Analysis Geometrica Propria

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G.W. Leibniz, 1698

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A *properly Geometric Analysis*, and the *calculus of situation* connected with it, we will in some measure attempt here as a sample, lest perchance the idea be lost, which so far as I know has not come to mind for others, and which will provide far different applications from those furnished by Algebra. What should be understood is that magnitude and situs are different considerations. There is magnitude in things that don't have situs, such as number, proportion, time, velocity, and indeed wherever there exist parts of which an estimation can be made by number or i.e. by repetition. Therefore the doctrines of magnitude and of numbers are the same, and Algebra itself, or if you prefer Logistic, in treating of magnitude in general, actually treats of a variable or at least unspecified number. But situs adds a certain form over and above magnitude, or multitude of parts, as in figurate numbers. Hence, clearly, Algebra consists in Analysis properly and per se pertaining to Arithmetic, even if it is transferred to Geometry and situs, insofar as the magnitudes of lines and figures are treated. But then Algebra necessarily supposes much that is proper to Geometry or situs, which themselves also should be released [from it]. Our Analysis, therefore, effects this release assuming or supposing nothing more, and being not so much accommodated to magnitude as to situation itself per se. Moreover, for explaining the matter of situs we now use nothing but *congruence*, reserving *similarity* and *motion* to another place.

(1) Those are *Congruent* that can be substituted for one another in the same place, as ABC and CDA (fig. 65), which I designate thus: $ABC \simeq CDA$. Namely, for me, $\sim$ is the sign of similarity, and $=$ of equality, from which I compose the sign of congruence, since those that are simultaneously similar and equal are congruent.

(2) Those are reckoned to relate [se habere] in the same manner or *congruently* that correspond to each other in congruents. For example $AB.ABC \simeq CD.CDA$, for of course AB is to ABC as CD to CDA; and so $AB.AC \simeq CD.CA$ and $A.BC \simeq C.DA$, that is, the point A relates in the same manner to the line BC as the point C to DA. Indeed here it is done not so much by ratio or

proportion, but by any relation whatever.

(3) *Axiom*: If the determiners are congruent, so too will be those determined, supposing of course the same mode of determining. For example, if $A.B.C \simeq D.E.F$, then also the circumference of the circle through $A.B.C$ is congruent to the circumference of the circle through $D.E.F$, since the circumference of a circle is determined by three given points. And in general, in the calculation of congruents, determined things may be substituted for those sufficient for determining, just as equals are substituted for equals in common calculation. See below §26, §30, §32.

(4) It should be observed, so that the calculus is better understood, that when we say $A.B.C \simeq D.E.F$, it is the same as if we said simultaneously $A.B \simeq D.E.$ and $B.C \simeq E.F$ and $A.C \simeq D.F$, so that these can be made from that by *disjoining*, and that from these by *conjoining*. See below §26, 27, §29, §30, §31, §32.

(5) A *common boundary* is a locus which is-in two loci, such that it is not a part of them. In this way a point E is a locus which is-in the lines AE, CE, but is a part of neither. Therefore it is called their common boundary.

(6) A *section* is the entire common boundary of two parts constituting a whole and not having a common part. Thus AC is the entire common boundary of triangles ABC, CDA constituting the whole $ABCD$ and not having a common part.

(7) We will express this by the calculus through which geometry is traced back to logic thus: Every point existing in a proposed locus is designated by a common mark or letter (for example) X, and the locus itself is designated by $\bar{X}$, drawing a line over the letter. If some points of the locus are Y and Z, [their] loca will be $\bar{Y}$ or $\bar{Z}$. Let, therefore, the whole be $\bar{X}$, the constituting parts be $\bar{Y}$ and $\bar{Z}$, and $\bar{V}$ be the section, then these propositions can be formed: Every Y is X, every Z is X, since $\bar{Y}$ and $\bar{Z}$ *are-in* $\bar{X}$. But also, whatever is neither Y nor Z is not X, having set $\bar{Y}$ and $\bar{Z}$ to be *parts constituting* or i.e. exhausting the whole $\bar{X}$. Further, every V is Y, and every V is Z, since $\bar{V}$ is *in common* to $\bar{Y}$ and $\bar{Z}$, or i.e. is-in both. Finally, whatever is simultaneously Y and Z is also V, since $\bar{V}$ is the *section* or i.e. the entire common boundary, namely that contains whatever is common to both, and indeed they don't have a common part (or i.e. something besides a boundary). Hence all subalterns, conversions, oppositions, and inferences of logic have a place here, sometimes fruitfully, while other times they seem to be precluded from real [use] by human failing, not their own deficiency.

