* are decomposed by the same measure introduces the possibility of a constraint. We could require that both of the similar figures be *parts* of the same bigger figure (so they can be decomposed simultaneously) and that one part is decomposed *by* a further decomposition of the decomposition of the other. That is, the decomposition of the larger must be a refinement of the decomposition inherited from the smaller in virtue of being similar.

Leibniz later makes a claim that ‘congruence is equality plus similarity’. The dialectic structure of my interpretation of the relationship between these two definitions of homogeneity is analogous or parallel to the dialectic of my interpretation of this claim about congruence given in Chapter 3. Since that claim is more controversial, it will be useful to explain in a few more words here the relationship between the two definitions of homogeneity.

The second definition casts homogeneity as a kind of equivalence relation in the same class as equality and similarity. Equality and similarity are formalised as logical connectives taking as arguments two composite objects (bodies or figures).¹⁹ The first definition is a bit looser: it makes sense to apply the term cast this way not only to two figures, but to all the various parts of a single composite whole. It can be read without emphasising the aspect of homogeneity as a binary relation between wholes.

What is the relation between these two definitions? I suggest the two definitions are linked by the possibility of taking two *parts* of a whole figure as *wholes* in their own

¹⁹GM, V.155, *Characteristica Geometrica* §35.

right, with their own parts corresponding to the complexion they had originally. One may ask of these wholes whether they are similar or equal, or (according to the second definition) homogeneous. That is, one may check whether they stand in one or the other equivalence relation. They are similar if they share abstract structure, and equal if when decomposed according to the same measure (given in the original perceptual field in which both figures were parts), they are expressed by the same number. According to the first definition they are homogeneous if they can be decomposed by the same measure and expressed by (possibly) different numbers, and according to the second definition they are homogeneous if they are equal to figures sharing abstract structure.

The definitions agree where they apply but they differ in scope. The second definition of homogeneity has application as broad as similarity since the question whether equal figures can be found sharing abstract structure is a function only of the abstract structure. Similar figures are trivially homogeneous. The first definition of homogeneity has application as narrow as equality since the question whether two figures can be decomposed and expressed by the same measure is a function of the choice of measure that is restricted to a particular perceptual field.

1.5 Ratio

Two things are said to have a *ratio* when a single abstract quantity provides a sufficient means to know any one given only the other. Abstract quantities, or numbers, have ratios since the quotient of two numbers can be used to reconstruct one given the other. Any two homogeneous concrete quantities have ratios in the sense that their corresponding abstract quantities have ratios.

Concrete quantities can also have ratios in a stronger sense. This happens when the concrete quantities have the same abstract structure, or building plan out of ‘simple’ parts, and these ‘simple’ parts can be compared quantitatively (for which they are no longer treated as simples). Then, knowing one object, its decomposition into these

simple parts, and the quantity, the other can be produced. There is a method for this: one would first produce a new unit or ‘simple’ part. It should have the same nature as the original unit, but when decomposed by the original unit it should be expressed by the number of the given ratio. Then one would construct the second object with the same arrangement (*situs*) as the first, but substituting the new unit in place of the old.

It follows that two concrete objects have a ratio in the stronger sense when they can be decomposed into units in the same manner, these units having the same relative *situs*, but where the unit of one can be further decomposed by the unit of the other, and the quantity of the first unit relative to the other unit is given by the number called their ‘ratio’. Two figures will have a ratio in this sense if they are similar and parts of the same whole, so that their units can be compared in quantity.

This theory of ratio is outlined in *De Ratione et Proportio* from 1682:

A *Ratio* is had between two things if there is a form of comparison of them according to quantity, and conversely. Thus two quantities may be unknown, but nevertheless their ratio can be known; suppose for example that we don’t know how many noses or eyes there are in this city, nevertheless I know the number of noses twice repeated to be the number of eyes.

To *compare* two things according to quantity is to look for a mode of discovering the quantity of one being given only the quantity of the other. It is indeed the comparison of two things, such that later it suffices in practice to have one manifest and to remember the comparison; thus it works as well to have one by sense and the other by memory, as to have both by sense (memory coming at a small price however to sense because memory is sometimes absent).

Therefore if there is a ratio between two things, it is because there is a way of knowing one being given only the other.²⁰

²⁰‘Duarum rerum *Ratio* inter se habebitur, habita forma comparationis earum secundum quantitatem,

Notice an ambiguity in the first paragraph, whereby it might be thought that two things have a ratio whenever they can be decomposed according to the same measure, and expressed by that measure and some number. But this would just repeat the (first) definition of homogeneity. From the second paragraph and the third it seems more likely that what Leibniz had in mind is a comparison according to quantity such that another thing *with the same quality*, or situs, but different quantity, can be known. It seems clear from the transition between the first and second sentences of the second paragraph, and then continuing into the third paragraph, that he assumed in the first sentence that the *form* of the two things is the same, and only the quantity differs (since the units differ). Otherwise, of course, it would not be possible to know one thing given only the other and a number.

Since ratio in the abstract is the ratio between abstract quantities, or numbers, it is true that ratios of concrete things only provide the possibility of intellectual transition from the number of the first to the number of the second. For this to provide full knowledge of the second concrete object, either the form of the second and a measure must already be given (as with the eyes in the town), or we must assume that the first and second have the *same* form.

Three years earlier, in the *Characteristica Geometrica*, Leibniz wrote the following about ratio:

(28) *The ratio of A to B* is nothing but the number by which the magnitude of the same A is expressed if the magnitude of the same B is held to be the unit. From this it is clear that Magnitude differs from ratio as

et contra. Hinc etsi ambarum quantitates sint incognitae, potest tamen ratio earum esse cognita; licet enim ignorem exempli causa, quot sint nasi aut quot sint oculi in hac civitate, scio tamen numerum nasorum bis repetendum esse ut fiat numerus oculorum. *Comparare* duas res secundum quantitatem est quaerere modum inveniendi quantitatem unius ex data sola quantitate alterius. Hic enim finis est comparationis duarum rerum, ut postea sufficiat saltem unam in promptu habere et comparationis meminisse; ita sensu unius et memoria alterius tantum efficimus, quantum sensu utriusque, minore autem pretio constat memoria quam sensus, quia memoria etiam absentium est. Itaque rationem duarum rerum inter se habere, idem est quod habere modum cognoscendi unam ex data altera sola.' (GM, VII.40, my translation.)

concrete number from abstract number; magnitude is indeed the number of things, of course of parts; ratio is really the number of unities. Clearly now the magnitude of a thing remains the same, no matter what measure we want to assume through which to express it; the ratio becomes one or another if one or another measure is assumed. It is clear even (from the definition of division) that if the number expressing the magnitude of the same A is divided by the number expressing the same other B (in this way the same measure or unity is applied to both), the number that is the ratio of one to the other is produced.²¹

It is somewhat tricky, but still possible, to assimilate this passage with the previous one and the interpretation I gave.

In the first sentence, the ratio is a number, as we have said it should be. It is the number that expresses the magnitude of A if *the magnitude* of B is taken as the unit. But what *is* it to take *the magnitude* of B as a unit? Previously only simple parts of concrete things were taken as units. Suppose Leibniz had said ‘if the same B is held to be the unit’. This would be nicely consistent with the logic of quantity. But it would raise another problem: is it really possible to take *any figure*, or any part, as a unit? It must make sense to pretend units are *simples*, lacking parts, and it must make sense to express all the other parts of a figure as repetitions of them. Yet neither pretence need make sense for an arbitrary B.

I suggest that what Leibniz had in mind was to take a new unit for the quantitative expression of A which, when decomposed according to any measure, has the same quantity as B decomposed by that measure. Then expression of A according to this

²¹(28) *Ratio ipsius A ad B nihil aliud est quam numerus, quo exprimitur magnitudo ipsius A, si magnitudo ipsius B ponatur esse unitas. Unde patet Magnitudinem a ratione differre ut numerum concretum a numero abstracto; est enim magnitudo numerus rerum, nempe partium; ratio vero est numerus unitatum. Patet etiam rei magnitudinem eandem manere, quacunque assumpta mensura per quam eam exprimere volumus; rationem vero aliam atque aliam fieri pro alia atque alia mensura assumpta. Patet etiam (ex definitione divisionis) si numerus magnitudinem exprimens ipsius A et alius numerus magnitudinem exprimens ipsius B (modo utrobique eadem mensura seu unitas adhibita sit) dividatur, provenire numerum qui est ratio unius ad alteram.* (GM, V.152-3, my translation.)

unit gives the ratio. Equivalently, one can decompose A and B by the same unit, and divide the number expressing A by that expressing B. (The last quoted sentence.)

What do the second and third sentences mean? I suggest that the shift between concrete–abstract number and magnitude–ratio is the shift between a context with a *natural* unit of measure to a context without one. This shift engenders a change in logic. It doesn't make sense to speak of a concrete ratio (since the nature of ratio externalises the unit), and it doesn't make sense to speak of an abstract magnitude (since there is no given unit, there is no determinate relation from which to abstract).

Let us consider in turn each of the four concept combinations. Concrete number is a concrete quantity, and is just a thing in which repetition can be discerned. Metaphysically, a concrete quantity is a whole having an affection relative to its parts in a context where a unit of measure is naturally determined by the nature of the whole. No additional act is necessary to give the unit. Such is the case for a basket of five eggs, for which '5 eggs' expresses an affection of the whole basket, and the unit of one egg is clear.

Abstract number is the (external) relation between totalities and concrete unities, or single individual things that are parts of those totalities, as could be had identically between different totalities and their respective units.

Concrete magnitude is also a concrete quantity in which repetition can be discerned, and metaphysically is a whole with an affection relative to its parts, but in a context where a unit of measure is *not* naturally determined by the nature of the whole. A unit of measure must be provided by an additional act. Such is the case for a line segment, which obviously has parts that are repeated, but no decomposition is more natural than another.

Abstract magnitude does not make sense. If the unit is not given before the abstraction, there can be no abstract relation since there is no relatum; if the unit is given before the abstraction, the abstraction would collapse to abstract number.

Ratio is the abstract (external) relation between a whole and a unit, where the unit

is given separately to the whole. If this choice of unit is substituted for the natural unit of a concrete number, then ratio expresses the same thing as abstract number. But the presumption in ratio is that the unit needs to be given separately. For this reason abstract ratio is logically *two-place*, while abstract number is logically *one-place*, despite the fact that both are (external) relations.

In this sense ratio is an abstract relation that *externalises* the unit. Recall Leibniz's remark in *De Arte Combinatoria*, (p. 16 above):

Quantity is therefore the number of parts. Hence quantity and number obviously coincide in the thing itself, but quantity is sometimes interpreted extrinsically, as it were, in a relation or ratio to another quantity, to aid us, namely, when the number of parts is unknown.²²

I take it that 'in a ratio' implies that a ratio is a kind of quantity, and 'another quantity' means another object or *concrete* quantity. The 'extrinsic' interpretation of quantity is not the interpretation that quantity is an abstract (external) relation, lying outside its relata and repeated between various pairs of relata, although that is no doubt the case for ratio. 'Extrinsic' quantity is here juxtaposed with 'intrinsic' quantity, which is the relation of a whole to its *internal* parts, not some external part. Quantity is normally the affection of a whole toward the inside. Ratio can be specified for one whole and something else *outside* that whole, but this only works when there is some way to express the first whole as a repetition of the other, and *this* makes sense only in virtue of the possibility of placing that other inside the first in some way.

It is worth considering in this context Leibniz's remarks on the nature of ratio in his debate with Clarke. There he presents ratio as the 'simplest' relation, using it to explain what he means by an abstract relation in order to explain what he meant by saying that *place* is an abstract relation:

²²L, 76.

The ratio or proportion between two lines L and M may be conceived three ways: as a ratio of the greater L to the lesser M, as a ratio of the lesser M to the greater L, and lastly, as something abstracted from both, that is, the ratio between L and M without considering which is the antecedent or which the consequent, which the subject and which the object. And thus it is that proportions are considered in music. In the first way of considering them, L the greater, in the second, M the lesser, is the subject of that accident which philosophers call ‘relation’. But which of them will be the subject in the third way of considering them? It cannot be said that both of them, L and M together, are the subject of such an accident; for, if so, we should have an accident in two subjects, with one leg in one and the other in the other, which is contrary to the notion of accidents. Therefore, we must say that this relation, in this third way of considering it, is indeed out of the subjects; but being neither a substance nor an accident, it must be a mere ideal thing, the consideration of which is nevertheless useful.²³

Interestingly, Leibniz and Clarke become involved here in a dispute over whether ‘relations have their quantity’: Leibniz thinks that place is an abstract external relation but can still be quantified. He cites ratios as an example of quantified relations, remarking that they are quantified by logarithms. Clarke objects that ‘what they call *bearing a greater or less proportion* is not bearing a greater or less quantity of proportion or relation, but bearing the proportion or relation of a greater or less quantity to another’.²⁴ What do they both mean by quantity here? For Leibniz it is the same general principle: since ratio ‘wholes’ can be decomposed into smaller ratios and these simply counted by use of logarithms, they can be expressed quantitatively. In this way a musical interval, at least in a 12-tone equal tempered tuning convention, is composed of ‘simple’ parts which are half-steps in the ratio $1 : 2^{1/12}$. Clarke’s concept of quantity is a confused

²³L:CLC, §5.47.

²⁴L:CLC, §54 of Clarke’s Fifth Reply.

mixture of concrete and abstract, ultimately closest to the idea of ‘amount of stuff’ held as something like a primitive, whereby one imagines it makes sense to say ‘a quantity [*simpliciter*] of X’ and ‘greater or lesser quantities of X’ for any stuff-kind X.

1.6 Number

The foundation of *number* is repetition, but number extends beyond simple integers by inverting repetition (fractions), or taking roots (irrational algebraic numbers), and ultimately considering series (for the transcendentals). Since abstract numbers abstract from quantities, it is important that the concept of quantity likewise be extended beyond simple repetition.

(Since a caveat to this effect could be added to every mention of repetition without altering the structure of any of my reasoning, though not without somewhat obscuring the point, it is generally elided.)

In *De Quantitate* Leibniz defines number thus:

Number is something homogeneous to Unity and thus may be compared with unity and added and subtracted. It is either an aggregate of unities which is called *integer*, thus 2 (or [seu] $1 + 1$), likewise 3, 4 (or [seu] $2 + 1$ or [sive] $1 + 1 + 1$), or an aggregate of parts of a number of unities, which is called *fraction*, ...²⁵

In the *New Essays*, correcting Locke on the numbers, he emphasises that number extends beyond whole numbers:

For *number in the broad sense* – comprising fractions, irrationals, transcendental numbers and everything which can be found between two whole numbers – is analogous to a line, and does not admit of a minimum any

²⁵*Numerus* est homogeneum Unitatis adeoque comparari cum unitate eique addi adimique potest. Estque vel aggregatum unitatum qui dicitur *integer*, ut 2 (seu $1 + 1$), item 3, 4 (seu $2 + 1$ sive $1 + 1 + 1$), vel aggregatum partium aliquotarum unitatis, qui dicitur *Fractus*, ...’ (GM, VII.31, my translation.)

more than the continuum does. So this definition of ‘number’ as a multitude of units is appropriate only for whole numbers.²⁶

Notice then that numbers share with the continuum the property of *infinite divisibility*, and in this way neither the numbers nor a line have smallest parts. Even the ‘simple’ number 1 can be divided into halves and quarters by use of fractional numbers.

This infinite divisibility is largely unrelated to the *indeterminacy* of the continuum. For the numbers have a most *natural* unit, namely unity or 1, while a line has no natural unit. Any choice of proper part of a line is arbitrary.²⁷ This issue is not *entirely* unrelated, for obviously if a thing has a finite number of parts which cannot be further divided (a discrete lattice, say), then provided the parts are homogeneous, they will be good candidates for natural units by which to express quantity.

²⁶NE, II.xvi, p. 156.

²⁷‘When I said that unity is not further analysable, I meant that it cannot have parts whose concept is simpler than it. Unity is divisible but not resolvable, for fractions, which are parts of unit, have less simple concepts because whole numbers (which are less simple than unit) always enter into the concepts of fractions. Many who have philosophised about the point and about the unit in mathematics have become confused by failing to distinguish between analysis into concepts and division into parts. Parts are not always simpler than wholes though they are always less than wholes.’ (G, III.583, tr. Cox p. 133-4) I take it this point is in response to a looming confusion resulting from the fact that there is an indeterminacy in the ‘decompositions of unity’. On what grounds should $1 = 1/2 + 1/2$ be preferred to $1 = 1/3 + 1/3 + 1/3$? But this is no ‘indeterminacy’ like that of the choice of unit in a continuum. For the concept of 1 is *contained in* the concept of $1/2$, first through repetition in 2, and then again in the concept of $1/2$ containing the concepts of 1 and 2. (Similarly for $1/3$.) The *concept* of 1 is simple, despite the fact that many unities have parts.

Chapter 2

Quantity in geometry

Thus far we have considered quantity under its most general metaphysical aspect, noting in passing that Leibniz clearly had the general concept in mind even when he came to the geometric context, where he introduced his new definitions of quality and quantity in terms of perception. Both the old general concept and the new definitions occur side by side in the *Characteristica Geometrica*, where he indicates clearly, if tersely, that they all ‘come to the same thing’. After that period, and indeed throughout his later life, he maintains the older metaphysical definition even while continuing to invoke his definition involving perception in geometric contexts.

Coming then to geometric quantity, I would like to suggest that Leibniz’s work can be explained in terms of three distinct themes or pivotal ideas, and that the variety of his remarks on quantity in specifically geometric contexts can be considered variations on these themes. The themes are:

- (1) (The prerequisite of quantity.) In geometry, the act of synthesis providing the composite whole required for quantity involves perception.
- (2) (The ground of abstraction.) Abstract places and extents are grounded in the situs and complexions, respectively, of the parts of some more concrete and determinate composite.
- (3) (The process and result of abstraction.) By considering only a finite portion of the infinite detail in real phenomena, we abstract the homogeneous continuum,

which is ideal, incomplete, indeterminately composed, and possibly (not actually) divided.

I provide a novel interpretation of Leibniz's remarks on the definition of quantity through 'co-perception' as variations on the first theme: co-perception is the act of synthesis providing the 'perceptual field', which is just the composite required for quantity. In the first part of what follows I will justify this interpretation largely from writings on the *Analysis Situs*, since the co-perception concept appears mainly in these writings. The second theme is based on the story Leibniz gives Clarke in order to explain his view that space is ideal. The third theme is related to the continuum composition problem, and I'll substantiate it mainly from Leibniz's letters to De Volder.

My interpretation of Leibniz's view of geometric quantity is an attempt to give a single story justifying and unifying these three themes. So they are the pillars of my interpretation. I take it as a consequence of this architecture that each pillar should be defended and criticised directly from extant texts, and the picture as a whole should be defended and criticised in virtue of its capacity to make sense of the three themes without unnecessary addition.

The most important posit of my story is the perceptual field and its role. This postulate is the initial step toward unifying these themes. The perceptual field is the composite whole of (1) that is prerequisite for quantity, both internal (quantity proper) and external (ratio). The perceptual field is the more real composite of (2) whose parts have determinate situs and complexion. The same perceptual field contains the detail ignored by the abstractions of (3) and it is real (albeit phenomenal), complete (though confused), determinately composed, actually divided.

2.1 The perceptual field and co-perception

Leibniz was very proud of his 'discovery' of perceptual definitions of geometric quality and quantity. Quality is that grasped by individual perception of a figure considered

alone. Quantity can be grasped only in two figures perceived together; for this Leibniz coined the term ‘co-perception’.

The definitions combine two ideas. One idea is that *perception* serves as the mode of discovering geometric content both qualitative and quantitative. Another idea is that the divide between quantity and quality maps somehow onto the divide (with regard to perception) between *two things together* and *one thing separately*.

These definitions are most clearly outlined in the late *Initia rerum mathematicarum metaphysica* from 1715:

[(a)] *Situs* is a mode of coexistence. Therefore it involves not only quantity but also quality.

[(b)] *Quantity* or magnitude is that in a thing which can only be known through co-presence (or through simultaneous perception). Accordingly we cannot know what a foot is, or a span, unless we have something guiding us as a measure, which thus can be applied to each thing. Neither can a foot be sufficiently revealed by any definition which does not indeed involve again the same sort of thing. Namely, although we say a foot is twelve inches, the question is the same for inches: neither do we acquire from that old light, nor are we able to say whether the notion of an inch or a foot is naturally prior, when it is arbitrary which of the two we want to take as the bottom.

[(c)] *Quality* however is that in things which may be known when they are observed individually, and co-presence is not needed. Such are the attributes which are explained by definition or rather through the various modifications that they involve.

[(d)] *Equals* are those of the same quantity.

[(e)] *Similar*s are those of the same quality. Hence if two similars are different, they cannot be distinguished except by co-presence.

Hence, plainly, for example, two equiangular triangles having proportional sides or vice versa. Namely if the sides are proportional, the triangles are similar, since they are determined in the same way.¹

Observe first from (a) the implication that *situs* *also* involves quantity. This will become important in Chapter 3 when we consider suggestions that Leibniz meant to eliminate quantity from his new characteristic.

In (b) we see the idea that a unit of measure for geometric quantity is arbitrary, meaning that it cannot be defined abstractly. It is still possible, of course, to indicate concrete quantities.

In (c) notice the remark that qualities ‘are the attributes which are explained by definition’. This wording suggests that qualities and definable attributes are at least extensionally equivalent. In his letters to De Volder, Leibniz explicitly links abstraction from phenomena to the possibility of supplying definitions:

Indeed, there is as much difference between substance and mass as there is between complete things, things as they are in themselves, and incomplete things, things as we grasp them through abstraction. In the phenomena we can define through abstraction whatever we want to ascribe to each part of mass, and everything can be distinguished and explained rationally, something that necessarily requires abstractions.²

¹*Situs est coexistentiae modus. Itaque non tantum quantitatem, sed et qualitatem involvit. Quantitas seu Magnitudo est, quod in rebus sola compraesentia (seu perceptione simultanea) cognosci potest. Sic non potest cognosci, quid sit pes, quid ulna, nisi actu habeamus aliquid tanquam mensuram, quod deinde aliis applicari possit. Neque adeo pes ulla definitione satis explicari potest, nempe quae non rursus aliquid tale involvat. Nam etsi pedem dicamus esse duodecim pollicum eadem est de pollice quaestio, nec majorem inde lucem acquirimus, nec dici potest, pollicis an pedis notio sit natura prior, cum in arbitrio existat utrum probasi sumere velimus. Qualitas autem est, quod in rebus cognosci potest cum singulatim observantur, neque opus est compraesentia. Talia sunt attributa quae explicantur definitione aut per varias modificationes quas involvunt. Aequalia sunt ejusdem quantitatis. Similia sunt ejusdem qualitatis. Hinc si duo similia sunt diversa, non nisi per compraesentiam distingi possunt. Hinc patet exempli causa, duo Triangula aequiangula habere latera proportionalia vel vicissim. Nam sint latera proportionalia, similia utique sunt triangula, cum simili modo determinentur.* (GM, VII.18-9, my translation, emphasis original.)

²AG, 177.

Inspired by this comment to De Volder, I suggest that we read the second part of (c) to the effect that quality in geometry is attributable to the *abstract form* of a figure, and indeed the abstract form *just is* that defined in terms of qualities. The question remains how we can reconcile this understanding of geometric quality with the idea that qualities can be known by observation of an individual, while quantities require simultaneous co-presence.

I suggest that we understand the difference between quality and quantity like this: qualities are ascribed to figures after they are abstracted from the perceptual field. So a figure's qualities are independent of the particular perceptual field from which the figure is abstracted. Quality is thus a function only of the abstract, or the 'definable', as that is read off of but removed from the *whole* perceptual field. Quantity, on the other hand, requires a determinate part for its unit. Since the abstract figures of geometry (e.g. lines) do not have determinate parts, and indeed no reason could be given for the choice of one part over another since they are all similar, quantity is *dependent on* and *restricted to* the concrete parts of a particular perceptual field.

I suggest that the terms 'co-perception' and 'simultaneous perception' and 'through co-presence' in the definition of quantity are ways of saying that subjects (figures or bodies in space) must be abstractions from the same perceptual field if they are to be comparable quantitatively. Only then is it possible to say that they are decomposed by the *same* measure.

An implication of this reading is that the denotation of 'co-perception' is no different from that of 'perception'. Every instance of perception is an instance of co-perception in which the parts or subfigures of one perceptual field are all perceived together. The terms differ in connotation, though, since 'co-perception' implies or presupposes that the perceptual field is divided into two dominant parts, and it is *these parts* which are perceived 'together'. Mere 'perception', on the other hand, implies no such preferred division into subfigures. Nonetheless any perceptual field will *have* parts, and technically the perceptual field involves co-perception of all its parts, whatever they

may be. This implication is consistent with the *use* to which Leibniz puts the term, since he uses ‘co-perception’ in contexts that *presuppose* two (or sometimes more) given figures.

This last observation would seem to cut both ways, however, and may be associated with a worry about this reading. Indeed, a worry that is linked to a more common way of reading the passage. The worry, in the abstract, is that this reading mistakes the multiplicity of application of the term, since Leibniz only writes of *whole figures* that they can be co-perceived or individually perceived. Applying the term to the myriad (even infinity) of parts of every single figure would stretch it out in an unintended way.

There is, I think, a superficially appealing reading of his definitions that takes a hard line on this objection. It supposes that quantity is an attribute of an abstract figure, like quality, but whereas quality can be known immediately upon observation, to know the quantity of a figure one needs to bring another figure into proximity. Let us call this the ‘hidden attribute’ reading of the co-perception definition of quantity.

The ‘hidden attribute’ reading cannot be correct since it is very obviously incoherent. Let us make only the plausible assumptions that figures can be constructed out of smaller subfigures, and that the quantity of a figure is determined by the quantities of its subfigures. Then, if the quantity of *one* figure is revealed merely by bringing another figure into proximity, the quantity of each *subfigure* of the same figure should be revealed by its proximity with the other *subfigures*. By our assumption, the quantity of the total figure would be known. In this way the quantity of any composite figure should be always manifest, contradicting the hypothesis that quantity is hidden.

The worry stems, I think, from the priority the passage seems to give to ‘figures’. Let us ignore the division between the abstract figure and its perceptual ground, and suppose that ‘figures’ (*simpliciter*) are the fundamental objects, whatever they may be. (Leibniz does not, after all, mention any ‘perceptual field’ here.) According to this line of thought, figures have two kinds of properties. Qualitative properties can be known by considering a figure alone, while knowledge of quantitative properties requires the

presence of another figure for comparison. From here we find ourselves very quickly in the ‘hidden attribute’ reading, which suffers the difficulty just indicated.

But I believe we can account for the emphasis on whole figures much more simply. Leibniz wanted to propose equivalence relations of quality and quantity, namely similarity and equality, and an equivalence relation takes two arguments. So for reason of logical form alone, the multiplicity of application was restricted to whole figures. We need not be leery of saying that *any* two parts of a figure in a single perceptual field can be compared in quantity (provided they are decomposed by the same measure).

Something like a perceptual field enters more clearly in Leibniz’s much earlier discussion of the same ideas, in the *Characterista Geometrica* of 1679. There is no published translation so I will quote the passage at length:

[(a)] *Similar*s are those which cannot be discerned by considering them individually in themselves [*singula per se*], as two equilateral triangles (in fig. 44 [omitted]). We can discover no attribute or property in one that we cannot also find in the other. And calling one of them *a*, the other *b*, we notate *similarity* $a \sim b$. If nevertheless they are simultaneously perceived, immediately the difference appears, that one is bigger than the other. The same thing can be done even if they are not perceived together, only assuming something as an intermediary or a measure that can be applied first to the one, or something in it, and notating how it or its part is congruent to the measure or its part, then later applying the same measure to the other. Therefore I customarily say that *similar*s cannot be discerned except by co-perception.

[(b)] On the other hand if I would see two equilateral but unequal triangles in succession, I would nevertheless not properly discern them. But it is known, [since] I speak here of discerning, that undoubtedly another mind is able to note something in the one that does not proceed in the

other, [although] not by sense and imagination. The ratio however, which the eyes discern in the two similar but unequal things, is manifest; for we are left with images of the things previously perceived, which, applied to the images of a thing newly perceived, reveal the same difference of these two images by co-perception. And the same fund of the eye, whose bigger and smaller parts are occupied by the image, does the duty of a specified measure.

[(c)] Finally we are accustomed to perceiving always together the other things that we perceived at first, whence the newest perceived thing is referred to them, in order that referring the first to these same things we might notice the distinction without difficulty.

[(d)] If we truly imagine God to shrink the appearance of everything in us and around us in some room, keeping the same proportions, everything would appear in the same way. We would not be able to discern the difference between the first and last state, unless of course we emerged from our room to the sphere of things undiminished by that proportion; then indeed by co-perception, bringing them before the unshrunk things, the distinction would appear.

[(e)] From this it is manifest indeed that *Magnitude* is that which may be discerned in things only by co-perception, that is, either by immediate application, the things being either actually congruent or coincident, or else mediate, by the intervention of a measure, which now to one and now to the other is applied, this sufficing for the things to be congruent, that is, to be able to be made congruent by action.³

³(31) *Similia* sunt quae singula per se considerata discerni non possunt, velut duo triangula aequalatera (in fig. 44), nullum enim attributum, nullam proprietatem in uno possumus invenire, quam non etiam possimus reperire in altera; et unum ex ipsis appellando a, alterum b, *similitudinem* ita notabimus: $a \sim b$. Si tamen simul percipiantur, statim discrimen apparet, unum alio esse majus. Idem fieri potest, etsi non simul percipiantur, modo aliquod velut medium assumatur sive mensura quae primum applicetur uni, aut aliquo in ipso, notatoque quomodo prius vel pars ejus cum mensura vel ejus parte congruat, postea eadem mensura etiam applicetur alteri. Itaque dicere soleo, *similia* non discerni nisi per comper-

Many ideas from this passage were repeated in the 1715 passage quoted previously, but a few require further explanation.