(8) The loca $\bar{X}$ and $\bar{Y}$ are *coincident* if every X is Y, and every Y is X. This I denote thus: $\bar{X} \cong \bar{Y}$.

(9) A *point* is the locus in which no other locus can be taken; therefore if the locus ☽ is taken in a point ⊙, then ☽ is coincident with ⊙, and if in turn ☽ is-in ⊙ and from this alone it is deduced that ⊙ and ☽ coincide, then ⊙ is a point.

Absolute space is the opposite of a point, for in space every other locus can be taken, as none [can be] in a point, so that the point is the simplest with respect to situation, and as it were the minimum, while indeed space is the most diffuse, and as it were the maximum.

(10) A *body* (mathematical, of course) or i.e. *solid* is a locus in which there is more than boundary. And this is what we mean when we attribute depth to a solid. Conversely, whatever is in a surface or curve can be understood to be a boundary, and is in common to something and another thing not having a common part with the first. There is an analogy here too, between point and solid, inasmuch as whatever is-in a point, is a point; and conversely whatever a solid is-in, is a solid. Likewise, a point cannot be-in something as a part; but a solid cannot be-in anything otherwise than as a part.

(11) A *plane* is a section of a solid having the same relation on both sides to what does not touch the boundaries of the solid, or i.e. having the same relation on both sides to what ends up in one part as in the other. If you slice an apple by a plane, the faces [extrema] of the two pieces [segmentorum], where they were conjoined, cannot be distinguished from each other.

(12) Therefore if it is an unbounded solid, it is absolutely true that the dividing plane has the same relation to both sides. If, on the contrary, the solid is bounded, it's enough that the boundaries do not enter the reckoning. And something done the same way on both sides will indeed have the same relation to the section itself.

(13) A *line* is a section of a plane having the same relation on both sides to what does not touch the boundaries of the plane.[1]

(14) If a plane AA is unbounded (fig. 66), and its section BB has the same relation on both sides, then BB will be an unbounded line.

(15) But if a plane CC is bounded (fig. 67), no matter what its shape, if we nonetheless cover the boundaries so that they don't appear or we in no way take them into account, then we will discover a dividing line DD to have the same relation on both sides, and it will be bounded.

(16) A curve, however, has a different relation on each side, since it is concave from one side, and convex from the other.

Now, everything that follows should be understood in the plane.[2]

(17) If there is a line (fig. 68) in which there are points A and B, and outside of it a point C on one side, then necessarily there is another D on the other side, which has the same relation to A and B that C has to them. For otherwise, since these points are in the line by hypothesis, one side would not relate to the line in such a way as the other, contrary to the definition of a line. Therefore, given $C.A.B$, one can find D such that $C.A.B \simeq D.A.B$.

(18) Therefore if there is a point X unique with its relation to two points $A.B$, it couldn't be on either side of the line through $A.B$, otherwise, by the preceding, it has some twin, contrary to hypothesis. Hence it necessarily falls on the same line on which what are twins elsewhere merge into one, since the

[1]Here Leibniz makes the following remark: What if someone doubted whether the plane could be so divided? Is it preferable, then, to form a line by section of two planes?