Note first from (a) that despite the appearance of a difference when a and b are co-perceived, ‘*we can discover no attribute or property in one that we cannot also find in the other*’. I take it that the attributes or properties we could ‘discover’ are *abstractions*, and that the difference we discern by co-perception does not provide any attributes or properties of that kind.

Paragraph (a) introduces the idea of *mediate* co-perception, whereby some intermediary is co-perceived first with one figure and later with another figure; this suffices, Leibniz reports, for us to discern their respective quantities. How do we make sense of this from the ‘perceptual field’ reading? It would seem that mediate co-perception allows us to compare quantities of figures even if they aren’t grounded in the same perceptual field.

The answer I recommend is to emphasise the phrase ‘only *assuming* something as an intermediary’ in (a). Ordinarily, quantitative comparison is restricted to a single perceptual field. But if we *assume* that something in one field has the same quantity as something in another, so that the units of the two fields can be related, things decomposed by the unit in one field can be compared quantitatively with things decomposed by the unit in the other. From an abstract perspective, the assumption adds information

ceptionem. At, inquires, ego etsi successive videam duo Triangula aequilatera inaequalia, ea nihilominus probe discerno. Sed sciendum est, me hoc loco loqui de intelligent, ut nimirum mens aliquid notare possit in uno, quod non procedat in altero, non de sensu et imaginatione. Ratio autem, cur oculi duas res similes, sed inaequales discernant, manifesta est; nam supersunt nobis rerum prius perceptarum imagines, quae rei nove perceptae imaginibus applicatae discrimen ipsa comperceptione harum duarum imaginum ostendunt. Et ipse fundus oculi, cujus partem majorem minoremque occupat imago, mensurae cujusdam officium facit. Denique alias res semper simul percipere solemus, quas etiam cum prioribus percepimus, unde rem novissime perceptam ad eas referendo, ut priorem ad easdem retulimus, discrimen non difficulter notamus. Si vero fingeremus, DEUM omnia in nobis ac circa nos in aliquo cubiculo apparentia proportionem eadem servata minuire, omnia eodem modo apparerent neque a nobis prior status a posteriore posset discerni, nisi sphaera rerum proportionaliter imminutarum, cubiculo scilicet nostro egredieremur; tunc enim comperceptione illa cum rebus non imminutis oblata discrimen appareret. Hinc manifestum est etiam, *Magnitudinem* esse illud ipsum quod in rebus distingui potest sola comperceptione, id est applicatione vel immediata, sive congruentia actuali sive coincidentia, vel mediata, nempe interventu mensurae, quae nunc uni nunc alteri applicatur, unde sufficit res esse congruas, id est actu congruere posse.’ (GM, V.153-4, my translation, emphasis original.)

to what was ‘given’. From a concrete perspective, this assumption may be grounded in the fact that all real perceptual fields are subsumed under God’s point of view, or at least agree as a matter of pre-established harmony.

Notice the thought early in (b), which grows into paragraph (d), that another mind is able to discern things that we cannot. The central point of (d) is just an illustration of the main idea of this passage. But the illustration appears to raise a small problem for my interpretation. For it seems to imply that the things outside our room in ‘the sphere of things undiminished’ have determinate quantity despite not being part of our perceptual field.

I take it that ordinary ‘bodies’ outside our perceptual field can have quantities too, but only in virtue of the way our perceptual field *would* appear in future or hypothetical circumstances. Ultimately these facts are determined by our metaphysical perception, whereby we do express everything about the universe, including its future, in ourselves. But perhaps we can say a little more. In one place Leibniz remarks that:

Between the appearance of bodies to us and their appearance to God, there is the same kind of difference as between a scene and an ichnograph. For scenographs are different according to the situation of the spectator, while the ichnograph, or geometrical representation, is unique.⁴

Perhaps this means we can usefully think of ‘God’s perceptual field’ as something that subsumes all others. Ours excludes others, since the ‘herenesses’ that form ours are constructed by our finite natures and are local. The manner of confusion in our perceptual fields separates them from each other. But the full detail of all parts is perfectly clear in God’s perceptual field. By our internal reflection of the divine we understand that in some way our fields can be joined with those of every other mind, all subsumed in a divine perceptual field that grounds the quantity for the real world of objects, namely phenomena that really are well-founded in the metaphysical reality of the universe of

⁴G, II.438; R, 253.

monads. This sort of consideration is in the end probably how Leibniz would have to explain how our quantitative expressions of *ordinary objects* are well-founded.

All of the remarks so far have posited the perceptual field without explicit textual support. It is, I hasten to add, largely a postulate of my interpretation of Leibniz's concept of quantity. But we can see at the end of (b) a thought that tends very much toward this idea. Leibniz remarks that 'the same fund of the eye, whose bigger and smaller parts are occupied by the image, does the duty of a specified measure'. What do we make of this? Is Leibniz really explaining the phenomenal roots of quantity by appealing to a particular piece of that phenomenal world, namely a *retina*?

Recall that a body and its operation *express* the state and changing perception of a monad. It should make sense that the cognitive faculty producing the 'perceptual field' is expressed in a bodily feature or mechanism. What exactly *is* expressed by images falling on the 'fund of the eye'? This is not obvious. For this story to work (indeed to license Leibniz's immediate application of the fund), the images on the fund need to contain figures as parts. This means it would not be appropriate to consider an image on the fund as the aggregate of intensity signals from the optical nerves spread out in the fund: some conceptualising ('processing') of that information must take place before the image is an image *of* various figures and bodies in their respective regions.⁵

I suggest that we read the content expressed by the image falling on the fund of the eye as what I have called the 'perceptual field'. (Or, alternatively, to define the perceptual field as whatever Leibniz took to be expressed by the image falling on the fund of the eye.) It follows that the perceptual field is the synthesis required for quantity.

⁵It may be suspected that the 2-dimensional nature of the images on the fund implies that the perceptual field we have been postulating should be such as is abstracted first into a 2-dimensional space. At some level I suppose there's no harm hypothesising about the layers of psychology, but it should be also clear that the perceptual field doesn't *need* to be abstracted first into 2-dimensional space. We have, after all, two eyes, and cognitive processing that represents objects 'here' and 'there' in a field that includes depth. It is most convenient for our story to set the perceptual field as the most-cognised composite, or the highest layer, before the abstractions of place and extent set in. Then these abstractions attempt to leave perspective altogether, but do not, as it were, 'stitch together' an aggregate of two dimensional images; that happens at a lower level.

2.2 The perceptual field grounds place and extent

The second pillar of my interpretation proposes some composite with determinate parts having determinate situs and complexion from which the relations of place (part to other parts) and extent (part to its own parts) are abstracted. I will take it that this composite is the same perceptual field as that invoked by the notion of co-perception, but I don't justify the link here.

In the next section I will consider in more detail the natures of these abstractions, and (to the extent possible) the process of abstraction itself. Here I want to focus on Leibniz's talk of a thing more determinate and real than these abstractions: something in the 'well-founded phenomena' that precedes abstract places in space.

Early in his fifth letter to Clarke, Leibniz distinguishes the abstract or ideal things, which can be (posited to be) repeated without difference, from concrete ones, which cannot:

The parts of time and place, considered in themselves, are ideal things, and therefore they perfectly resemble one another like two abstract units.

But it is not so with two concrete ones, or with two real times, or two spaces filled up, that is, truly actual.⁶

Concrete things are never perfectly alike; this is a well-known theme in Leibniz's thought. But the germane point is that Leibniz allows himself to speak of 'real times' and 'actual spaces' in a mode in which they are not abstract. These real times and places, which are well-founded phenomena, I wish to call the 'perceptual field'.

Later in the letter, in the long §47 in which he tries to 'show how men come to form to themselves the notion of [abstract] space', he distinguishes the abstract place with its merely postulated and 'extrinsic' identity from the 'real situation' of a body:

And here it may not be amiss to consider the difference between place

⁶L:CLC, §V.27.

and the relation of situation of the body that fills up the place. For the place of A and B is the same, whereas the relation of A⁷ to fixed bodies is not precisely and individually the same as the relation which B (that comes into its place) will have to the same fixed bodies; but these relations only agree. For two different subjects, as A and B, cannot have precisely the same individual affect, it being impossible that the same individual accident should be in two subjects or pass from one subject to another. But the mind, not contented with an agreement, looks for an identity, for something that should be truly the same, and conceives it as being extrinsic to these subjects; and this is what we here call *place* and *space*. But this can only be an ideal thing, containing a certain order, wherein the mind conceives the application of relations.⁸

The real situation that Leibniz supposes here is an affection inhering in bodies through which they express their surroundings, and from which place is abstracted. Place is an external relation that does not inhere in its relata.

By ‘these relations only agree’ I do not think Leibniz had in mind the agreement of approximate identity (an approximation that could be made as close to identity as desired by varying only extrinsic facts). That the relations only ‘agree’ is to say again that they are not ‘individually’ the same, and that is because they inhere in different things. The relations could not be identified without identifying the things, no matter the degree of precision with which a body’s place is specified. Agreement is a similarity of relation of a higher type. Suppose, for example, X is the father of Y, so there is a predicate of ‘having Y for son’ that is expressed in X, and ‘having X for father’ expressed in Y. These predicates ‘agree’ in some way with similar predicates in Q and P when Q is the father of P. But there is no question here of *approximate* identity. Even if $X = Q$,

⁷In this discussion B names some body that is imagined to come into the place once occupied by A, meaning it’s imagined to come into a (nearly) identical situation as A with respect to some other bodies C, E, F, G, etc.

⁸L:CLC, §V.47.

the expression of ‘son of X’ in Y would approximate the expression of ‘son of Q’ in P only in the limit as Y becomes P, and that is not a relevant limit to consider.

Since situation is a real relation, it must be expressed in the body itself. Leibniz made this point explicit in his previous correspondence with De Volder:

[(a)] Also, things that differ in place must express their place, that is, they must express the things surrounding, and thus they must be distinguished not only by place, that is, not by an extrinsic denomination alone, as is commonly thought.

[(b)] Hence bodies, like the atoms of the Democriteans or the perfect globes of the Cartesians, cannot, as commonly understood, be found in nature, nor are they anything but the incomplete thoughts of philosophers who have not inquired sufficiently well into the natures of things.⁹

Note first that the bodies of (b) are incomplete (abstract) things. I think this remark, together with that above to Clarke,¹⁰ entails an ambiguity in the senses with which Leibniz uses the term ‘body’. In some contexts a ‘body’ is a concrete thing (an actual well-founded phenomenon). Bodies in this sense express in themselves an infinitely detailed situation. In other contexts he uses ‘body’ in a sense closer to ‘mathematical body’, which is the abstract thing corresponding to (properly having) the abstraction of place.

Notice a similar ambiguity of usage with ‘place’ appearing in (a). The second half makes clear that place is really an extrinsic denomination. But in the first line place is not purely extrinsic because the expression of ‘place’ that he means is precisely what is

⁹AG, 174; L:CLDV.

¹⁰Not just to Clarke, as though he might have changed his forms of expression between the correspondences. Leibniz also uses ‘body’ in a concrete sense with De Volder:

But in real things, namely, in bodies, the parts are not indefinite (as they are in space, a mental thing), but are actually assigned in a certain way, ... (AG, 178; L:CLDV).

Moreover, I also put corporeal forces where I put bodies, namely, among the phenomena, if they are understood as adding something over and above simple substances or their modifications (AG, 181; L:CLDV).

not a matter of difference of ‘place’ in the third line. What Leibniz meant by ‘express their place’, he apparently thought, is better explained by ‘express the things surrounding’. This is, I take it, really to have a relative situation.

Unfortunately Leibniz also uses ‘situation’ with a parallel ambiguity. Most often it is the concrete situation that expresses the relative affection of one part to others, and as such is an expression in itself (in a certain way) of their complete concepts. But other times he is willing to use it for the abstraction of place, as when he says that a mathematical ‘point’ is pure situation and ‘space’ is pure extension.¹¹

Since it is the business of the present work to draw a sharp line between the abstract and the concrete (at least at a certain layer of cognition), I have chosen to restrict usage of ‘situation’ and ‘complexion’ to the concrete parts of the perceptual field and ‘place’ and ‘extension’ to the abstract. I have also tried to restrict ‘body’ to the abstract, in the sense of ‘mathematical body’, since it is these bodies which have place in space, and have used ‘region’ (in the perceptual field) as its concrete counterpart. This last choice is probably most dubious since I can discern no tendency for Leibniz to use ‘body’ predominantly for either the abstract or the concrete. And it also threatens to mislead into the idea that a region is part of a two dimensional image, when it should really be identified with the concrete phenomenal body *just before* the abstractions of place and extent are applied, just before the perspectival phenomenon is abandoned for general description. The main, and sufficient, advantage of the term is that it emphasises the perspectival aspect.

Let us consider in a little more detail what it could mean (from (a)) for a region (concrete body) to express in itself its situation, which is to say, what it means that it has an affection relative to the surrounding regions. A convenient feature of the concept of ‘perceptual field’ is that it allows us to provide a not insubstantial story about how this might be.

Consider a thought experiment. Imagine deleting a region from the perceptual

¹¹GM, V.144; §§9-10 of *Characteristica Geometrica*.

field.¹² What follows? It is not enough to imagine creating an empty space where this region was, since an empty space is still full of situations. The parts of the perceptual field are determined by their ‘herenesses’, and ‘empty space’ still has herenesses. A region is not a body that can be deleted without change to the ‘background’ space; a region is itself the thing separating other regions. The nature of a region is logically linked to the nature of other regions: *what it is* to be ‘here’ is linked to *what it is* to be ‘not there’ (for another region). Indeed it’s tempting to *define* ‘here’ as:

$$\text{‘here’} := \bigwedge_{x \text{ is a region, } x \neq \text{‘here’}} \text{‘not } x\text{’}.$$

Of course this isn’t adequate since ‘here’ appears in the definiens. But the point is clear: the perceptual field is inflated by its own content.

This explanation of the way a region expresses its situation should be compared with Leibniz’s contrast between the concrete *plenum* and abstract empty space, for it is really the same contrast. The parts of the plenum are to be understood as separated *by each other*. The plenum is not a plurality of bodies existing *in* some background space, separated *by* that space. It does not even *make sense*, logically speaking, to imagine removing something from the plenum without the others coming closer together, as it were. The logic of mathematical bodies in an abstract space allows them to be moved and removed without affecting the others, provided they don’t overlap. The logic of the parts of the plenum, or the regions of the perceptual field, does not allow them to be moved or removed without change to the others.

I don’t say that matter and space are the same thing. I only say there is no space where there is no matter and that space in itself is not an absolute reality. Space and matter differ as time and motion. However, these things, though different, are inseparable.¹³

¹²This is meant to be parallel to a case Leibniz famously considers: imagine a father X living in England while his son Y is annihilated in India. How is the change expressed in X?

¹³L:CLC, §V.62. Here Leibniz is speaking of concrete matter in the plenum, not mathematical bodies

The situation of a concrete body (region) to its surrounding is not the same as the body itself, but since it inheres in the body itself it cannot (logically) be changed without change in the body itself.

I should emphasise that Leibniz does not make this explicit. He does not give much intuition or explanation for the way in which he thinks a concrete body expresses in itself its situation with respect to its surroundings. Clearly he thinks mathematical bodies *do not* express in themselves their places in space, since place is a purely extrinsic denomination. It is also clear that he thinks there is some phenomenal substrate whose parts do express their situation. The story I offer identifies that substrate with the ‘perceptual field’.

2.3 Abstracting from the perceptual field

In this section I discuss the process and result of abstraction from the perceptual field. In the introductory sketch (p. 9) I suggested that place is abstracted by ignoring internal detail in a pretence that everything lies outside, and conversely for extension. That was an attempt to manifest the ‘process’ of abstraction. I also made substantive claims about the ‘nature’ of the abstraction, for example I assumed a connection between the incompleteness of detail, an indeterminacy of parts, the externality of abstract relations, and the abstract as possible rather than actual. I will discuss now a few texts supporting my claims on the nature of the result of abstraction, and then some others supporting my explanation of the process.

2.3.1 The product of abstraction

The various facets of abstract space can be understood in terms of a basic tenet that abstract things are incomplete mental creations that arise by ignoring all but a finite number of constituent concepts of concrete things. Let us focus first on the case of

in abstract space.

extension, since Leibniz says some things more clearly about that, and because it is slightly harder to understand. Then I will summarily apply the same reasoning to place.

I think we can find a thread of reasoning running through and linking together these facets of abstract space, as follows. First, Leibniz has an argument that continuously extended bodies are ideal. The argument is that they are neither real simples (they are divisible) nor real pluralities (they lack simple parts). This argument is premised on the *possibility of division*. This leads to a general thought that extended bodies are ‘possibly’ but not ‘actually’ divided: divisions are not given in a body’s concept but its concept can be *augmented*. We can generate a more complex concept containing the old but with additional data, additional ‘givens’, of certain further divisions.¹⁴ Now it follows from this incompleteness that a body’s concept can be augmented in a plurality of ways compatible with the original partial concept but potentially incompatible with each other. This is precisely the *indeterminacy* (or *indefiniteness*) of decomposition. To summarise: an extended body can be divided into parts yielding a more complex body, these divisions are possible but not actual, some divisions are incompatible with others, and any choice of division must be arbitrary. The relation of extension lies *outside* the extended thing, meaning it is an extrinsic denomination, because the incomplete concept of the extended does not contain the data of its division into parts.

Place is abstracted from situation by ignoring enough detail that the remaining partial concept could be augmented into the real situations of a variety of bodies. In this way place is ‘possible’ situation. It follows that a body’s place is indeterminate, and a choice of place arbitrary. The relation of place is external to a body simply because the

¹⁴Leibniz employs this concept of possibility more generally: an individual *a* is possibly P if an incomplete concept exemplifying *a*’s most salient traits can be augmented (infinitely) to become the complete concept of *a*, but could also be augmented into the concept of another individual *b* such that *b* is P. This is most clear in Leibniz’s correspondence with Arnauld. To say that ‘Adam’ might not have fallen is to say that a partial concept, Adam considered ‘*sub ratione generalitatis*’ as ‘the first man, set in a garden of pleasure, from whose side God fashioned a woman’ etc., could be augmented into a complete concept (determining a complete world) either as the actual Adam or as an ‘Adam’ who doesn’t fall (AG, 72-3; L:CLA).

same identical partial concept can be extended to one or another complete situation, so a situation is not contained in the partial concept of the place.¹⁵

Leibniz most clearly and helpfully expresses these ideas in his letters to De Volder. His argument for real unities, or monads, is quite famous. He thought a parallel argument proves that extended bodies are not real:

... FROM THE FACT that a mathematical body cannot be resolved into first constituents we can, at any rate, infer that it isn't real, but something mental, indicating only the possibility of parts, not anything actual. Indeed, a mathematical line is like the arithmetical unit [i.e., the number 1]: for both, the parts are only possible and completely indefinite. ...

But in real things, namely, in bodies, the parts are not indefinite (as they are in space, a mental thing), but are actually assigned in a certain way, in accordance with how nature has actually instituted divisions and subdivisions as a result of various motions; ... phenomena can always be divided into lesser phenomena, phenomena that can be seen by other smaller animals, and we will never arrive at the least phenomena.¹⁶

The full argument that bodies are not real draws on the continuum composition problem. It goes like this: An extended mathematical body can be divided so it cannot be simple. Suppose it is real. Since it is not simple it must be a plurality. Every real plurality is a composite of real simples, so a mathematical body must be composed of real simples. Suppose those simples are not extended. Then their composite can't be

¹⁵It may be illuminating to connect the indeterminacies of place and extent with some concepts from modern geometry. The volume form generated by the metric on a Riemannian manifold would respect the indeterminacy of extent only if instead of mapping a region to a number (setting a determinate unit), it mapped a region to a segment of a line, with bigger regions mapped to longer segments. As the coordinate numbers break the indeterminacy of place in space, the real number valued metric tensor breaks the indeterminacy of extent. Of course these indeterminacies are only dialectically separable. For a sufficiently complex metric that acts as an individuating field, the metric also breaks the indeterminacy of place. And the natural Euclidean metric generated by one chart of the manifold can be extended from any specified point in that chart to the rest of the manifold by using it to determine the units of the global metric (given previously in the form of a map to line segments): in this way the coordinates break the indeterminacy of size. But both breakages are arbitrary.

¹⁶AG, 178; L:CLDV.

extended, which is a contradiction. Suppose they are extended. Then they can be further divided in the same way they were originally divided (every line segment is similar to every other), another contradiction. So the original supposition is false: extended bodies are not real.

The passage moves from the thought that an extended body is ideal to the thought that its parts are possible but not actual. Leibniz recapitulates this later, more clearly and firmly, in connection with the thought that a body's parts are indeterminate, meaning that they are not actually given in a thing:

It is also obvious from what I have said that, in actual things, there is only discrete quantity, namely a multitude of monads or simple substances, indeed, a multitude greater than any number you choose in every sensible aggregate, that is, in every aggregate corresponding to phenomena. But continuous quantity is something ideal, something that pertains to possibles and to actual things considered as possible.

The continuum, of course, contains indeterminate parts. But in actual things nothing is indefinite, indeed, every division that can be made has been made in them. Actual things are composed as a number is composed of unities, but ideal things are composed of fractions: there are actually parts in a real whole, but not in an ideal whole. As long as we seek actual parts in the order of possibles and indeterminate parts in aggregates of actual things, we confuse ideal things with real substances and entangle ourselves in the labyrinth of the continuum and inexplicable contradictions.¹⁷

Place is abstracted from situation in a complementary process. Though his statement to Clarke on place is less explicit on the link between ideality, possibility, and indeterminacy, he does connect the indeterminacy with conceptual incompleteness:

¹⁷AG, 185; L:CLDV.

This supposition of two indiscernibles [in different places], such as two pieces of matter perfectly alike, indeed seems to be possible in abstract terms, but it is not consistent with the order of things, nor with the divine wisdom by which nothing is admitted without reason. The vulgar fancy such things because they content themselves with incomplete notions. And this is one of the faults of the atomists.¹⁸

Let us summarise these connections. To consider a body having extension and a place in space is to consider an incomplete concept. The *incomplete* concept does not contain a particular situs and complexion, meaning a body does not express in itself its surroundings or its parts. So place and extent are *external relations* whose application corresponds to the *possibility* of augmenting the concept of a body with the information of a particular situs and complexion. Since a particular completion is not determined by the partial concept it completes, it is *indeterminate* and requires a choice that must be grounded outside the abstractions.

2.3.2 The process of abstraction

Leibniz recognised that certain abstract concepts are inseparable.¹⁹ Extension and place are, in particular, logically internally related: extension is distance between places, and places are distinguished by intervening extension. So my goal is not to give proper definitions of these abstractions independently from each other; my goal is rather to locate the roots of these abstractions in the more concrete perceptual field.

Many or most of the features of my general metaphysical characterisation of abstract place and extent apply to other abstractions. This means the generic description, in terms of incompleteness, ideality, and indeterminacy, cannot be specially illuminating

¹⁸L:CLC, §V.21.

¹⁹‘I don’t entirely approve of the doctrine of attributes which they are formulating today, as if a single simple and absolute predicate (which they call an attribute) constitutes a substance, for I don’t find among notions any predicates that are entirely absolute, predicates that don’t involve a connection with others. Certainly none are less so than those attributes, thought and extension, which they commonly put forward as examples, as I have often shown.’ (AG, 174; L:CLDV)

for the case of space. I hope to illuminate place and extent by showing how they in particular are abstracted from something more concrete.

I will give that story in three and a half conceptual moves or points, supported by three passages. First I would establish Leibniz's belief that extended bodies are abstracted by ignoring details, pretending that different things with different natures have the same nature (differing numerically by stipulation). In a second move I'll present Leibniz's clearest remarks on the nature of extension. Leibniz says that extension is a *diffusion* of the nature of a part into a strictly greater whole (or vice versa), meaning that proper parts are assumed to have identical natures to the whole (from which it follows that the parts are infinite and indefinite: given any part another smaller can be generated). In a third move I'll explain how this dynamic generates an abstract (external) relation between a whole and its parts: the real relation of complexio inhering in and determining the whole is taken out of the whole, reducing the concept of the whole to that of a part. The half-move is an ancillary consideration identifying the third point with the idea that there is nothing *outside* the whole. This consideration gives insight into the abstraction itself by letting us cast extension as the pretence that the whole is all of space. This move is supported by Leibniz's claim that space, that in which all things have place, is also *absolute extension*.

Leibniz makes a dual claim that a point, that which properly contains nothing, is *absolute place*. I will assume again that the rest of the story applies *mutatis mutandis* to place: rather than a diffusion of nature from the whole to its parts *inside* itself, we have a diffusion of nature from one part to another *outside* itself. Intrinsic diffusion is repetition in a sequence changing in size, extrinsic diffusion is repetition in a sequence changing in place. Since the (partial) concept of the part is also found in another part, the place is again an extrinsic denomination. Since absolute place is a point, the corresponding ancillary consideration is that place is the pretence that there is nothing *inside* a region.

Begin with the first move. All universals are abstracted by ignoring innumerable

details, and extended bodies are no exception:

Individuals [*singularia*] involve infinity; in forming *universals* the soul only abstracts certain circumstances by concealing innumerable others. And so it is only in an individual that there is a complete notion, and thus that notion also involves changes. A *spherical body* complete in all respects is nowhere in nature; the soul forms such a notion by concealing aberrations. And it is the same for any other shape that a finite mind can grasp, namely, such a shape can never exist exactly.²⁰

Extension is an abstraction representing repetition of the nature of a whole in its parts, like whiteness in milk or the segments of a line (the whiteness in a part is qualitatively identical with the whiteness in the whole; any part of a line segment is similar to any other and to the whole).

Indeed, ... extension is an abstraction from the extended thing, and it is no more a substance than number or multitude can be considered to be a substance; it represents only a certain nonsuccessive ([unlike] duration) and simultaneous diffusion or repetition of a certain nature, or what comes to the same thing, it represents a multitude of things of the same nature, existing simultaneously, with a certain order among themselves. It is this nature, I say, that is said to be extended or diffused.²¹

...

The diffusion that I conceive of in extension and that seems to have put into you the suspicion of some hidden paradox, I know not what, is, I claim, nothing but the continuity [*continuatio*] in which a part is similar to the whole, as, for example, we conceive of whiteness as diffused in milk, the same direction as diffused everywhere in a straight line, and equal

²⁰ AG, 183; L:CLDV.

²¹ AG, 179; L:CLDV.

curvedness as diffused in the circumference of a circle.²²

So extension is an abstraction generated by ignoring certain details given in the phenomena, and the nature of that abstraction is a diffusion whereby the quality of a whole is considered identical to that of some part(s).

My third move is to infer an external relation between the whole and its parts, a relation that does not inhere in the whole. This indeed follows: since by supposition the whole and its parts have the same nature, whatever affection is found in the whole (in particular the affection relative to its parts) must also be found in the parts. So the whole-part difference is not reducible to the separate concepts of the whole and the parts, implying that the composition relation is external.

The additional half-move, what I called the ancillary consideration, is that the pretence that the nature of a whole is diffused into its parts can be identified with a pretence that the whole is all of space. The pretence of extension is the pretence that a region has no situs.

The purpose of this consideration is to allow us some way of indicating *what* details are ignored to generate the abstraction. It gives a succinct characterisation. The consideration is ancillary because the third move (that extension is the external relation corresponding to a diffusion) was the desired conclusion. It is admittedly also more speculative than the three primary steps, and can be dropped if considered implausible or unhelpful.

As a first step toward this claim, consider that space is *absolute extension* and a point is *absolute place*:

(9) Treating everything in orderly fashion, we know that *Space* is the first thing to be considered. It is pure absolute Extension: *pure*, without matter and change, *absolute* moreover, since it is unlimited and contains every extended thing. Therefore all points are in the same space, and may

²²AG, 183; L:CLDV.

be referred to each other. But whether this space is a thing distinct from matter, or else only a constant appearance or phenomenon, does not matter here.

(10) Next is the consideration of Point, which is the simplest among all things pertaining to space or extension; since Space contains absolute extension, point expresses that which in extension is maximally limited, namely simple situation. It follows that point is the minimum and lacks parts, and all points are congruent to each other (or can be made to coincide) and thus are similar and, if you will, equal.²³

Note that the term ‘absolute’ should be contrasted with ‘mixed’, where a mathematical body or figure has both place and extent, but is still completely abstract. In context Leibniz is treating abstract geometrical concepts and isn’t explicitly considering the process of abstraction from concrete phenomena. Nevertheless there is a connection between the extrema and the abstraction process. The abstraction of a figure, I suggest, pulls a region into two extrema, from inside and from outside, into place and extent, and then builds the detail back by mixing the two while retaining their abstract natures. More refined, or more accurate, mathematical representations of actual phenomenal bodies are simply more complex mixtures of abstract place and extent, and the abstractions are generated by ignoring in turn the infinitely complex external or

²³‘(9) Verum ut omnia ordine tractemus, sciendum est primam esse considerationem ipsius *Spatii*, id est Extensi puri absoluti: *puri*, inquam, a materia et mutatione, *absoluti* autem, id est illimitati atque omnem extensionem continentis. Itaque omnia puncta sunt in eodem spatio, et ad se invicem referri possunt. An autem spatium hoc a materia distinctum res quaedam sit, an solum apparitio constans seu phaenomenon, nihil refert hoc loco. (10) Proxima est consideratio *Puncti*, id est ejus quod inter omnia ad spatium sive extensionem pertinentia simplicissimum est; quemadmodum enim Spatium continet extensionem absolutam, ita punctum exprimit id quod in extensione maxime limitatum est, nempe simplicem situm. Unde sequitur punctum esse minimum et partibus carere, et omnia puncta congruere inter se (sive coincides posse) adeoque et similia atque si ita loqui licet, aequalia esse.’ (GM, V.144, my translation, emphasis original). The same duality appears in the very late *Initia rerum mathematicarum metaphysica* (GM, VII.21, my translation):

Point (of space of course) is the simplest locus, or the locus of no other locus. [*Punctum* (spatii scilicet) est locus simplicissimus, seu locus nullius alterius loci.]