[2]It appears necessary for some property of the plane to enter the argument, such as that two planar figures with the same boundary are congruent, or i.e. that the interior is uniform. Marginal note by Leibniz.

line is also the common boundary of either side.[3]

(19) A straight line (bounded, of course) is consequently the locus of all points unique with their relation to two points in itself. If $X.A.B \simeq Z.A.B$ and for that reason X coincides with Z, then $\bar{X}$ is the line (unbounded) through $A.B$.[4]

(20) From this it is now obtained that two lines could not intersect each other except in one point, or that two lines, which have two points A and B in common, coincide with each other when extended, since either one is the locus of all points (and for that reason of the same [points]) unique with their relation to the points A and B. And so, given two points, a line is determined on which they fall.

(21) Hence, again, two lines which, extended of course, do not coincide, cannot have a segment in common. For if they have a segment AB in common (fig. 69), then they also have at least the two common points $A.B$; therefore, [being] extended, they coincide.

(22) Similarly, two lines cannot enclose space, otherwise they will intersect each other twice, and thus they will have two common points (fig. 70).

(23) Thus the Axioms that Euclid assumes without demonstration concerning the line are demonstrated from our definition of a line.

(24) A *circle* is produced by the motion of a line around a single point remaining stationary in the plane. The stationary extremum is the center, and the curve described by the other extremum is the circumference.

(25) Therefore (fig. 71) every point of the circumference, say X, will have the same relation to the center C, or i.e. every X will be to C as A is to C. As expressed in our calculus, this is: if $X.C \simeq A.C$, then $\bar{X}$ will be the *circumference of the circle.*

(26) The locus of all points having the same relation to two points is a *line.* Let C and D be two points (fig. 72), and the locus be $\bar{X}$, any point X of which has the same relation to C that it has to D; I claim that $\bar{X}$ is a line. In order to demonstrate this, let two points A and B be taken in the locus $\bar{X}$, and the line $\bar{Z}$ is drawn through A and B; it is determined from $A.B$ by §20. Now $A.B.C \simeq A.B.D$ by hypothesis, since A and B fall in $\bar{X}$; therefore (by an axiom in §3) the line through $A.B$ or i.e. $\bar{Z}$ also will have the same relation to C as to D, or i.e. $\bar{Z}.C \simeq \bar{Z}.D$; and now again $X.C \simeq X.D$ by hypothesis, and then by conjoining, $X.\bar{Z}.C \simeq X.\bar{Z}.D$. Therefore the point X cannot be on one side of the line $\bar{Z}$, such as (if you will) on the side of D; then indeed it would relate

[3]Leibniz remarks: It still needs to be demonstrated, conversely, that every point in a line is unique of its relation to two points in it.

So in turn, every point in the line through $A.B$ is unique of its relation to those two points. Let such a point be X; if it is not unique, then there exists Ω which is to $A.B$ as X is to $A.B$; since X is in the line through $A.B$ by hypothesis, then so too is Ω; there is therefore some order in the straight line among the four points $A.B.X.\Omega$, therefore $A.B.X$ and $A.B.\Omega$ are not $\simeq$. Supposing that the locus is a *curve*, there is not an order of all points in the (linear?) curve. Therefore there are two points in it having the same relation to two points in the same [curve], and so the locus determined with respect to the two is a curve.

[4]In general, every point in a curve not returning onto itself is unique with its relation to two points taken in it. Remark by Leibniz.

differently to the line $\bar{Z}$ and to D, than to the line $\bar{Z}$ and to C; and so necessarily X falls in $\bar{Z}$ or i.e. every X will be Z, whence also $\bar{X}$ falls in $\bar{Z}$, which is what we wanted to prove.

(27) Here we have a not inelegant sample of the calculus following the rule of §4. Indeed, since $X.\bar{Z} \simeq X.\bar{Z}$, which is an identity, and $X.C \simeq X.D$ by hypothesis, and $\bar{Z}.C \simeq \bar{Z}.D$, which we proved from the nature of the line, from all these congruent doubles considered individually, it follows that the triples fused from them are also congruent, or that by conjoining, $\underbrace{X \underset{\smile}{.} \bar{Z} \underset{\smile}{.} C} \simeq \underbrace{X \underset{\smile}{.} \bar{Z} \underset{\smile}{.} D}$.