Absolute space is the fullest locus or the locus of all other loci. [*Spatium absolutum* est locus plenissimus seu locus omnium locorum.]

internal detail.

Now the pretence that the nature of a whole is diffused into its parts is identified with the pretence that the whole is space because the situs relations of the parts with respect to determinate things outside is what actually differentiates the parts from each other and from the whole. Equivalently, a whole is distinguished from its parts by *boundaries per se*, and the pretence that there are no boundaries to a whole is the pretence that ignores the qualities of the whole which aren't left in its parts. Let us unpack this idea.

Consider what is entailed by supposing the nature of a whole repeated in the nature of its part. This nature is such as to support a part by concept augmentation, but it is also the same nature which is such as to support being a part of a bigger whole by (an inverse) concept augmentation. It follows that in order to suppose that the nature is diffused into a thing's parts one must also suppose the nature diffused into a greater whole, meaning it is itself the possible subdivision of an as yet undetermined whole. The supposition of infinite divisibility is the same as the supposition of infinite extensibility.

An illustration may help with intuition (see Fig. 2.1). In (a) the rings are distinguished from each other by the number of their layer, counting from the outside, where the boundary or outermost ring is defined as the complement of Y. In (b) there is no possibility of distinguishing one ring from another on the basis of its qualities: all the rings contained in any chosen ring are similar to each other and to the chosen ring. But this is in virtue of the possibility of continuing to draw larger and larger rings, ultimately encompassing all of space.

The abstraction of extension erases the difference between a whole and a smaller part; the same act removes the boundaries of the whole. All that remains is an indeterminate concept needing further determination by the addition of place, which means the addition of expressions in the concept of different natures outside.

The abstraction of extension is an external relation given by ignoring the difference

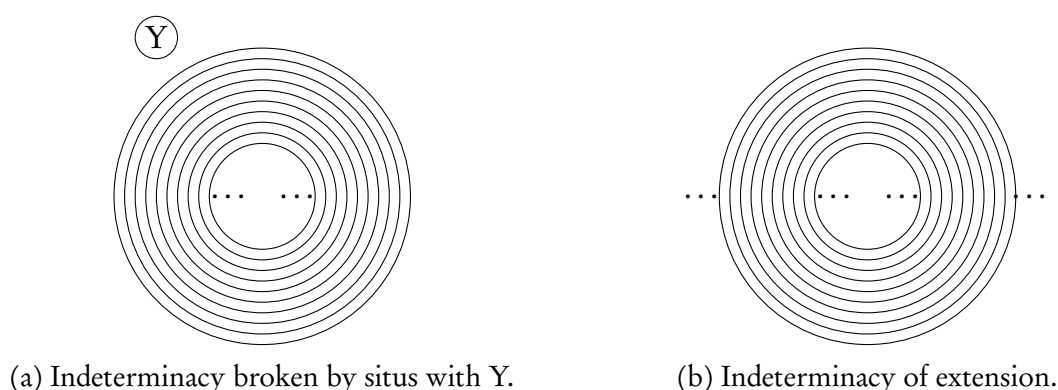


Figure 2.1: Extension pretends nothing is outside.

between a whole and its parts in a pretence that one nature is repeated by diffusion to its parts. This pretence is equivalent (I speculate) to a pretence that the whole has no boundaries (since a boundary is the principle distinguishing a whole from the aggregate of its parts), and this is equivalent to the pretence that the whole is space.

Reciprocally, place is the external relation given by ignoring the difference between one part and another in a pretence that one nature is repeated by diffusion to another nature (outside, not a part). This pretence is equivalent (I speculate) to a pretence that the part has no internal detail, since internal detail is the principle distinguishing one part from another outside, and this is equivalent to the pretence that the part is a point.

2.4 Is space Euclidean?

As an aside from the main presentation of Leibniz's logic of geometric quantity, the framework sketched above lets us treat the question whether Leibniz's concept of space was inescapably Euclidean in the sense that it is of the nature of the abstraction of bodies with place and extension that they inhabit a Euclidean space. We could ask a superficial question whether Leibniz was aware of the possibility of spherical geometry (he was), or whether he believed Euclidean geometry to be necessary, or a necessary background for other geometries (perhaps he did). But a deeper question worth asking concerns how deep the roots of a commitment to Euclidean space were: *why* did he

think space is Euclidean, if he did? Can that belief be dropped without troubling the rest of the picture developed above?

There is one not implausible reason to think the abstract space constructed from the perceptual field is necessarily Euclidean. It seems credible that the abstraction of place is generated by the possibility of *rigid motion*, and the abstraction of extent by the possibility of *homothety (scale) transformation*. These transformations are only well-defined if space is Euclidean.

The abstraction of place comes from the pretence that there is a body which can have a variety of places and remain the same body. This is not implausibly associated with the idea that the concept of space is such that a body can be *moved* in space while preserving all its internal structural properties. As is well known, the requirement that such a transformation be well-defined is equivalent to the requirement that space have a constant background curvature.

The abstraction of extension comes from the pretence that there is a body which can have a variety of extents and remain the same body. This is not implausibly associated with the idea that the concept of space is such that a body can be *expanded* (or at least another, larger, body can be defined) in space while preserving all its internal structural properties. As is well known, the requirement that such a transformation be well-defined, combined with the requirement above that rigid motion be well-defined, is equivalent to the requirement that space have a background curvature identically zero, hence that space is Euclidean.

This reasoning is not implausible, but I suggest that it is nevertheless mistaken. One faulty assumption is that the possibility of setting different relata (bodies) in one slot of a relation (place or composition) entails the possibility of the respective symmetry transformations of space. But this is not true. On the contrary, I argue that this assumption betrays a misunderstanding of the abstractions of place and extent, and indeed those abstractions *per se* do not constrain the curvature of space at all. Let us build to this in steps.

The first suggestion that something is wrong in the above argument for Euclidean space is the appearance of an asymmetry between place and extent with respect to the transformations. The possibility of *motion* is alleged to ground the abstraction of place. Motion is something that actual bodies undergo. The possibility of *expanding* is alleged to ground the abstraction of size. But expansion is something bodies do not (usually) undergo.

The second sign of trouble is that the abstraction of place was supposed to arise out of the pretence that *a body is a point*. Yet the ‘motion’ of a *point* is not associated with the curvature constraints that are associated with motion of *rigid bodies*. Similarly, the abstraction of extent was supposed to arise out of the pretence that *a body is space*. Yet the change in ‘size’ of space—the whole—is not associated with the curvature constraints that are associated with a homothety transformation *within* space. Indeed, one can move a point by any arbitrary transformation, even a permutation of points; and one can change the ‘size’ of space simply by adding an arbitrary coefficient to the metric. But neither of these ‘transformations’ constrains or even involves the curvature profile.

I suggest that the abstractions of place and extent arise not through imagined motion or inflation of bodies, but through imagined *constancy* of bodies despite change in *perspective*. The ‘regions’ of the perceptual field that bodies occupy *do* move *and* change size as the perspective on them changes. Then the fact that a body can occupy *any* place is simply the nature of place as a reified external relation, and the fact that a body can be composed in any of a variety of ways is simply the nature of the extended continuum as a reified external relation.

To put the point another way, the suggestion is that the indeterminacies of place and extent are not indeterminacies of the abstract structure of space, which would vary from one metric field to another. They are indeterminacies of the connection between that abstract structure and the parts of our perceptual field. There is no indeterminacy in perception, which manifests determinate situs and complexio; but the situs and

complexion cannot be changed without changing the nature of the perception. The determinate perceptual field serves to coordinate the abstract geometric structure of space onto the world.

If this is right, then the Euclidean character of space cannot be deduced from the considerations above regarding the generation of space as an ideal abstraction from perception. Whether it can be deduced from other Leibnizian principles I cannot address. I would venture a conjecture that the question should be for Leibniz a contingent one that couldn't be established in any finite analysis.

Chapter 3

Recovering the Analysis Situs

In this chapter I show how trenchant objections to Leibniz's project are dissolved by the account of his concept of quantity developed in the first two chapters.

3.1 Couturat

Criticism. Couturat's study of *The Logic of Leibniz* includes a chapter devoted to the Analysis Situs. In that chapter he employs a naïve concept of geometric quantity, analysed simply as a dyadic relation of length. He supposes, based on texts like the introduction to Huygens, that Leibniz sought entirely to eschew quantity in a purely qualitative geometric form.

Couturat does not observe the distinction and interplay between abstract figure and concrete perceptual field. So he takes similarity and congruence as simple equivalence relations that apply to abstract figures. Congruent figures differ only in position, similar figures might differ in position or scale.

Now Couturat concludes that Leibniz's whole project failed because the geometric forms he characterised with similarity do not exclude quantity after all. He ends his chapter on the Analysis Situs with biting criticism:

For this [investigation of the 'reasons for his failure'], it is only necessary to ask whether Leibniz actually did what he wanted to do, that is, whether he freed geometry from any consideration of magnitude and managed to express situation directly.

The answer to this question can only be, no. We have seen that Leibniz neglected relations of similarity in order to study almost exclusively the relation of congruence, the least general and most complex of all the geometrical relations. Moreover, this relation in no way abstracts from magnitude, since it involves, along with the relation of similarity, that of quantitative equality. . . .

To establish the properly geometric calculus of which he dreamed, Leibniz would have had, on the contrary, to separate relations of situation from relations of magnitude, and to abstract from any *metrical* consideration. It would not have been enough to substitute similarity for congruence as the fundamental relation, for as he showed, similarity still involves a relation of magnitude—namely proportionality. It would have been necessary to complete the analysis of situation by reducing figures to projective relations and properties.¹

A premise of this criticism, taken as an argument, is that Leibniz wished to design a geometric science entirely independent of geometric quantity.² In this vein Couturat reads the transition from congruence to similarity as a transition that discards one source of quantity, namely the external or global quantity pertaining to a figure as a whole. But that is not enough: similar figures also contain internal quantities corresponding to the relative positions and sizes of their parts. So on this reading the ‘discovery’ of similarity was only a small step on a long path.

Couturat thought Leibniz might have achieved a thorough-going rejection of quantity in geometry had he developed projective geometry or smooth topology instead. Projective geometry is the geometry of projective space, which is the space generated

¹Couturat, *La logique de Leibniz*, §9.20, Rutherford’s translation.

²It is worth remarking that Couturat cannot possibly have supposed Leibniz to have aimed for a geometry without quantity of *any* kind. We have no such science today, and I cannot imagine what it would be like. Even topology involves quantity in e.g. the number of connected components of a space, and projective geometry in other ways. The key is *geometric* quantity: length, area, and volume.

from ordinary space by identifying points along an equivalence relation of scale: two vectors in ordinary space are the same in projective space if they differ only by a scalar multiple. Smooth topology is the study of smooth manifolds, which are mathematical spaces identified by an equivalence relation of *smooth transformation*. A smooth transformation, i.e. a diffeomorphism, is an invertible map that is infinitely differentiable in both directions. In both of these branches of geometry the mathematical structures preserved by the equivalence relations do not include a metric or distance relation between points.

But Leibniz did not invent topology. In fact, between similarity and congruence, only the latter ended up taking central place whenever Leibniz illustrated the geometrical ‘situations’ that his calculus captured. Similarity and co-perception tended to be relegated to metaphysical or conceptual definitions, or else used to introduce his project to others. When it came to working out the project in any detail, he relied on congruence. In developed writings on the project a point would be represented by a letter, A , another point by B , and their joint situation by AB . The same situation would obtain between any other pair of points CD such that a path from C to D could be moved rigidly to a path from A to B . Rigid motion is the foundation of congruence.

On Couturat’s reading, the introduction of similarity was supposed to be a move toward ridding geometry of metrical properties, but Leibniz’s intuition for the nature of geometric figure did not follow suit. Eventually Leibniz had to retreat to a fully metrical geometry, and with that he lost the motivating principle with which he began the project.

Response. Couturat has misread Leibniz’s project. The kernel of his mistake was a failure to notice the role of perception in geometric quantity, and concomitantly the distinction between the fully determinate (complete, infinite) to which we have access in the perceptual field, and the indeterminate abstractions (incomplete, finite concepts) used in the mathematical sciences.

Failing to understand the real significance of Leibniz's perception-based definitions of similarity and congruence, Couturat supposed that the difference was adequately captured in the fully abstract notion of homothety or scale transformation: similar figures are those which can be made congruent by change of scale factor. What changes when a figure is scaled up or down? The *distance* between each point of the figure; the *lengths* of the lines and the *areas* and *volumes* of regions. Since Leibniz certainly identified the geometric forms of similar figures (rather: defined similarity as identity of form), it seems natural to conclude that Leibniz wanted to create a geometry without distances or 'metrical considerations'. This, of course, is precisely the judgement of the last paragraph from Couturat quoted above.

Had he suggested instead that Leibniz wanted to define a geometry without *units* of distance (meters), Couturat would have struck much closer to the mark.

Leibniz's project can be defended from the account I offered, beginning with a re-assessment of the definition and purpose of the concept of similarity, and continuing with a reconsideration of each further step in the reasoning of Couturat's criticism.

The definition of equality through co-perception was meant to indicate that geometric quantity (intrinsic, whole-part, not ratio) is restricted to the figures abstracted from parts of the same perceptual field. This is what it means for figures to be perceived together. The definition of similarity served to indicate that geometric form is precisely that which is abstracted from the perceptual field, and is thus independent of the perceptual field.

Geometric quantity is restricted to the abstractions from a single perceptual field because the unit it requires is not given in the abstract concepts. Quantity is a whole considered as a repetition of a certain part called the measure. Since the constituents of abstract figures have incomplete concepts such that they are similar to their parts, their concepts do not contain natural units or the conceptual basis to ground a choice of unit. Such a choice must be based instead on the more determinate parts of the perceptual field, or, in the context of geometry rather than physics, in the parts of the imagined

perceptual field that grounds a single mathematical space.

From these considerations it follows that the abstract forms of geometric figures cannot contain intrinsic quantities. But they certainly can contain extrinsic quantities: as we saw in §1.5, there is an abstraction from geometric quantity that explicitly externalises the unit, meaning it is an extrinsic denomination holding between two parts of an abstract figure. This was called *ratio*. Of course ratio is abstract (it is the exemplar external relation), not expressed in the natures of its relata, so the parts of a purely abstract geometric figure could still have determinate ratio.

The purpose of the definition of similarity was not, I suggest, to provide an equivalence relation on purely abstract figures that eliminates quantity from geometry, for of course there are quantities in abstract figures, namely the extrinsic quantities or ratios between their parts, just as there are distances or relative places between parts. Instead the purpose was (as I shall argue more fully in the conclusion) to expose or bring into clear view *that there is a distinctively geometric form*. Geometric form is whatever can be perceived in a figure taken individually, and this is just whatever pertains only to the abstraction from a single perceptual field. Leibniz's definition of similarity is an attempt to isolate and give intuition for abstract geometric structure.

It follows from this that Couturat was wrong to think Leibniz meant to design a kind of geometrical science that did not involve any kind of quantity, for there is no doubt that the abstract figures Leibniz had in mind contained ratios between their constituents. The lengths of all parts of any figure can be expressed numerically as soon as one part is chosen as the unit. (For instance we could choose the shortest side of a triangle, which is a determinate constituent of the abstract concept of the triangle, and which has determinate ratio to the other sides. Of course the parts of the shortest side are indeterminate.) This is just the ordinary procedure in geometry, where parts of figures are assigned only numbers and never lengths in (e.g.) metres. So the introduction of similarity was in no sense a move in the direction of projective geometry or smooth topology.

Let us grant now that the primary function of the concept of similarity was to isolate abstract geometric structure, which means we should expect the concept to have had a primarily introductory role in Leibniz's work, where it was used to identify that structure for his audience. We should likewise expect it to have had a less significant role when he comes to working out the details of his project, and indeed that is what we find.

In order to give a more complete and satisfying response to Couturat's argument, even granting the role I propose for similarity, we should also ask about the role of similarity (in contrast to congruence) in the context of a *single* mathematical space, such as the context of working out the details of his project. In the abstraction from a single imagined perceptual field in which figures can be decomposed by the same measure and compared in quantity, both relations of congruence (difference only in place) and similarity (difference in place and quantity) are operative. Why does it seem Leibniz preferred to use congruence when defining his situations?

I would answer that this is an illusion. The illusion comes from the division in situs between absolute and relative situs in contrast to the monolithic quantity, and this contrast in turn stems from the plurality of a plurality (the parts outside a part) in contrast to the unity of a unity (the oneness of a whole).

The illusion is that Leibniz appears to switch from similarity to congruence when he provides characters for geometric forms. Couturat thought Leibniz switched to congruence because the situation expressed by AB is said to be the same as that expressed by CD provided curves between the end points come in congruent pairs, one between A and B for each one between C and D . But this is not a use of congruence to determine geometric form at all. It is the use of congruence to define relative situs.

Place draws things apart, separates them, because place is generated by pretending things are simple points. Since place draws things apart, difference in place is the principle giving difference in portions of a figure, or subfigures of a figure, or parts of the whole figure. (These parts are in different places.) From this it follows that the division

of the whole into a plurality of parts which *one might take to be subfigures* is a division operating by cutting out place. In contrast, ratio is a principle that unifies or brings together separated things; in distinguishing different parts of a figure we need not cut their ratios. A consequence is that quantity itself is monolithic, whereas place itself is as plural as the plurality of parts: there is a place of each part relative to each other part, a place relative to each other collection taken as a whole, and then finally a place relative to the main whole (absolute place).³

With these considerations in mind we can see that Leibniz was not giving undue attention to congruence, since he was not using congruence to determine geometric forms. When he writes that AB is the same situation as CD , he means the relative situation. Obviously, if all four points are in the same space, each *absolute* place is unique, just as the absolute quantity (given a measure) of each line or area is unique. But certain *relative* places may be non-unique: the place of A relative to B (expressed ' AB ') is in fact the same as the place of C relative to D . If the points are not in the same space but the pairs are in two different spaces, the relative situation AB also happens to be an absolute situation, and likewise for BC . But now the question whether they are congruent lacks a determinate answer (it is ill-posed). The question whether they are similar, on the other hand, does not; they are.

A last consideration is that it is useful to consider partial places, or relative places, in order to build up or construct a geometric figure from various parts. One wants to consider the 'situation' of some parts taken together, while denying that the situation of just those parts is the entire figure. One would like to imagine adding in or removing further parts, and these must come from somewhere. Leibniz appears to work with congruence because ratios are fixed by the other parts which are left 'off the screen', as it were. So his use of relative place was accordingly no preference for congruence over

³It is tempting but mistaken to compare ratio to relative situs, and suggest accordingly that quantity also is not monolithic. But ratio externalises only a *unit*. It is possible to extend the concept of ratio to similar figures in the same space by reducing them to units and decomposing one unit by the other unit. But figures in general, and obviously collections of figures, will not have ratios even though they will have relative places.

similarity in the foundation of his project. In the end he was interested in the forms of entire figures, and these include ratios between parts.

3.2 De Risi

Vincenzo De Risi's 2007 study of the *Analysis Situs* is the only comprehensive and detailed critical work on the subject. De Risi treats the family of interrelated concepts of similarity, congruence, and equality, as well as magnitude, scale, and measure. What follows is my best attempt to distil his criticism into a few specific thoughts, and then to show how Leibniz might have responded on the basis of the foregoing interpretation.

Like Couturat, De Risi does not distinguish the conceptual layers of geometric abstraction and concrete perceptual field. When characterising geometric quantity, he considers geometric figures under a single univocal aspect, and consequently supposes that quantities and qualities are equally intrinsic properties of these figures and their parts. Also like Couturat, he thinks the definition of similarity ought to be analysed in mathematical contexts in terms of homothety transformations on a vector space:

As we have seen, according to this definition two objects are similar when they are indiscernible if separately considered. Such a phenomenological proposition should be articulated into the usual definition of Euclidean similarity as an angle-preserving transformation, and Leibniz even claims he can give a demonstration of the equivalence between his own perceptual definition and the Euclidean one.⁴

[In a footnote to this paragraph:] A little definition exercise on this topic will perhaps be useful. Well, apparently it comes as natural to define the similarity of two geometric figures by multiplying their coordinates by

⁴De Risi, *Geometry and monadology*, 140.

a given real, non-null constant λ , that is to consider the group of homotheties having the origin as their center.⁵

(I do not think De Risi means to read into Leibniz, anachronistically, the modern concept of vector space this definition presupposes, but I do think his comparison demonstrates the concept of quantity he has in mind.)

Now the set of criticisms that De Risi brings relevant to our work on the concept of quantity can be divided into two classes, with three specific challenges in each class. The first class represents challenges to Leibniz's attempts to *define* geometric quantity. The second class represents challenges to the *purpose* of Leibniz's definition(s), both that he didn't and that he couldn't succeed in using his definitions toward the ends he advertised.

In the first class he considers (1a) the definition of quantity through co-perception, objecting that its negativity renders it useless. Then he considers (1b) a definition of quantity in terms of congruent simple units, and concludes that it would engender a vicious circle with Leibniz's formula 'congruence = similarity + equality'. Finally he concludes that Leibniz (1c) eventually acquiesced in the doctrine that quantity can have no metaphysical definition.

In the second class De Risi considers the concept of similarity (in contrast to equality) under the aspect of an attempt to develop a geometry free of geometric quantity. Here we return to thoughts already found above in Couturat's work. De Risi objects (2a) that the form of a triangle cannot include the ratios of side lengths without invoking the side lengths themselves, (2b) that in applications of the definition of similarity, the concept of congruence does most of the work, and finally (2c) the Analysis Situs is in the end undoubtedly a *metric geometry* so there can be no real hope of extracting quantity from geometric form.

If the reading I have presented is correct, all of these criticisms are misguided. Let

⁵Ibid., 140.

us consider them in turn.

3.2.1 Definitions of quantity

Criticism 1a. According to the main definition we considered above, quantity or magnitude is ‘that in things which can be known only through their compresence (or by their simultaneous perception)’. Quality is ‘what can be known in things when they are observed singly, without requiring any compresence’.⁶ De Risi suggests that these definitions are *complementary*, in that quality involves taking things separately in the same sense in which quantity involves taking them together. In this way the two kinds of attributes should be mutually exclusive, and jointly exhaustive, of the set of attributes of a figure. The dichotomy supposedly ‘holds qualitative and quantitative determinations to *finish off* all possible determinations of any given phenomenon’.⁷

While this dichotomy is attractive philosophically, De Risi thought it couldn’t bear significant mathematical weight:

This last “*analysis situs*”, here intended however as the analysis and dissolution of the situation *concept* into its two (qualitative and quantitative) components, probably created more problems than it solved, so that it is certainly easier to find an accomplished Leibnizean geometrical theory before than after such a dramatical inversion.⁸

Now De Risi thinks such general definitions need to be provided with ‘theorems of characterisation’ in order for them to be of use in mathematical proofs. A theorem of characterisation would establish that the general definition reduces (in a more specific

⁶De Risi, *Geometry and monadology*, 146, from (GM, VII.18-19) = (L, 1084).

⁷De Risi, *Geometry and monadology*, 147. Cf. also the passage he cites: ‘Imaginatio generaliter circa duo versatur Qualitatem et Quantitatem, sive magnitudinem et formam; secundum quae res dicuntur similes aut dissimiles, aequales aut inaequales’ (C, 348). It is not clear that De Risi is justified in reading more into this passage than that Leibniz thought quality and quantity are the two main concepts involved in geometric imagination, each having their equivalence relations.

⁸De Risi, *Geometry and monadology*, 160.

mathematical context) to a definition in abstract mathematical terms. This definition would be employed in mathematical practice.

In order to preserve the complementarity of quantity and quality, it follows that either (1) each class should be given a theorem of characterisation independently of the other, and then an argument provided that they are indeed jointly exhaustive, or (2) one class could be explained in mathematical terms, and the other *defined* as its mathematical complement. This latter approach is ‘negative’ in the sense that geometric quantity would be characterised mathematically as the negation of mathematical quality.

De Risi supposed that Leibniz took the second approach. He thinks Leibniz’s *quality* was adequately characterised mathematically in terms of the angles and ratios of the parts of a figure, and that the definition of *quantity* was supposed to be whatever is ‘leftover’ (in the geometric figure) after quality has been considered. But this approach doesn’t yield a mathematically practicable concept:

Just because of the merely negative character of his phenomenological definition of quantity, thus of equality as well, Leibniz never succeeded in finding a really satisfactory geometrical theorem of characterization that could serve as a bridge between the philosophical and geometrical formulations of the problem.⁹

His failure to find a bridge for quantity between metaphysical foundations and abstract mathematics (while respecting its complementarity with quality) constituted a failure of the project. If Leibniz was to have had a working mathematical theory he would need to have abandoned his perception-based definitions of quantity and quality.

Response 1a. De Risi’s criticism turns on a tension between (1) the desire for complementary concepts of quantity and quality, and (2) a need for characterisations of these concepts in terms suitable for mathematical practice. But this mistakes the quantity-quality division. The abstractions in geometry from concrete quantity and quality are

⁹Ibid., 147.

not complementary, and quantity can only be partially characterised abstractly through ratio and extension.

According to my reading, the definition of quality and quantity in terms of co-perception should be understood as a way to invoke the act of synthesis generating a concrete perceptual field. Qualities can be known without ‘co-perception’ because qualities are elements of the *abstract* figure. In this way the qualities of two figures can be compared without the abstract forms having come from the same perceptual field (we can ask whether they are isomorphic without them having been ‘co-perceived’). Everything necessary for qualitative comparison is abstracted from figures perceived ‘singly’. Quantity requires co-perception simply because quantity only applies in the context of the abstraction from a (single) perceptual field, and that is because geometric quantity requires the possibility of specifying a measure to decompose each part of the whole field.

From this it follows that quantity (internal quantity, of course, not ratio) *should never* receive a ‘theorem of characterisation’. Abstract geometry should not have the resources to distinguish (internal) quantities because the abstract concepts from which abstract figures are constructed lack the requisite detail (the complete concepts given in the perceptual field) to distinguish them from each other and from the whole.

Since Leibniz is not interested in bringing (internal) quantity into abstract geometry, he does not employ strategy (2), and it does not matter whether there is a sense in which quantity and quality are complementary.

Criticism 1b. The most vexing criticism De Risi offers is an observation, simply stated, that a second definition for quantity that can be found in Leibniz’s writings is incompatible with a claim to which Leibniz was quite attached. The definition is that quantity is the number of congruent parts in a figure, and the incompatible claim is that congruence is the combination of similarity and equality. It is circular to define quantity in terms of congruence and congruence in terms of equality.

De Risi draws this second definition from passages like (27c-d) of the *Characteristica Geometrica* (p. 23 above). Quantity is there called the attribute that ‘remains the same through substitution of congruents’. Or from the *Specimen Geometriae luciferae*: ‘Equals arise from congruents. Those which are congruent of course, or if necessary may be made congruent by transformation, are called equal.’¹⁰

The antagonistic line of thought appears in the same *Characteristica Geometrica* (though it is not limited to that essay):

From this, moreover, we can see that simultaneous similars and equals are congruent.¹¹

De Risi was motivated to consider the formula

$$\text{Congruence} = \text{Similarity} + \text{Equality}$$

as a definition of congruence by a line of thought that isn’t at all implausible. Recall from the previous section his notion that quantity and quality partition the set of attributes of abstract geometric figures. Similarity is identity of quality, and equality is identity of quantity. It follows that similar and equal figures should have *all* attributes in common. This makes congruence a good candidate for the *foundational* or *finest* equivalence relation between figures. The goal of the Analysis Situs would be to provide a characterisation of figures *up to congruence*. Two strings of letters $A_1A_2 \dots A_n$ and $B_1B_2 \dots B_n$ designate the *same situation* if and only if the figures determined by the points named by each letter sequence are congruent. De Risi was motivated to consider the formula as a definition because it shows how the foundational relation of the Analysis Situs derives from combination of the geometric species of two very general metaphysical denominations.

¹⁰‘Ex congruis oriuntur aequalia. Nempe quae congrua sunt, aut transformatione si opus sit congrua reddi possunt, ea dicuntur aequalia.’ (GM, VII.266, my translation.)

¹¹‘Ex his autem intelligi potest, similia et aequalia simul esse congrua. (GM, V.154, my translation.)

On the other hand, De Risi was motivated to consider the description of quantity in terms of reduction to congruents as a definition of quantity on account of a line of reasoning familiar from the last section. The definition in terms of co-perception is useless in mathematical practice, and this characterisation in terms of reduction to congruents is of the utmost utility; indeed it can serve as the foundation of a robust theory of measurement and all manner of geometric demonstrations may be carried out by its use. If, as he assumed, Leibniz was in the business of looking for a bridge from the metaphysical quantity to a mathematical species, this would be the perfect candidate. It is only natural to consider it an alternative definition of quantity in geometry.