(28) Hence if $X.C \simeq X.D$, then $\bar{X}$ will be a *line*, which congruence is of the utmost utility in our calculus. And conversely, if $\bar{X}$ is a line, there should exist points of such kind as C and D, so that a congruence has [that] locus.

(29) It cannot happen that a line has the same relations to three points of the plane, or i.e. that $X.C \simeq X.D \simeq X.E$ (fig. 73). For if this were so, then by conjoining, also $X.C.E \simeq X.D.E$ would hold, and thus E cannot be to either side. But the same E cannot be in $\bar{X}$, for indeed then also C and D would be in $\bar{X}$, and moreover, they would coincide with E, otherwise some point of the line (of course, E itself) would relate differently to C and to D than to E, therefore a point E cannot be found in addition to C and D.

(30) A circle does not intersect a circle in more than two points. If two circular circumferences be $\bar{X}$ and $\bar{Z}$ (fig. 74), I say they cannot cross except in two points, such as L and M. Indeed, let the center of $\bar{X}$ be A, the center of $\bar{Z}$ be B. Since now L is X and M is X, then $L.A \simeq M.A$, and since L is Z and M is Z, then $L.B \simeq M.B$, both of them from the nature of the circumferences to which the points are common, by §25. Then by conjoining, $L.A.B \simeq M.A.B$. Suppose $\bar{Y}$ is the line through $A.B$, so that now $L.\bar{Y} \simeq M.\bar{Y}$ by the axiom of §3; but if there is some additional point N common to the two circumferences, we would have $L.\bar{Y} \simeq M.\bar{Y} \simeq N.\bar{Y}$, or i.e. the line $\bar{Y}$ would have the same relation to the three points L, M, N, which cannot happen by the preceding. Hence it follows that by three given points a circumference is determined, in which they in-are; since they cannot simultaneously in-be in multiple [circumferences].

(31) If a circle is tangent to a circle (fig. 75), the point of contact is in the same line as the centers. Let the centers be A and B, and the point of contact C, where of course two points of intersection merge. Thus (by the preceding) the circle does not intersect the circle in addition [to C], otherwise there will be three intersection points. Therefore a single point of contact is common to the two circles. I claim it is in the line through $A.B$. This is apparent from §19, if it is shown to be unique of its relation to $A.B$. Let there be another, if it can happen, F, and $F.A.B \simeq C.A.B$ should hold; therefore by disjoining, both $F.A \simeq C.A$, and likewise $F.B \simeq C.B$; therefore F falls in both circumferences, and thus either they coincide with C, or C is not alone in common, which is absurd.

(32) A line and a circle cannot intersect each other in more than two points. If L and M (fig. 76) are in line $\bar{X}$, and likewise in circumference $\bar{Z}$ around C, I say that there could not be another point N besides L and M. Let D be taken

with the same relation to the line $\bar{X}$ as C on the other side, by §17. On account of the circle, $L.C \simeq M.C \simeq N.C$ holds, and therefore since the points L, M, N in the line have the same relation to D as to C, then also $L.D \simeq M.D \simeq N.D$; therefore by conjoining, $L.C.D \simeq M.C.D \simeq N.C.D$. Let $\bar{Y}$ be the line through $C.D$; then (by the axiom of §3) $L.\bar{Y} \simeq M.\bar{Y} \simeq N.\bar{Y}$ will hold, or i.e. the line $\bar{Y}$ will have the same relation to the three points L, M, N, which cannot happen by §29.

And so we have expressed the fundamentals of the line and the circle, that is to say, how these loca can intersect: a line with a line, a circle with a circle, a line with a circle, by the intersections of which others are determined. The consequence of this is that the others can also be handled by our calculus.