De Risi sharply denounced an attempt to follow both lines of thought simultaneously:

The very extrinsicity of this definition [things which can be overlaid] might not have satisfied Leibniz. He preferred therefore to use it as a (practical) characterization in a more general definition able to eschew all extrinsic considerations. ... Refining and deeply reworking the preceding definition, this one will first assume another primary relation, namely equality, and then define congruents as figures being *both similar and equal*.¹²
...

However, *never* in his life did Leibniz get to an adequate metaphysical definition for quantity. This very shortcoming caused him in his young years repeatedly to hide this imposing problem under the operational definition of quantity through the equality concept, and define the latter through congruence. Which, being a gigantic *petitio principii*, seems to me the greatest obstacle we are faced with when trying to interpret Leibniz's theory of measure (not to speak of several metaphysical questions).¹³

The definition of quantity through congruence was question begging, indeed begging

¹²De Risi, *Geometry and monadology*, 143.

¹³Ibid., 146.

of *the* question, because *the* question was how to define the foundational equivalence relation of the Analysis Situs.

Response 1b. Again, De Risi's challenge collapses if we acknowledge that geometric quantity is not an attribute of the abstract alone, but also a function of its perceptual ground. According to the theory of quantity I have developed, both lines of thought leading to the circularity stand in need of correction. After they are corrected the circle disappears.

Consider first the line of thought that 'reduction to congruents' is a definition of quantity that replaces 'known through co-perception' on account of its superior mathematical utility. I suggest that this line cannot be reconciled with the texts, both in the same neighbourhood and elsewhere. Instead, I suggest, both 'definitions' of geometric quantity are partial characterisations of the very same concept, each emphasising a different aspect.

Recall (27e) of the *Characteristica Geometrica* (p. 23 above) in which Leibniz notes the definition in terms of co-perception *immediately* after explaining the matter of reduction to congruents, and after that he remarks explicitly that both considerations 'come to the same thing'. It is difficult to believe that he thought they were competing definitions. Moreover, if we look a little beyond the passage cited above from the *Specimen*, we find a passage in which they are directly combined:

Equals arise from congruents. Those which are congruent of course, or if necessary may be made congruent by transformation, are called equal.
...

We may say that things having the same magnitude are equal. And magnitude is the attribute of a certain thing, whose definite appearance cannot be determined by any definition or thorough examination, but rather it is necessary to have a certain fixed measure that we may consult. Hence if God restored the universe and all its parts to their former state, preserving

the same proportions, nothing would be noticed. Nevertheless the magnitude of each other thing can be known by taking one thing to be fixed, indeed as a measure, which is applied to other things by counting the number of repetitions. And thus magnitude is determined by the number of parts that are equal among themselves, or unequal according to a certain rule. ... And hence equals are those which have the same mode to the same measure with respect to repetition, by which fact it is clear that they can be made congruent by decomposing them individually into parts in the same manner.¹⁴

The idea that ‘nothing would be noticed’ and the requirement that we be able to ‘consult’ a fixed measure correspond to quantity as ‘known through co-perception’. The idea that magnitude is the number of congruent parts (decomposed by the same measure) is the supposedly alternative definition of quantity through ‘reduction to congruents’. On my interpretation these are aspects of the same thing, meaning that in the context of geometry both considerations come together.

A (concrete) quantity is in general a thing in which repetition may be discerned. In geometry that thing is just the composite perceptual field considered together with the abstractions of place and extent. Co-perception describes geometric quantity under the aspect of the generation of the necessary composite presupposed in any quantity ascription. Reduction to congruents describes quantity under the aspect of the repetition one must discern in order to express a whole in terms of a given part. Thus they

¹⁴‘Ex congruis oriuntur aequalia. Nempe quae congrua sunt, aut transformatione si opus sit congrua reddi possunt, ea dicuntur aequalia. ... Aequalia etiam dici possunt quorum eadem est magnitudo. At magnitudo est attributum quoddam rerum, cuius certa species nulla definitione potest determinari nullisque certis notionibus, sed opus est fixa quadam mensura quam liceat consulere, et proinde si Deus universum orbem cum omnibus partibus proportionem eadem servata redderet maiorem, nullum esset principium id notandi. Una tamen re fixa sumta, tanquam mensura, hujus applicatione ad alias res adhibitisque repetitionum numeris magnitudo quoque aliarum cognosci potest. Atque ita magnitudo determinatur per numerum partium, quae inter se sunt aequales, vel certa quadam regula inaequales. ... Et proinde aequalia sunt quae eodem modo se habent ad eandem mensuram respectu repetitionis, eaque eo ipso patet iieri posse congrua, cum in partes congruentes singulas singulis eodem modo resolvantur.’ (VII.266 GM, my translation.)

both concern quantity, but from different aspects, and are completely compatible.

The second line of thought, that the foundational equivalence relation of the Analysis Situs is generated by combining the equivalence relations of equality and similarity, also needs correction.

According to my analysis, qualities exhaust the abstract concepts of geometric figures. The isomorphism class of a figure is a function only of qualities. Since the geometric characteristic is aimed at a characterisation of abstract geometric figures, which is to say, isomorphism classes, similarity is the foundational relation.

The relation of congruence only arises when attention is restricted to the parts of a single mathematical space so that everything can be decomposed by the same measure taken as unit. The relation of equality also applies only in this restricted context. One could never communicate a complete perceptual field together with an expression of the geometric characteristic (e.g. $A_1 A_2 \dots A_n$), so neither congruence nor equality could be the fundamental relation.

This response is sufficient to undermine the whole line of thought, since with it we lose the motivation to read the formula as a definition. This is a good thing, since to my knowledge no texts explicitly support such a reading.

On the contrary it seems quite possible, and apparently appropriate, to take the formula instead as a statement of fact, not as a definition. The formula has application only in the restricted context of a single mathematical space, but we can still ask what it means in that context. I suggest that in such a context it reduces to a tautology, as follows. Let A and B be two co-perceived figures. The judgement that they are equal is the judgement that they both can be decomposed according to the same measure as unit, call it o , after which they are expressed by the same number. The judgement that A and B are congruent is the judgement that they can be made to overlap by a rigid transformation, meaning a transformation that preserves the relative places of the parts within A and B separately. That means they have the same structure or ‘building plan’ in the sense that the relative places of the repeated o units correspond. This, in turn,

means they are similar: abstract structure is represented in the building plans given in terms of the simple unit *o*, so A and B must have the same abstract structure.¹⁵

This argument rests on two observations. First is that the abstract structure of a figure which is decomposed by some given unit reduces to the relative places of its parts among themselves. The second observation is that subfigures are congruent whenever their parts differ in absolute place but not place relative to just the other parts of the same subfigures. Equality ensures merely that the same parts, the simple building blocks of type *o*, are used in each case.

There is no longer a vicious circle. The formula is not a definition of congruence, but a tautologous statement of fact with application in the restricted context of a single mathematical space. The basic definition of quantity in the context of geometry is that in terms of congruent parts, but since the constituents of the abstract figures don't have determinate parts, the same concept of quantity is incomplete without the perceptual field.

Criticism 1c. Concluding that neither a definition in terms of co-perception nor a definition in terms of congruence would succeed, De Risi proposed that Leibniz eventually abandoned all hope, committing himself to the doctrine that quantity cannot be defined. A famous passage of the *New Essays* is particularly relevant:

... it is impossible to have the idea of an exact determinate length: no one can say or grasp in his mind what an inch or a foot is. And the signification of these terms can be retained only by means of real standards of measure which are assumed to be unchanging, through which they can always be re-established. An English mathematician, Mr Greaves, wanted to make use of the Egyptian pyramids, which have lasted for a rather long

¹⁵Note that the argument goes equally well in reverse, since I assume discontinuous transformations are allowed. The permissible transformations are any changes in the absolute places of the building blocks of type *o* in A and B that do not alter their relative places within A and B, respectively. Mirror inversions are allowed, so there are none of Kant's famous 'incongruent counterparts'.

time and are apparently going to last a while longer, to preserve our units of measurement, by indicating to posterity the proportions of the units to certain designated lengths on one of these pyramids.¹⁶

Since he thinks Leibniz only resigned himself to this after slowly accepting previous failures, De Risi casts Leibniz's alleged abandonment of his previous attempts to define quantity as a revelation he had long after beginning the project:

The breakthrough, in my opinion, comes only in Leibniz's ripe thought (in the late nineties, I would say), and consists not so much in his devising an *ad hoc* definition of quantity as in elaborating an articulated and rich doctrine of knowledge meant to show how intrinsically impossible it is to define the quantity concept in metaphysical terms. ... In fact, he will assume the phenomenological definition of quantity and quality as primitive, and the metaphysical definition of quality as a merely secondary characterization, while an analogous characterization proves impossible for quantity.¹⁷

De Risi links this idea that quantity is a real property which cannot be defined to his broader programme of bringing Leibniz into conversation with Kant. De Risi thinks that Leibniz was led, toward the end of his life, in the direction of a kind of transcendental turn. Quantity is a real property of a thing given in perception, but cannot be known through abstract definitions. The contrast is between determinate knowledge and the residue in perception, rather than the phenomenal and noumenal, but in both cases there is some significant way of knowing by which some real things cannot be known.¹⁸

¹⁶NE, II.xiii, p. 147; G, V.134.

¹⁷De Risi, *Geometry and monadology*, 146-7.

¹⁸Ibid., §3.

Response 1c. If my rejoinders to De Risi's criticisms (1a) and (1b) are accepted, we don't need to suppose Leibniz felt pushed by initial failures into a third view, a view that quantity cannot be defined. Furthermore, it is easy to incorporate Leibniz's later claims within the account of geometric quantity developed above without any suggestion that he significantly changed his view. I will try to show this in two parts. First I'll indicate that De Risi was not licensed to conclude that the *concept* of quantity is indefinable from statements that *particular* quantities are indefinable. Second, I'll explain how Leibniz's statements that *particular* quantities are indefinable fits into the account of geometric quantity I have presented and that I believe he maintained from 1679 to the end of his life.

De Risi explicitly affirms that Leibniz eventually thought it impossible 'to define the quantity concept in metaphysical terms'. But Leibniz does not say that the *concept* of quantity cannot be defined, he says that we cannot have the idea of a determinate length, meaning a particular concrete geometric quantity. It does not follow from the impossibility of defining or grasping in the mind the signification of 'one foot' that it is impossible to define or grasp in the mind the signification of 'length' or of 'geometric quantity'.

On the contrary, I take it Leibniz held the view that it is possible to define 'length' and 'geometric quantity' but impossible to define 'one foot' or any other specific length. Geometric quantity is defined as the affection of a whole or part of the perceptual field relative to its own parts. Using the abstractions of place and extent, a concrete quantity can be expressed as the repetition of a certain part called the unit or measure. When it comes to particular quantities, however, the question changes entirely. The question is now whether it is possible to give an abstract characterisation that distinguishes one specific length from all others.

The answer to this question is surely no. A pivotal feature of my account of Leibniz's concept of geometric quantity is that the information needed to give a characterisation of one specific length that distinguishes it from all others is not contained in

the abstract concepts, but is found only in the determinate parts of the perceptual field. The *abstract concept* of a foot, taken individually, is identical with the abstract concept of an inch, taken individually. (This follows from the fact that every line segment is similar.) There is no possibility of distinguishing them on purely abstract grounds.¹⁹ It is easy to distinguish them in the perceptual field, one simply points to an old ruler.

Regarding the chronology De Risi suggests, I will admit that I don't know of any passage dating from before 1695 in which Leibniz explicitly says that a unit of length cannot be defined. But three considerations would seem to weigh against De Risi's conclusion that the explicit statements after 1695 correspond to a change of mind.

First, it must be acknowledged that something very *close* to the idea *is* explicit in all his early writings since the 1679 *Characteristica Geometrica*. When Leibniz writes concerning similars (p. 45) that 'We can discover no attribute or property in one that we cannot also find in the other', he is only a very short step away from the 1695 statement from the *Specimen* (p. 83) that 'magnitude is the attribute of a certain thing, whose definite appearance cannot be determined by any definition or thorough examination'. When we examine two figures taken separately, and note their features, we are providing (at least nominal) definitions. As he said already in 1679, no list of properties or attributes, no *definitions*, we might say, can distinguish one unit of length from another.

Second, the 1695 passage not only does not treat the indefinability of a length unit as a novelty, it seems to assume that it is just the same consideration he has illustrated many times before by the thought experiment in which God shrinks everything around us — we couldn't notice without comparing to unshrunk things via 'co-perception'.

Lastly, it should be noted that Leibniz's relatively pointed statement in the *New Essays* is the immediate response to Locke's suggestion (through Philalethes) that in 'addition to what nature provides, men have settled in their minds the ideas of certain

¹⁹Unless, it should be added, they are taken together, in which case the extrinsic ratio of 1 : 12 for feet to inches can be added.

determinate lengths, such as an inch and a foot'. It is plausible that the weight placed in that passage on the impossibility of defining a unit of length is due to the fact that it serves to correct a specific claim to the contrary. The weight needn't suggest that Leibniz abandoned his older views.

3.2.2 Quantity and similarity

Criticism 2a. We discussed the idea that Leibniz needed bridge 'theorems of characterisation' to enable him to use the concepts of quantity and quality in mathematical practice since the definitions in terms of perception do not lend themselves to proofs. Quantity is supposed to be characterised in terms of reduction to congruents. De Risi takes it that similarity is characterised in terms of the ratios of straight lines and their angles of intersection; he draws this from the reasoning Leibniz employs when presenting his novel proof that equiangular triangles are equilateral and vice versa.

[H]ere we meet one of the greatest difficulties of the Leibnizean geometrical investigation. When Leibniz professes to deduce angular equality from the phenomenology of coperception, since he does not want directly to refer to the width of the angle he cannot but refer to the ratio between the sizes of the segments. But then, the operational characterization of similarity ends up for him by depending on the consideration of the length of the segments. Thus, the problem will become, how originally to define linear measure.²⁰

De Risi understood that similarity was meant to be identity of quality, and the thought that quality and quantity are supposed to be independent and exhaustive classes of attributes of abstract figures. He concluded that the operational characterisation of similarity should not depend on any geometric quantity. His criticism is an allegation

²⁰De Risi, *Geometry and monadology*, 142.

that the operational definition of similarity depends indirectly on quantity by helping itself to ratio.

It does not matter whether the criticism amounts to the accusation that Leibniz failed to define quality *independently* of quantity, or simply that the definition of quality is parasitic on quantity so the failure to define quantity undermines the definition of quality. My response is the same in either case.

Response 2a. As Leibniz explained to Clarke (p. 35 above), ratio is the abstraction from a real relation of situs that actually inheres in a thing. Ratio ‘is indeed out of the subjects’, an abstract, ideal thing. It is the number expressing one thing if the other is taken as unit, or the number by which one may be known if only the other is given (p. 32). Since ratio is an abstract quantity (a number), a ratio can be a constituent of the abstract concept of a mathematical figure. It follows that De Risi was mistaken to think ratio is illegitimately non-qualitative.

Why does De Risi think ratio depends on ‘the lengths of the segments’? Wouldn’t that make ratio reducible to the internal qualities of its relata? Isn’t that idea exactly what Leibniz was at pains to deny in his last letter to Clarke?

I do not claim to know what led De Risi to this, but one consideration may partially explain his statement. For although a ratio may be a constituent of the abstract structure of a figure, the *justification* for a ratio must be found in the actual determinate lengths of line segments in the perceptual field. The abstract concepts of two line segments are perfectly similar so their ratio, as an element of the abstract structure, cannot be *explained* on the basis of those concepts. The explanation of the presence of a ratio in the abstract structure will rely on the actual lengths of two parts of one perceptual field. I suppose this is *some* sense in which ratio depends on the lengths of segments. But of course this dependence is not a problem for Leibniz.

Here is an abstract mathematical figure: ‘Consider a triangle with side lengths in the ratio 3 : 4 : 5.’ We can prove that this is a right triangle. This is the sort of thing

mathematicians regularly do. We do not need first to define one metre or to render the side lengths in terms of metres.

Criticism 2b. In a footnote on congruence and mathematical form in subsequent mathematical history, De Risi writes:

... they [end-of-the-nineteenth-century geometers] must have thought ... that the determination of similarity was most unsuitable to constitute a primitive principle. The consideration of mathematical *form*, in fact ... was more and more distancing itself from the one elementarily identified through similarity, or the equality of angles, to come closer to more general concepts, which were rather to be found in the field of projective geometry (according to most of these scholars), or later on in the very general idea of topological transformation (according to which two figures have the same form provided there is a bicontinuous transformation that takes one into the other). We will have in fact to understand just why, among all possible geometrical relations, Leibniz intended the similarity relation as suitable to represent the formal aspect of geometry.²¹

This consideration is not set outright as a criticism. He expresses some puzzlement regarding the choice of similarity as the equivalence relation for geometric form. Especially in light of forthcoming developments in mathematics, it seemed to De Risi uncharacteristically short-sighted for Leibniz to have given such attention to the preservation of angles and ratios in a figure.

Response 2b. If Leibniz's concept of similarity really *meant* equality of angles, he apparently would have developed a precursor to *conformal* geometry. But his concept of similarity does *not* reduce to equality of angles.

²¹De Risi, *Geometry and monadology*, 153.

If my arguments are accepted, that similarity is identity of geometric form and geometric form is whatever can be characterised abstractly, then any mathematical framework purporting to capture the abstract structure of our actual perceptual field will have by that very fact its own claim to interpret the concept of similarity. Some modern mathematical structures, like indeed conformal spaces, are not given as attempts to formalise actual perception, so they would have no such claim. Others, like the locally Lorentz manifolds of relativity theory, would have a claim on the concept of similarity insofar as they are attempts to abstract from phenomena.

De Risi suggests that the concept of form involved in projective geometry is more general than Leibniz's, and that in smooth topology more general still. In what sense more general? I think De Risi means that they are more general in the sense that they include strictly less structure. Every projective space and every metric space is an instance of a topological space, for example. In this sense De Risi is right, since the structure Leibniz aimed to characterise abstractly was no more or less than can be drawn from the perceptual field, and that is certainly more structure than one might imagine considering.

But there is another sense in which Leibniz's geometric form was more general than all of these: Leibniz's similarity was not restricted to any particular formalism or structure type. It serves rather as a guide to choosing the right structure. It says the structure should arise out of the phenomenal world of bodies in space.

Criticism 2c. The last of De Risi's criticisms that we will consider aligns with one of Couturat's:

However, the project was a different one yet — namely, to disengage the consideration of similarity from the measure problem and, as a result, arrive at a science of quality that might be able to found angular equality neither on specific axioms of congruence nor on the determination of lengths. Which was of course an impossible task, because even when it deals

with similarity and affinity, Leibniz's elementary geometry still remains a *geometria magnitudinum*.²²

The measure problem is the problem of defining geometric quantity, and the alleged disengagement is the attempt to draw apart quality and quantity. We have already discussed this issue; I wish to focus here on the last sentence. This sentence summarises the same sentiment we found in Couturat's final judgement. The basic criticism is this: Leibniz never imagined a geometry without a metric, and the essence of geometric quantity is contained in metric relations. Therefore Leibniz cannot have hoped to succeeded in a project of separating quantity from quality and characterising quality alone.

Response 2c. Leibniz could respond in the same way here as to Couturat. The structure of a metric space is perfectly qualitative. Leibniz wished to develop a characteristic for ordinary metric geometry. But the *unit* of the metric in an abstract metric space is given in the space without a definition. One must look to the world, point to the parts of the perceptual field, to give significance to the abstract elements called '1'.

²²De Risi, *Geometry and monadology*, 159.

Conclusion

We have seen that the concept of quantity Leibniz wished to unfold in geometry was a species of a concept with much broader application. His most general metaphysical definition of quantity is ‘the affection of a thing to its parts’, particularly expressed as the repetition of a certain part chosen as unit or measure. Concrete quantities come in two kinds. Concrete numbers are things in which repetition may be discerned and a natural unit also discerned. Magnitudes are concrete quantities in which repetition may be discerned but a unit must be supplied by the discerner. Since a magnitude is concrete, a choice of unit can be grounded in the various exigencies of real life. Abstract numbers are the abstractions instantiated in concrete numbers and have a natural unit, the number one. Magnitudes do not come with a natural unit, so they do not instantiate any single abstraction. There are instead two related concepts that might be considered partial, even complementary, quantitative abstractions from magnitudes. Ratio captures the aspect of repetition of *some* unit, and takes the unit as an additional external argument. Extension captures the aspect of ‘affection to its parts’ but with only possible and no actual division into parts. Ratio is an incomplete concept of a quantitative relational affection; an extended thing is an incomplete concept of a concrete thing.

The concept of quantity in geometry combines elements from the abstractions of place and extent with additional information originating in some more concrete substratum. For this substratum I postulated the ‘perceptual field’, and argued that it satisfies three interpretive needs. It is first the composite that has quantity, meaning it supports an affection to its parts expressible by repetition of a determinate part chosen as measure. Leibniz explained to Clarke and others that bodies with place and extent are abstracted from a prior concrete phenomenal thing whose parts internally express their surroundings and their composition (have real situs and complexion). The second role

of the perceptual field was to supply this thing. The third and most critical interpretive role played by the same perceptual field was to explain the meaning of ‘co-perception’. Two figures are co-perceived, rendering them comparable in quantity, only when they are parts (or are abstracted from parts) of the same perceptual field.

Quality is the attribute or set of attributes of a figure that is a function only of the abstraction and not of the additional information given in a particular perceptual field. The qualities of a figure comprise a distinct conceptual layer. Quality in geometry is therefore best represented in modern terminology simply as the ‘abstract structure’ of a body or figure. The independence of quality and the dependence of quantity on a particular perceptual field led to a concept of ‘mathematical space’: a particular space is an abstract structure together with the particular perceptual field (whether real or imagined) from which it is abstracted. It follows that figures are comparable in quantity only when they are parts of the same mathematical space.

We can return with this to answer a question we posed at the beginning. We hoped to understand geometric quantity in order to understand Leibniz’s project of the *Analysis Situs*, and we had this hope because Leibniz seemed to use the distinction between quantity and quality to define and advertise his project. Now that we have a clear account of his concept of geometric quantity, we can approach that question expecting new insights.

Critics have in the main assumed that the distinction Leibniz drew between quantity and quality appears in the domain of purely abstract geometry. They have assumed furthermore that Leibniz’s project involved dividing the abstract domain into quantitative and qualitative constituents, and involved recommending the creation of a qualitative geometry. Leibniz sought, they thought, an abstract geometry uncontaminated by abstract quantity. And clearly this view was a necessary presupposition of suggestions to the effect that Leibniz was, or should have been (given his advertising), moving toward a different kind of geometric structure altogether, a structure without geometric quantity, as for example the structure of smooth topology or projective geometry that

would be developed only much later.

If I am correct that quality in geometry just *is* abstract structure, then this assumption must be abandoned. Geometric quantity requires ‘co-perception’ because quantity is restricted to a single perceptual field and hence doesn’t reside fully in the more abstract conceptual layer of quality. Leibniz did not aim at a division in mathematical structure, with his quantity-quality distinction, in order to promote one kind of structure and demote the other. Indeed abstract ratio or number, such as the ratios between the sides of a triangle or the number of its vertices, may well be perfectly qualitative features of a figure. This much Leibniz takes for granted when he uses ratios in demonstrations about geometric forms (similar triangles have the same side ratios). It is only the internal geometric quantities, the determinate lengths, areas, and volumes, that are excluded from abstract geometry. And they are excluded by nature, not stipulation.

What *was* the point of the distinction? The quantity-quality distinction can’t have been meant to shape or constrain the structure Leibniz recommended for abstract geometry if the distinction doesn’t even arise within abstract geometry, and doesn’t partition abstract attributes into qualitative and quantitative kinds. What was it meant for?

Let me suggest an alternative. The role of the distinction was to manifest, or expose, or to *point to*, distinctively geometric structure. Leibniz wanted to persuade his readers that *there is such a thing* as geometric quality by drawing on their ability to recognise the structure common to figures in different places and *of different sizes* (in the perceptual field). What these figures share, the part we could characterise or write down considering them one at a time, is their abstract geometric structure. Leibniz wanted to persuade his readers of this because it would not be possible to gain support for a project to develop a language specially tailored to distinctively geometric structure if people were unaware or unconvinced that there is such distinctively geometric structure.

What is special about size? If Leibniz only wanted to draw into public view our

ability to recognise the same form in different figures, could he not have achieved the same effect with place? Perhaps in principle he could have, but in practice, I would suggest, we are so accustomed to the notion that figures in different places have the same form that using place would not have had the same rhetorical effect. The realisation that figures of different size have the same form *feels like* a more substantive realisation. It makes a stronger impression.

(In modern geometry the feeling is substantiated by spaces of constant but non-zero curvature, in which figures can be moved but not scaled. Incidentally, this shows that while Leibniz may have *presupposed* a Euclidean geometry, he needn't be taken to have been *asserting* one. Faced with today's Riemannian geometry, he might have responded, consistent with the account developed above, that other abstract geometric structures may be worth considering, provided they are understood as abstractions *from perception*.)

My proposal amounts to the suggestion that the quantity-quality distinction was part of a sort of advertising strategy. Leibniz offered this consideration in order to expose something his audience didn't see clearly, namely that geometric figures have a peculiar abstract form. He wanted to establish this because his project was to develop a language to express these natures. Why could he not just say as much? Why did he need to illustrate it with this business involving perception? I suppose he thought his audience needed a stark, pointed illustration because they were otherwise oblivious. Maybe he thought their vision was clouded by entrenchment in another characteristic, one which wasn't tailored to geometry, and they lazily acquiesced in the science that evolved around that characteristic. But it was a different science.

This last point broaches another matter on which we have thus far said nothing, but which is relevant to these broad questions about Leibniz's intentions. I mean of course the matter of Cartesian coordinates, and the science that developed around the expression of geometry in the characteristic of algebra in which coordinates are managed. Leibniz engages another quantity-quality distinction with regard to these *coordinate*

quantities in advertising his project to Huygens. I have not discussed his complaints about Cartesian geometry because (at least in details) they are largely unrelated; his objections involve considerations about senses in which a characteristic is suitably ‘fitted’ or ‘tailored’ to its subject matter, rather than considerations involving perception and determinacy. But it happens that with regard to the broader question of his purpose in drawing a distinction between quality and coordinate-quantity, the approach I recommended for the distinction between quality and perception-quantity is similarly illuminating.

The corresponding interpretive suggestion is that with the quantity-quality distinction involving coordinates, Leibniz was also trying to point toward distinctively geometric structure, ultimately in order to gain adherents to his new programme. We can see this by looking briefly at his complaints against coordinate quantities.

Leibniz’s complaints against the Cartesian method, in his introductory letter to Huygens, can (arguably) be reduced to two judgements. First, that insofar as it describes geometry, it must import geometric structure from outside. Second, if geometric structure were naturally contained in the structure of coordinate quantities, there should not be a strain or felt mismatch between the characteristic and subject matter.

Illustrative of the first judgement is a complaint Leibniz lodges that the Pythagorean Theorem must be *added* to the Cartesian quantities in order to produce useful demonstrations about geometric figures in its algebraic language. This theorem cannot be proved quantitatively; it must be derived from ‘geometric’ reasoning. It follows that the structure of abstract numbers does not contain (in the relevant sense of ‘contain’) the structure of geometry.

A second judgement is that the Cartesian characteristic is ill-fitted to the subject of geometry. Descartes’s method requires an arduous initial translation of a figure into a coordinate definition before it is possible to proceed with the algebraic reasoning, and then of course after a demonstration is completed one must translate the result back into geometric terms. It is often not possible to understand an expression in the middle

of a demonstration in geometric terms, and finally the demonstrations are themselves often circuitous and opaque. For these reasons Leibniz thought the characteristic for quantity, namely algebra, is too general, and not adequately tailored to geometric structure. But this fact implies that *there is* a distinctively geometric structure.

The role of the distinction between quality and perception-quantity was to point to geometric structure *from the concrete*, from experience, from perception, from something less abstract. It shows that there is an abstract structure from a perspective prior to abstract forms. The quality and coordinate-quantity distinction works differently: it points to geometric structure *from the abstract*, meaning from a formal characteristic, namely the theory of abstract numbers. It shows that there is a distinctively geometric abstract structure from the perspective of *another* kind of abstract form, that of quantities. The first points from prior to the abstraction, the second from a different abstract structure. The goal is the same: to convince his readers (i.e. Huygens) that there is really something there worth studying, worth providing with a new and better artificial technical language.

So it happens that the original proposal, an interpretation of the quantity-quality distinction that was naturally generated by our analysis, also reveals a parallel interpretation of a similar distinction that he similarly ties to the foundation of his programme. This is excellent evidence, I submit, both for the analysis of the concept of quantity that generated both explanations, and for the resulting explanation itself of the role of both distinctions in the beginning of his programme. In the end, both quantity-quality distinctions had a primarily rhetorical role. Leibniz's goal was to win followers for his project, and his project was to give a naturally tailored artificial language for geometric structure, so he needed to persuade us that such structure exists and that it really is feasible to develop such a new language.

This occasions a suggestion for future work. Without being pressured to read his concrete proposals for a characteristic of geometric structure through the lens that they are attempts to characterise a structure *without* quantity, but on the contrary acknow-

ledging that geometric structure may well include certain kinds of abstract quantity, we should expect to gain new insights from a fresh look at his proposals. What, according to Leibniz, really is the nature of geometric structure? If we learn to read in his admittedly inchoate language, perhaps we will find out.

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